Online Appendix for "Surrogate measures and consistent surrogates" by TJ VanderWeele

The following definitions and results were given in work on signed causal directed acyclic graphs (VanderWeele et al., 2008; VanderWeele and Robins, 2009, 2010). Lemma 1 below was stated in somewhat greater generality in VanderWeele and Robins (2009) but the special case given below will suffice here.

Definition 1. Suppose that variable A is a parent of some variable Y and let $\widetilde{pa}_Y$ denote the parents of Y other than A. We say that A has a distributional positive monotonic effect on Y if the survivor function $S = (y|, \widetilde{pa}_Y) = pr(Y > y|A = a, \widetilde{pa}_Y)$ is such that whenever $a_1 \geq a_0$ we have $S(y|a_1, \widetilde{pa}_Y) \geq S(y|a_0, \widetilde{pa}_Y)$ for all y and all $\widetilde{pa}_Y$; the variable A is said to have a distributional negative monotonic effect on Y if whenever $a_1 \geq a_0$ we have $S(y|a_1, \widetilde{pa}_Y) \leq S(y|a_0, \widetilde{pa}_Y)$ for all y and all $\widetilde{pa}_Y$.

Definition 2. An edge on a causal directed acyclic graph from A to Y is said to be of positive or negative sign if respectively A has a distributional positive or negative monotonic effect on Y; if an edge is neither positive nor negative, it is said to be without a sign. The sign of a path on a causal directed acyclic graph is the product of the signs of the edges that constitute that path. If one of the edges on a path is without a sign then the sign of the path is said to be undefined.

Lemma 1. Let X denote some set of non-descendents of A that blocks all backdoor paths from A to Y. If all directed paths between A and Y are of positive sign then $pr(Y > y|a, x)$ is non-decreasing in a for all y; if all directed paths between A and Y are of negative sign then $pr(Y > y|a, x)$ is non-increasing in a for all y.

Lemma 1 immediately implies Proposition 4 in the text, stated again below.

Proposition 4: In the causal diagram in Figure 2, if (a) $pr(Y > y|a, s, u)$ is non-decreasing in a and s for all y, u and (b) $pr(S > s|a, u)$ is non-decreasing in a for all s, u

then $pr(Y_a > y)$ is non-decreasing in a.

To prove Proposition 3 in the text we consider a variant of the definitions and results in VanderWeele and Robins (2009, 2010).

Definition 3. Suppose that variable A is a parent of some variable Y and let $\widetilde{pa}_Y$ denote the parents of Y other than A. We say that A has an average positive monotonic effect on Y if whenever $a_1 \geq a_0$ we have $E(Y|a_1, \widetilde{pa}_Y) \geq E(Y|a_0, \widetilde{pa}_Y)$ for all y and all $\widetilde{pa}_Y$; the variable A is said to have an average negative monotonic effect on Y if whenever $a_1 \geq a_0$ we have $E(Y|a_1, \widetilde{pa}_Y) \leq E(Y|a_0, \widetilde{pa}_Y)$ for all y and all $\widetilde{pa}_Y$.

Definition 4. A directed path which is of positive sign with the exception that the parent of the final edge may only have an average monotonic effect on the child, rather than a distributional monotonic effect, will be said to be a directed path with mean positive sign. A directed path which is of negative sign with the exception that the parent of the final edge may only have an average monotonic effect on the child, rather than a distributional monotonic effect, will be said to be a directed path with mean negative sign.

Lemma 2. Let X denote some set of non-descendents of A that blocks all backdoor paths from A to Y. If all directed paths between A and Y are of mean positive sign then $E(Y|a, x)$ is non-decreasing in a for all x; if all directed paths between A and Y are of mean negative sign then $E(Y|a, x)$ is non-increasing in a for all x.

The proof of Lemma 2 follows from the proof of Proposition 4 in VanderWeele and Robins (2009) by simply replacing "$E[1(Y > y)|...]$" by "$E[Y|...]$" wherever the former expression appears in the proof. Lemma 2 immediately implies Proposition 3 in the text, stated again below.

Proposition 3: In the causal diagram in Figure 2, if (a) $E(Y|a, s, u)$ is non-decreasing in a and s for all u and (b) $pr(S > s|a, u)$ is non-decreasing in a for all s, u then $E(Y_a)$ is non-decreasing in a.