Supplementary Material

A strategy for the design of skyrmion racetrack memories

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I. THIELE EQUATION OF MOTION FOR A MAGNETIC SKYRMION IN THE PRESENCE OF THE SPIN HALL EFFECT

In this section, we derive the Thiele equation (Eq. (1) in the main text [S1]), in the presence of the spin-Hall effect (SHE) and the interfacial Dzyaloshinskii–Moriya interaction (DMI) (Fig. 1, scenario B) by neglecting the STT contribution and confinement effects,

$$\mathbf{G}\times\mathbf{v} - \alpha_G \overleftrightarrow{\mathcal{D}}\cdot\mathbf{v} + 4\pi B\overleftrightarrow{\mathcal{R}}\left(\phi_0 = 0\right)\cdot\mathbf{j_{HM}} = 0, \tag{S1}$$

being $\mathbf{G}$ the "gyrocoupling vector", α_G the Gilbert damping, $\overleftrightarrow{\mathcal{D}}$ the dissipative tensor, $\overleftrightarrow{\mathcal{R}}$ the in-plane rotation matrix, $\mathbf{v}=\dot{\mathbf{R}}'(t)$ ("$\cdot$" denotes the time derivative) with $\mathbf{v}=\left(v_x,v_y\right)$ and $\mathbf{R}'=(X,Y)$ the drift velocity and the core position of the magnetic skyrmion, respectively and B is linked to the SHE. The magnetization distribution of the magnetic skyrmion can be represented in the form $\mathbf{m}(\mathbf{r}')=\sin\theta(\rho)\cos\phi_0\,\hat{\rho}+\sin\theta(\rho)\sin\phi_0\,\hat{\phi}+\cos\theta(\rho)\hat{z}$ with $\mathbf{r}'=(x,y)$ [S2]. The Thiele equation is written for the Néel skyrmion obtained by setting $\phi_0=0$, but it can be generalized also to the Bloch skyrmion by setting $\phi_0=\frac{\pi}{2}$ in $\mathbf{m}(\mathbf{r}')$ as discussed at the end of this section.

A. DERIVATION OF THE THIELE EQUATION IN THE PRESENCE OF THE SPIN HALL EFFECT

The Eq. (3) in the main text (dimensionless form) and in the absence of STT terms can be rewritten as:

$$\frac{\partial\mathbf{m}}{\partial t} = -\gamma_0\left(\mathbf{m}\times\mathbf{h_{eff}}\right)+\alpha_G\left(\mathbf{m}\times\frac{\partial\mathbf{m}}{\partial t}\right)-b'\mathbf{m}\times\left[\mathbf{m}\times\left(\hat{z}\times\mathbf{j_{HM}}\right)\right], \tag{S2}$$

where $b'=\gamma_0\dfrac{\hbar\theta_{SH}}{2eM_sL}$, being γ_0 the gyromagnetic ratio, $\hbar$ the reduced Planck constant, θ_{SH} the spin-Hall angle, e the electron charge, M_s the saturation magnetization, L the thickness of the ferromagnetic layer. $\mathbf{m}$ and $\mathbf{h_{EFF}}$ are the normalized magnetization and the effective field of the ferromagnet, $\hat{z}$ is the unit vector of the out-of-plane direction and $\mathbf{j_{HM}}=(j_{HM},\,0)$ is the in-plane current injected via the heavy metal. The corresponding Thiele equation for the magnetic skyrmion is obtained by projecting the equation of motion onto the relevant translational modes. This mathematical procedure is achieved in three steps:

1) by considering the cross product of both members of Eq. (S2) by **m**;

2) multiplying the result by $\hat{\mathrm{T}}^{\alpha}\,\mathbf{m}=\frac{\partial}{\partial r'_{\alpha}}\mathbf{m}$ with α =(x,y) where $\hat{\mathrm{T}}$ is the generator of the translational mode expressing the skyrmion motion;

3) integrating over a two-dimensional unit cell of the lattice [S3].

It is also assumed that the shape of the magnetic skyrmion remains constant (this is also confirmed by micromagnetic simulations) during its motion, taking into account, for the core position as a function of time, the form $\mathbf{m} = \mathbf{m_0}(\mathbf{r}' - \mathbf{R}'(t))$, where $\mathbf{m_0}(\mathbf{r}')$ is a static skyrmion profile centred at $\mathbf{r}' = \mathbf{0}$ and $\mathbf{R}'(t) = \mathbf{v}\,t$.

For steady motion, it is $\frac{\partial \mathbf{m}}{\partial t} = -\dot{\mathbf{R}}'(t)\cdot\nabla\mathbf{m}$, a relation also used for manipulating the terms depending on the time derivative of the reduced magnetization in the derivation of the Thiele equation. Due to the translational invariance, the term proportional to the effective field $\mathbf{h_{eff}}$ and to the generator of the translational mode integrated over the unit cell vanishes [S3].

The term linked to the time derivative of the reduced magnetization on the first member of the equation of motion gives rise to the well-known "gyro-coupling vector" $\mathbf{G}$ =(0,0,G) with $\mathbf{G} = -4\pi Q\hat{z}$ and

$$Q=\frac{1}{4\pi}\int_{\text{unit cell}} d\mathbf{r}'\,\mathbf{m}(\mathbf{r}')\cdot\left(\frac{\partial \mathbf{m}(\mathbf{r}')}{\partial x}\times\frac{\partial \mathbf{m}(\mathbf{r}')}{\partial y}\right), \tag{S3}$$

the winding (skyrmion) number, viz. Q =-1 [S2].

The dissipative tensor $\overleftrightarrow{\mathcal{D}}$, arising from the term proportional to the Gilbert damping coefficient, takes the form multiplying by $\frac{1}{4\pi}$:

$$\mathcal{D}_{\alpha\beta}=\frac{1}{4\pi}\int_{\text{unit cell}} d\tilde{\mathbf{r}}'\left(\frac{\partial \mathbf{m}(\mathbf{r}')}{\partial \tilde{r}'_{\alpha}}\cdot\frac{\partial \mathbf{m}(\mathbf{r}')}{\partial \tilde{r}'_{\beta}}\right), \tag{S4}$$

where $\tilde{r}'_{\alpha}$ is a rescaled length, $\tilde{r}'_{\alpha}\rightarrow\frac{r'_{\alpha}}{L_{\text{sc}}}$, with L_{sc} a typical scaling length. In particular, for a single skyrmion it is $\mathcal{D}_{\alpha\beta}=\mathcal{D}$ for $\alpha=\beta=x$ or $\alpha=\beta=y$ and $\mathcal{D}_{\alpha\beta}=0$ for $\alpha\neq\beta$.

The spin Hall term of Eq.(S2) is associated to the integral:

$$H_{\alpha\beta}=-\frac{1}{4\pi}\int_{\text{unit cell}} d\tilde{\mathbf{r}}'\left(\frac{\partial \mathbf{m}(\mathbf{r}')}{\partial \tilde{r}'_{\alpha}}\times\mathbf{m}(\mathbf{r}')\right)_{\gamma}\varepsilon_{\gamma\beta} \tag{S5}$$

where the summation over the repeated indexes is assumed and $\varepsilon_{\gamma\beta}$ is the Levi-Civita symbol. For a single magnetic skyrmion, $H_{\alpha\beta}=-I\, R_{\alpha\beta}\ (\phi_0)$ where $I=\frac{1}{4}\int_0^{\infty} d\tilde{\rho}\left(\sin\theta\cos\theta+\tilde{\rho}\frac{d\theta}{d\tilde{\rho}}\right)$ is a dimensionless integral in the dimensionless radial variable $\tilde{\rho}$ and $R_{\alpha\beta}\ (\phi_0)$ is the in-plane rotation matrix performing a counterclockwise rotation of an angle ϕ_0 [S2]. The coefficient B in Eq. (S1) is expressed by $B=b'L_{sc}I$. Combining together Eq. (S3-S5) for $\phi_0=0$ and, taking into account Eq. (S2), the corresponding Thiele equation for the Néel magnetic skyrmion (Eq. (1) in the main text and Eq. (S1)) is obtained.

B. SOLUTION TO THE THIELE EQUATION IN THE PRESENCE OF THE SPIN HALL EFFECT

The solution to the Thiele equation for a magnetic skyrmion in the presence of the spin-Hall effect can be easily derived. By expressing explicitly Eq. (S4) and Eq. (S5) in terms of components and by considering the electrical current flowing along the x-direction, namely $\mathbf{j}_{\mathbf{HM}}=\left(j_{HM},0\right)$, the Thiele equation can be written in the matrix form:

$$\begin{pmatrix} \varepsilon_{xx}-\alpha_G\mathcal{D} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{xx}-\alpha_G\mathcal{D} \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix}=-B\begin{pmatrix} R_{xx}(\phi_0) & R_{xy}(\phi_0) \\ R_{yx}(\phi_0) & R_{yy}(\phi_0) \end{pmatrix}\begin{pmatrix} j_{HM} \\ 0 \end{pmatrix}, \tag{S6}$$

with $R_{xx}(\phi_0)=\cos\phi_0$, $R_{xy}(\phi_0)=\sin\phi_0$, $R_{yx}(\phi_0)=-\sin\phi_0$, $R_{yy}(\phi_0)=\cos\phi_0$.

For the Néel skyrmion ($\phi_0=0$ in the skyrmion magnetization distribution) Eq. (S6) can be cast in the form:

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix}=-B\begin{pmatrix} -\dfrac{\alpha_G\mathcal{D}}{1+\alpha_G{}^2\mathcal{D}^2} & \dfrac{1}{1+\alpha_G{}^2\mathcal{D}^2} \\ -\dfrac{1}{1+\alpha_G{}^2\mathcal{D}^2} & -\dfrac{\alpha_G\mathcal{D}}{1+\alpha_G{}^2\mathcal{D}^2} \end{pmatrix}\begin{pmatrix} j_{HM} \\ 0 \end{pmatrix}, \tag{S7}$$

yielding (Eq. (2) in the main text):

$$\begin{cases} v_x = \dfrac{\alpha_G\mathcal{D}B}{1+\alpha_G{}^2\mathcal{D}^2}j_{HM}, \\ v_y = \dfrac{B}{1+\alpha_G{}^2\mathcal{D}^2}j_{HM}. \end{cases} \tag{S8}$$

In the calculations shown in Fig. 3b of the main text (green line with triangles) corresponding to scenario B we have considered α_G = 0.015, L_{sc} = 1000 nm, that is a scaling length equal to the strip dimension along *y,* where the Néel skyrmion motion takes place and corresponding to the unit cell size. From micromagnetic simulations we have obtained I = -0.010 and $\mathcal{D} \approx 1$, leading to $v_x << v_y$. Note that, in the units used, it is $B > 0$ and both components of the drift velocity have the same sign as found in micromagnetic simulations. The agreement with the numerical results is excellent.

Straightforwardly, the solution to the Thiele equation for a current density flowing along the *x*-direction can be obtained also for the Bloch skyrmion of the scenario D ($\phi_0 = \frac{\pi}{2}$ in the skyrmion magnetization distribution) in the form:

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = B \begin{pmatrix} -\frac{\alpha_G \mathcal{D}}{1+\alpha_G{}^2\mathcal{D}^2} & \frac{1}{1+\alpha_G{}^2\mathcal{D}^2} \\ -\frac{1}{1+\alpha_G{}^2\mathcal{D}^2} & -\frac{\alpha_G \mathcal{D}}{1+\alpha_G{}^2\mathcal{D}^2} \end{pmatrix} \begin{pmatrix} 0 \\ j_{HM} \end{pmatrix}, \tag{S9}$$

leading to:

$$\begin{cases} v_x = \frac{B}{1+\alpha_G{}^2\mathcal{D}^2} j_{HM}, \\ v_y = -\frac{\alpha \mathcal{D} B}{1+\alpha_G{}^2\mathcal{D}^2} j_{HM}, \end{cases} \tag{S10}$$

which gives $v_y << v_x$, confirming again our numerical predictions. Note that in this case the two components of the drift velocity have opposite signs as found in micromagnetic simulations.

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