Ecology Letters Supporting Information

Understanding patterns and processes in models of trophic cascades

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Appendix S1. Analytical steady state solutions to differential equations describing food chain model dynamics

Table S1. Analytical steady state solutions to equations 12 and 13 in the main text, for a Lotka Volterra system with density-dependent mortality regulation.

Rate equations	Steady state solutions for x_1 and x_2
$\frac{dx_1}{dt} = \sigma x_1 - x_1 x_2 - \lambda_1 x_1^2$	$x_1^* = \frac{(\delta + \lambda_2 \sigma)}{(1 + \lambda_1 \lambda_2)}$
$\frac{dx_2}{dt} = x_1 x_2 - \delta x_2 - \lambda_2 x_2^2$	$x_2^* = \frac{(\sigma - \lambda_1 \delta)}{(1 - \lambda_1 \lambda_2)}$

Table S2. Analytical steady state solutions to equations 14 and 15 in the main text, for a Lotka Volterra system with consumer density-dependent uptake regulation.

Rate equations	Steady state solutions for x_1 and x_2
$\frac{dx_1}{dt} = \sigma x_1 - \frac{x_1 x_2}{(\rho_2 + \upsilon_2 x_2)}$	$x_1^* = \frac{\delta \rho_2}{(1 - \sigma \upsilon_2)}$
$\frac{dx_2}{dt} = \frac{x_1 x_2}{(\rho_2 + \upsilon_2 x_2)} - \delta x_2$	$x_2^* = \frac{\sigma \rho_2}{(1 - \sigma \upsilon_2)}$

Table S3. Analytical steady state solutions to equations 16 - 18 in the main text, for a 3-level food chain (n=2) with a linear prey dependency of consumer uptake rate and top-level mortality regulation, with and without chemostat bottom-level regulation.

Rate equations	Steady state solutions for x_0, x_1 and x_2 without bottom-level regulation (φ = 0)	Steady state solutions for x_0, x_1 and x_2 with chemostat bottom-level regulation (φ = 1)
$\frac{dx_0}{dt} = \varphi(I - a_1 x_0 x)$	$x_0^* = x(0)$	$x_0^* = \frac{I}{a_1 x_1^*}$
$\frac{dx_1}{dt} = \varepsilon_1 a_1 x_0 x_1 - a_2 x_1 x_2$	$x_1^* = \frac{\delta + \lambda_2 x_2^*}{\varepsilon_2 a_2}$	$x_1^* = \frac{\varepsilon_1 I}{a_2 x_2^*}$
$\frac{dx_2}{dt} = \varepsilon_2 a_2 x_1 x_2 - \delta x_2 - \lambda_2 x_2^2$	$x_2^* = \frac{\varepsilon_1 a_1 x_0^*}{a_2}$	$x_2^* = \begin{cases} \frac{-\delta + \sqrt{\delta^2 + 4 \varepsilon_1 \varepsilon_2 I \lambda_2}}{2\lambda_2} & \lambda_2 \neq 0 \\ \frac{\varepsilon_1 \varepsilon_2 I}{\delta} & \lambda_2 = 0 \end{cases}$

Table S4. Analytical steady state solutions to equations 19 - 21 in the main text, for a 3-level food chain (n=2) with consumer uptake rate determined by linear prey-dependency and consumer density-dependence, with and without chemostat bottom-level regulation.

Rate equations	Steady state solutions for x_0, x_1 and x_2 without bottom-level regulation (φ = 0)	Steady state solutions for x_0, x_1 and x_2 with chemostat bottom-level regulation (φ = 1)
$\frac{dx_0}{dt} = \varphi(I - a_1 x_0 x)$	$x_0^* = x(0)$	$x_0^* = \frac{I}{a_1 x_1^*}$
$\frac{dx_1}{dt} = \varepsilon_1 a_1 x_0 x_1 - \frac{a_2 x_1 x_2}{\rho_2 + x_2}$	$x_1^* = \frac{\delta x_2^*}{\varepsilon_1 \varepsilon_2 a_1 x_0^*}$	$x_1^* = \frac{\varepsilon_1 I}{a_2}\left(\frac{\rho_2}{x_2^*} + \upsilon_2\right)$
$\frac{dx_2}{dt} = \frac{\varepsilon_2 a_2 x_1 x_2}{\rho_2 + x_2} - \delta x_2$	$x_2^* = \frac{\varepsilon_1 a_1 x_0^* \rho_2}{a_2 - e_1 a_1 x_0^* \upsilon_2}$	$x_2^* = \frac{\varepsilon_1 \varepsilon_2 I}{\delta}$

Table S5. Analytical steady state solutions to equations 22 - 24 in the main text, for a 3-level food chain (n=2) with a Type-II prey-dependency and top-level mortality regulation, with and without chemostat bottom-level regulation

Rate equations	Steady state solutions for x_0, x_1 and x_2 without bottom-level regulation ($\varphi = 0$)	Steady state solutions for x_0, x_1 and x_2 with chemostat bottom-level regulation ($\varphi = 1$)
$\frac{dx_0}{dt} = \varphi\left(I - \frac{a_1 x_0 x_1}{k_0 + x_0}\right)$	$x_0^* = x(0)$	$x_0^* = \frac{I k_0}{\left(a_1 x_1^* - I\right)}$
$\frac{dx_1}{dt} = \frac{\varepsilon_1 a_1 x_0 x_1}{k_0 + x_0} - \frac{a_2 x_1 x_2}{k_1 + x_1}$	$x_1^* = \frac{a_2 x_2^* (k_0 + x_0^*)}{\varepsilon_1 a_1 x_0^*} - k_1$	$x_1^* = \frac{\varepsilon_1 k_1 I}{\left(a_2 x_2^* - \varepsilon_1 I\right)}$
$\frac{dx_2}{dt} = \frac{\varepsilon_2 a_2 x_1 x_2}{k_1 + x_1} - \delta x_2 - \lambda_2 x_2^2$	$x_2^* = \frac{(\varepsilon_2 a_2 - \delta) \pm \sqrt{(\delta - \varepsilon_2 a_2)^2 + \frac{4\lambda_2 \varepsilon_2 \varepsilon_1 a_1 k_1 x_0^*}{(k_0 + x_0^*)}}}{2\lambda_2}$	$x_2^* = \begin{cases} \frac{-\delta + \sqrt{\delta^2 + 4\varepsilon_1 \varepsilon_2 I \lambda_2}}{2\lambda_2} & \lambda \neq 0 \\ \frac{\varepsilon_1 \varepsilon_2 I}{\delta} & \lambda = 0 \end{cases}$

Table S6. Analytical steady state solutions to equations 25 - 27 in the main text, for a 3-level food chain (n=2) with consumer density-dependent non-linear (Type-II) prey-dependency, with and without chemostat bottom-level regulation.

Rate equations	Steady state solutions for x_0, x_1 and x_2 without chemostat bottom-level regulation ($\varphi = 0$)	Steady state solutions for x_0, x_1 and x_2 with chemostat bottom-level regulation ($\varphi = 1$)
$\frac{dx_0}{dt} = \varphi\left(I - \frac{a_1 x_0 x_1}{k_0 + x_0}\right)$	$x_0^* = x(0)$	$x_0^* = \frac{I k_0}{\left(a_1 x_1^* - I\right)}$
$\frac{dx_1}{dt} = \frac{\varepsilon_1 a_1 x_0 x_1}{k_0 + x_0} - \frac{a_2 x_1 x_2}{k_1 + x_1 + \upsilon_2 x_2}$	$x_1^* = \frac{\delta x_2 (k_0 + x_0^*)}{\varepsilon_1 \varepsilon_2 a_1 x_0^*}$	$x_1^* = \frac{\varepsilon_1 I\left(k_1 + \rho_2 x_2^*\right)}{\left(a_2 x_2^* - \varepsilon_1 I\right)}$
$\frac{dx_2}{dt} = \frac{\varepsilon_2 a_2 x_1 x_2}{k_1 + x_1 + \upsilon_2 x_2} - \delta x_2$	$x_2^* = \frac{\varepsilon_1 \varepsilon_2 a_1 k_1 x_0^*}{\varepsilon_2\left(a_2 k_2 + a_2 x_0^* - \varepsilon_1 a_1 \rho_2 x_0^*\right) - \delta\left(k_0 + x_0^*\right)}$	$x_2^* = \frac{\varepsilon_1 \varepsilon_2 I}{\delta}$

Figure S1. Analytical steady states of a simple food chain system with non-linear Type-II prey-dependency (equations 22-27), to top-down (TD) and bottom-up (BU) forcing, depending on whether regulation is located at the bottom-level (BL), top-level (TL) or both, of the food chain. Parameters used in the analysis were $a_i = 1$; $\varepsilon_i = 1$, $\delta = 0.1$ (except when treated as the driving variable), $\lambda = 0.1$, $\upsilon = 0.1$. Solid lines for cases 2, 3, 5 & 6 show the response with top-level mortality regulation, dashed with top-level uptake regulation.

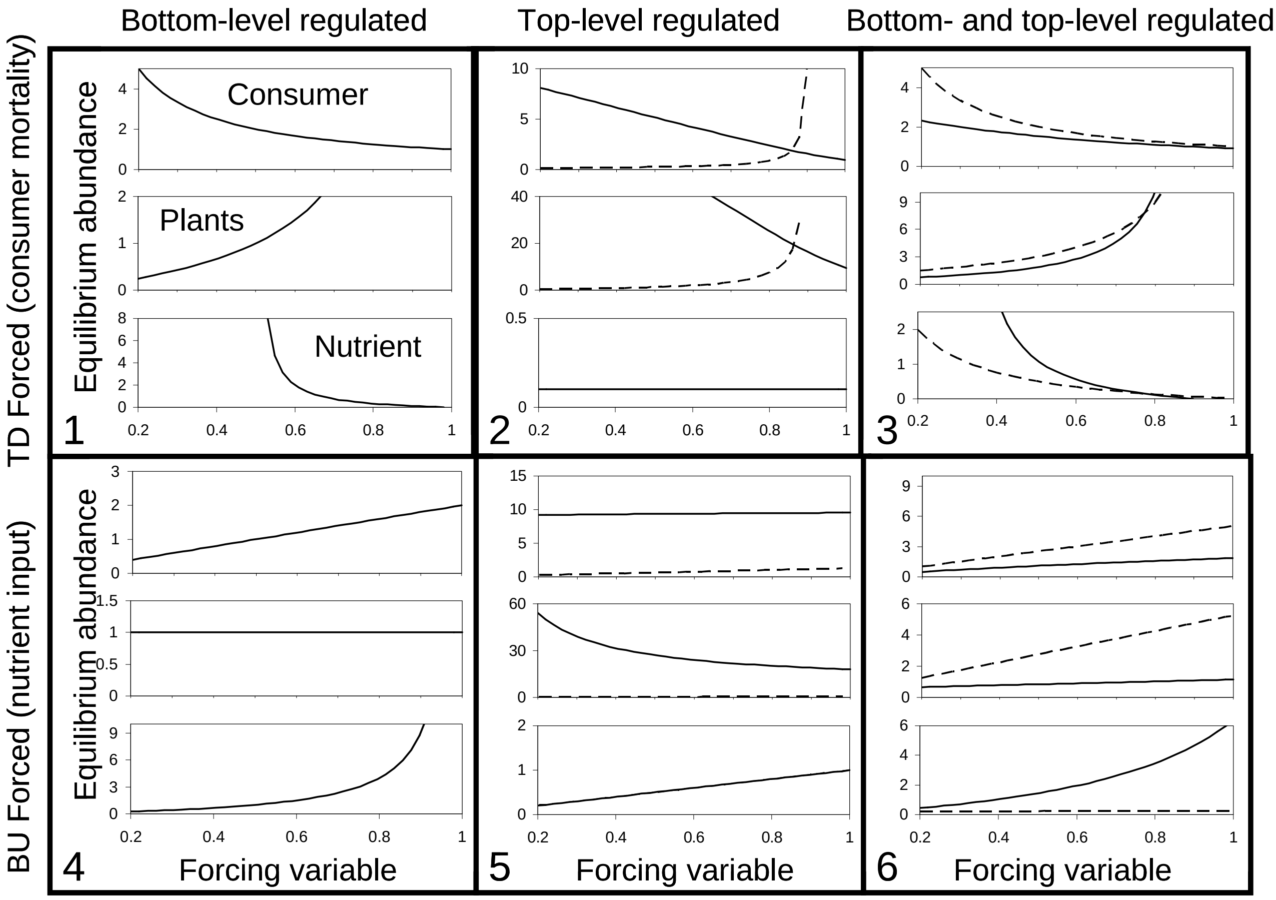

Figure S2. Proportional response of the 4-level food chain models to a doubling of top-down forcing (density independent mortality; top row), or bottom-up forcing (nutrient input rate; bottom row), with mortality or uptake regulation at the top and interior-trophic levels, i.e. as Fig.5 in the main manuscript, except with regulation at interior as well as at the top ($\lambda_i = 1$, $\upsilon_i = 1$, i = 1 to n). Left column, linear prey dependency; right column, Type-II non-linear prey dependency. Pale shaded bars, uptake regulation; dark shaded, mortality regulation. The proportional response is calculated as in eq. 28 of the main text, i.e. $\log_2$(abundance with altered forcing/abundance with default forcing).