Supporting Information

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SI Text

Interaction with Constant NGB Fields

Here we prove the formula $H_{\rm int}^{(1)} = -[i\pi^a Q_a, H_0]$ for constant NG fields π^a. More precisely, we prove the Lagrangian version, $L_{\rm int}^{(1)} = -[i\pi^a Q_a, L_0]$. (Here, commutation relations with Q_a and the Lagrangian density means the symmetry transformation of the fields contained in the Lagrangian, as we explain below.) They are equivalent as long as symmetry generators Q_i commute with the total Hamiltonian of the system.

In general, we can decompose the total Lagrangian density $L_{\rm tot}$ into three pieces, $L_{\rm el}(\psi^\dagger, \partial_\mu\psi^\dagger, \psi, \partial_\mu\psi)$, $L_{\rm NG}(\pi^a, \partial_\mu\pi^a)$, and $L_{\rm int}(\psi^\dagger, \partial_\mu\psi^\dagger, \psi, \partial_\mu\psi, \pi^a, \partial_\mu\pi^a)$. We define $L_{\rm int}^{(1)}$ and L_0 by

$$L_{\rm int}^{(1)} = \left.\frac{\partial L_{\rm int}}{\partial \pi^a}\right|_{\pi=0} \pi^a + \left.\frac{\partial L_{\rm int}}{\partial \partial_\mu \pi^a}\right|_{\pi=0} \partial_\mu \pi^a, \qquad \text{[S1]}$$

$$L_0 = L_{\rm el} + L_{\rm int}|_{\pi=0}. \qquad \text{[S2]}$$

For constant π^a, we can drop the second term of $L_{\rm int}^{(1)}$.

Internal Symmetries. Let us start with a general symmetry-breaking pattern $G \to H$ of internal symmetries. We introduce a NG field π^a for each broken generator Q_a ($a \in \{1, 2, \ldots, \dim G/H\}$) to describe low-energy fluctuations of the order parameter. Under the symmetry transformation $U = e^{i\epsilon^i Q_i}$, NG fields transform as

$$(\pi^a)' \equiv U_\epsilon \pi^a U_\epsilon^\dagger = \pi^a + \epsilon^i h_i^a(\pi) + O(\epsilon^2), \qquad \text{[S3]}$$

and its infinitesimal form is

$$\delta_i \pi^a \equiv (\pi^a)' - \pi^a = [iQ_i, \pi^a] = h_i^a(\pi). \qquad \text{[S4]}$$

In the standard parameterization introduced by refs. 1 and 2, $h_b^a(\pi) = \delta_b^a + O(\pi)$ for broken generators Q_b and $h_\rho^a(\pi) = O(\pi)$ for unbroken generators Q_ρ. Namely, a broken generator Q_a shifts π^a by a constant and an unbroken generator Q_ρ does not shift any NG fields by a constant amount.

Each component of the Lagrangian density is invariant under the symmetry transformation (up to total derivatives). Namely, $U_\epsilon L_A U_\epsilon^\dagger = L_A$ ($A \in \{el, NG, int\}$). Hence,

$$\begin{aligned} 0 &= [iQ_a, L_{\rm int}] \\ &= [iQ_a, \psi]\frac{\partial L_{\rm int}}{\partial \psi} + [iQ_a, \partial_\mu\psi]\frac{\partial L_{\rm int}}{\partial \partial_\mu \psi} + \left(\psi \leftrightarrow \psi^\dagger\right) + \left[iQ_a, \pi^b\right]\frac{\partial L_{\rm int}}{\partial \pi^b} \\ &\quad + \left[iQ_a, \partial_\mu \pi^b\right]\frac{\partial L_{\rm int}}{\partial \partial_\mu \pi^b}. \end{aligned} \qquad \text{[S5]}$$

We set $\pi^a = 0$ *after* using the relation $[iQ_b, \pi^a] = \delta_b^a + O(\pi)$:

$$\left.\frac{\partial L_{\rm int}}{\partial \pi^a}\right|_{\pi=0} = -[iQ_a, \psi]\frac{\partial L_{\rm int}}{\partial \psi} - \left.[iQ_a, \partial_\mu\psi]\frac{\partial L_{\rm int}}{\partial \partial_\mu \psi} + \left(\psi \leftrightarrow \psi^\dagger\right)\right|_{\pi=0}. \qquad \text{[S6]}$$

The right-hand side is nothing but $-[iQ_a, L_{\rm int}|_{\pi=0}]$. Hence, by multiplying π^a to both hand sides, we get

$$\left.\frac{\partial L_{\rm int}}{\partial \pi^a}\right|_{\pi=0} \pi^a = -\pi^a\left[iQ_a, L_{\rm int}\big|_{\pi=0}\right]. \qquad \text{[S7]}$$

Because $L_{\rm el}$ commutes with Q_a, we can add it to inside of the commutator. Therefore, for constant π^a, we get $L_{\rm int}^{(1)} = -[i\pi^a Q_a, L_0]$.

As the simplest example, let us discuss the spin–spin interaction in ferromagnets,

$$L_{\rm el} = i\psi^\dagger \partial_t \psi - \frac{|\vec\nabla \psi|^2}{2m}, \quad L_{\rm int} = J\vec n \cdot \vec s, \qquad \text{[S8]}$$

where $\vec n$ is the normalized ferromagnetic order parameter, ψ is an electron field with the spin degree of freedom, $\vec s \equiv \psi^\dagger(\vec\sigma/2)\psi$ is the electron spin, and $\vec\sigma$ is the Pauli matrix.

We introduce fluctuation $\pi_{x,y}(\vec x, t)$ as $\vec n = (\pi_y, -\pi_x, 1)^T + O(\pi_{x,y}^2)$. By expanding the interaction to the linear order in fluctuation, we find

$$L_0 = i\psi^\dagger \partial_t \psi - \frac{|\vec\nabla \psi|^2}{2m} + Js_z, \qquad \text{[S9]}$$

$$L_{\rm int}^{(1)} = J\left(\pi_y s_x - \pi_x s_y\right). \qquad \text{[S10]}$$

Using the commutation relation $\left[s_i(\vec x), s_j(\vec x')\right] = i\epsilon_{ijk} s_k(\vec x)\delta^d(\vec x - \vec x')$, it can be readily shown that

$$L_{\rm int}^{(1)} = -\pi^x[iQ_x, L_0] - \pi^y\left[iQ_y, L_0\right] \qquad \text{[S11]}$$

for $\vec Q = \int \mathrm{d}^d x(\vec s + m\vec n)$.

Equivalently, in terms of the Hamiltonian,

$$H_0 = \frac{\vec p^2}{2m} - Js_z, \quad H_{\rm int}^{(1)} = -J\left(\pi_y s_x - \pi_x s_y\right), \qquad \text{[S12]}$$

and it is straightforward to check

$$H_{\rm int}^{(1)} = -\pi_x[iQ_x, H_0] - \pi_y\left[iQ_y, H_0\right]. \qquad \text{[S13]}$$

Translation. Now we move on to space–time symmetries. As we will see, the above derivation applies with only some minor changes.

Let us discuss translation $\vec x' = \vec x + \vec a$ as the easiest example. The displacement field $\vec u(\vec x, t)$ obeys the transformation rule,

$$\vec u'(\vec x, t) \equiv e^{i\vec a\cdot\vec P}\vec u(\vec x, t)e^{-i\vec a\cdot\vec P} = \vec u(\vec x - \vec a, t) + \vec a, \qquad \text{[S14]}$$

$$\delta_j u^i(\vec x, t) \equiv u'^i(\vec x, t) - u^i(\vec x, t) = \left[iP^j, u^i(\vec x, t)\right] = \delta_j^i - \partial_j u^i. \qquad \text{[S15]}$$

Computing $\delta L_{\rm int} = [i\vec P, L_{\rm int}]$ in the same way as in Eq. **S5**, we have

$$\delta L_{\rm int} = \left[i\vec P, \psi\right]\frac{\partial L_{\rm int}}{\partial \psi} + \cdots. \qquad \text{[S16]}$$

Using Eq. **S15** first and then setting $\vec u = 0$, we get a relation between $\delta L_{\rm int}$ and $\partial L_{\rm int}/\partial \vec u$. The only difference from the previous case is that $\delta L_{\rm int}(\vec x, t)$ does not exactly vanish but changes by a surface term $-\vec\nabla L_{\rm int}(\vec x, t)$. Hence we get

$$\left.\frac{\partial L_{\rm int}}{\partial \vec{u}}\right|_{\vec{u}=0}\cdot\vec{u}=-\vec{u}\cdot\left[i\vec{P},L_{\rm int}\Big|_{\vec{u}=0}\right]-\vec{u}\cdot\vec{\nabla}L_{\rm int}\Big|_{\vec{u}=0}. \qquad \text{[S17]}$$

Adding $0=-\vec{u}\cdot\left[i\vec{P},L_{\rm el}\right]-\vec{u}\cdot\vec{\nabla}L_{\rm el}$ to the right-hand side, we get

$$L^{(1)}_{\rm int}=\left.\frac{\partial L_{\rm int}}{\partial \vec{u}}\right|_{\vec{u}=0}\cdot\vec{u}=-\left[i\pi^a Q_a,L_0\right]-\vec{\nabla}\cdot\left(\vec{u}L_0\right) \qquad \text{[S18]}$$

for a constant NG field $\vec{u}$. The last term is just a total derivative and can be dropped.

This derivation does not change at all even for the magnetic translation, because the displacement field is real and its transformation rule does not involve phase rotation. All characteristic features of the magnetic transformation are hidden in the commutation relation $\left[i\vec{P},\psi\right]$.

Rotation. In the case of the spatial rotation $\vec{x}'=R_\epsilon\vec{x}$, where

$$R_\epsilon=\begin{pmatrix}\cos\epsilon & -\sin\epsilon\\ \sin\epsilon & \cos\epsilon,\end{pmatrix}, \qquad \text{[S19]}$$

the NG field θ transforms as

$$\theta'(\vec{x},t)\equiv e^{i\epsilon L_z}\theta(\vec{x},t)e^{-i\epsilon L_z}=\theta\left(R_{-\epsilon}\vec{x},t\right)+\epsilon, \qquad \text{[S20]}$$

$$\delta\theta(\vec{x},t)\equiv\theta'(\vec{x},t)-\theta(\vec{x},t)=\left[iL_z,\theta(\vec{x},t)\right]=1-\left(x\partial_y-y\partial_x\right)\theta(\vec{x},t). \qquad \text{[S21]}$$

In this case, the change of the Lagrangian density is $\delta L_A(\vec{x},t)=-\partial_y(xL_A)+\partial_x(yL_A)$. Hence, in exactly the same way as above, we get

$$\left.\frac{\partial L_{\rm int}}{\partial\theta}\right|_{\theta=0}\theta=-\theta\left[iL_z,L_0\right]-\partial_y(\theta xL_0)+\theta\partial_x(\theta yL_0). \qquad \text{[S22]}$$

Again, the second and third terms can be dropped.

The most general case should now be obvious. The keys are the transformation rule $\delta_a\pi^b(\vec{x},t)=\left[iQ_a,\pi^b(\vec{x},t)\right]=\delta^b_a+\cdots$ and the fact that the Lagrangian density can change only by total derivatives.

Singularities in the Matrix Element

Here we demonstrate the divergence of the matrix element $\langle\vec{k}'|Q_a|\vec{k}\rangle$, using several examples.

When an operator Q_a does not commute with $\vec{P}$, it does not commute with the lattice translation either, $[Q_a,e^{i\vec{P}\cdot\vec{a}_i}]\neq0$. By further assuming that $\langle\vec{k}|[Q_a,e^{i\vec{P}\cdot\vec{a}_i}]|\vec{k}\rangle\neq0$ (here we omit the band index n), which is generically true except for some high-symmetry points in the Brillouin zone, the expectation value $\langle\vec{k}'|Q_a|\vec{k}\rangle$ is inversely proportional to (k'_j-k_j).

Using commutation relations $[x_i,p_j]=i\delta_{ij}$, $[\ell_z,p_i]=i\epsilon_{ij}p_j$, and $[P_i,P_j]=-i\epsilon_{ij}eB$, one can show

$$\left[\vec{x},e^{i\vec{p}\cdot\vec{a}_i}\right]=-\vec{a}_i e^{i\vec{p}\cdot\vec{a}_i}, \qquad \text{[S23]}$$

$$\left[\ell_z,e^{i\vec{p}\cdot\vec{a}_i}\right]=-\hat{z}\cdot\vec{a}_i\times\vec{p}\,e^{i\vec{p}\cdot\vec{a}_i}, \qquad \text{[S24]}$$

$$\left[\vec{P},e^{i\vec{P}\cdot\vec{a}_i}\right]=-eB\hat{z}\times\vec{a}_i e^{i\vec{P}\cdot\vec{a}_i}. \qquad \text{[S25]}$$

Evaluating the matrix element of both hand sides, using the definition $e^{i\vec{p}\cdot\vec{a}_i}|\vec{k}\rangle=e^{i\vec{k}\cdot\vec{a}_i}|\vec{k}\rangle$, one finds

$$\begin{aligned}\left\langle\vec{k}'\middle|\vec{x}\middle|\vec{k}\right\rangle&=-\frac{e^{i\vec{k}\cdot\vec{a}_i}}{e^{i\vec{k}\cdot\vec{a}_i}-e^{i\vec{k}'\cdot\vec{a}_i}}\vec{a}_i\delta_{\vec{k}',\vec{k}}\\&=\frac{i\vec{a}_i}{\left(\vec{k}-\vec{k}'\right)\cdot\vec{a}_i}\delta_{\vec{k}',\vec{k}}+O\left(\left(\vec{k}-\vec{k}'\right)^0\right),\end{aligned} \qquad \text{[S26]}$$

$$\begin{aligned}\left\langle\vec{k}'\middle|\ell_z\middle|\vec{k}\right\rangle&=-\frac{e^{i\vec{k}\cdot\vec{a}_i}}{e^{i\vec{k}\cdot\vec{a}_i}-e^{i\vec{k}'\cdot\vec{a}_i}}\hat{z}\cdot\vec{a}_i\times\vec{k}\delta_{\vec{k}',\vec{k}}\\&=\frac{i\hat{z}\cdot\vec{a}_i\times\vec{k}}{\left(\vec{k}-\vec{k}'\right)\cdot\vec{a}_i}\delta_{\vec{k}',\vec{k}}+O\left(\left(\vec{k}-\vec{k}'\right)^0\right),\end{aligned} \qquad \text{[S27]}$$

$$\begin{aligned}\left\langle\vec{k}'\middle|\vec{\mathcal{P}}\middle|\vec{k}\right\rangle&=-\frac{e^{i\vec{k}\cdot\vec{a}_i}}{e^{i\vec{k}\cdot\vec{a}_i}-e^{i\vec{k}'\cdot\vec{a}_i}}eB\hat{z}\times\vec{a}_i\delta_{\vec{k}',\vec{k}}\\&=\frac{ieB\hat{z}\times\vec{a}_i}{\left(\vec{k}-\vec{k}'\right)\cdot\vec{a}_i}\delta_{\vec{k}',\vec{k}}+O\left(\left(\vec{k}-\vec{k}'\right)^0\right).\end{aligned} \qquad \text{[S28]}$$

Comoving Frame of NGBs

In the main text, we explained how to make derivatives in the electron–magnon coupling obvious, using local $SU(2)$ transformation. Here we show that the same argument can be applied to the electron–phonon interaction in the absence of the magnetic field, although it fails for space–time symmetries that do not commute with momentum.

Phonons in Crystals. The electron–phonon interaction in $H_{\rm int}=V(\vec{x}-\vec{u})\psi^\dagger(\vec{x},t)\psi(\vec{x},t)$ does not contain derivatives acting on the displacement field $\vec{u}(\vec{x},t)$. Their vertex still vanishes in the limit of small energy-momentum transfer, as can be argued in the same way as in the main text. To see the vanishing vertex more explicitly, we convert the nonderivative coupling $V(\vec{x}-\vec{u})$ into derivative ones by going to the comoving frame of the crystal lattice. That is, we change the integration variable of the Lagrangian from $\vec{x}$ to $\vec{x}'=\vec{x}-\vec{u}$ and redefine the electron field by $\psi'(\vec{x}',t)=\psi(\vec{x},t)$. Then the potential $V(\vec{x}-\vec{u})=V(\vec{x}')$ can no longer fluctuate. Instead, all of the electron–phonon interactions come from rewriting the volume element and derivatives:

$$\mathrm{d}^dx\mathrm{d}t=\mathrm{d}^dx'\mathrm{d}t'\left(1+\vec{\nabla}'\cdot\vec{u}\right)+O\left(\left(\partial\vec{u}\right)^2\right), \qquad \text{[S29]}$$

$$\partial_\mu=\partial'_\mu-\left(\partial'_\mu u^i\right)\partial'_i+O\left(\left(\partial\vec{u}\right)^2\right). \qquad \text{[S30]}$$

It is now clear that all electron–phonon interactions vanish for a constant $\vec{u}$.

NGBs Originating from Rotation. If possible, we want to eliminate all nonderivative couplings in the interacting Lagrangian

$$\int\mathrm{d}^dx\mathrm{d}t\left|\vec{n}\cdot\vec{\nabla}\psi\right|^2=\int\mathrm{d}^dx\mathrm{d}t\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\cdot\vec{\nabla}\psi^\dagger\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\cdot\vec{\nabla}\psi \qquad \text{[S31]}$$

by performing a local transformation as above. If we change the integration variable from $\vec{x}$ to $\vec{x}'=R_\epsilon\vec{x}$ (Eq. **S19**), we get

$$\int\mathrm{d}^dx'\mathrm{d}t\begin{pmatrix}\cos(\theta-\epsilon)\\ \sin(\theta-\epsilon)\end{pmatrix}\cdot\vec{\nabla}'\psi'^\dagger\begin{pmatrix}\cos(\theta-\epsilon)\\ \sin(\theta-\epsilon)\end{pmatrix}\cdot\vec{\nabla}'\psi \qquad \text{[S32]}$$

for a constant angle ε. Therefore, this manipulation effectively shifts θ by $-\epsilon$. One may thus expect that setting $\epsilon(\vec{x},t)=\theta(\vec{x},t)$ locally eliminates all θ dependence without derivatives, but it does not work for the following reason. If we define $\vec{x}'=R_{\theta(\vec{x},t)}\vec{x}$ and rewrite derivative $\vec{\nabla}$ in terms of $\vec{\nabla}'$, we find

$$\partial_i=\left(\partial_i x'^j\right)\partial'_j=\partial_i\left[(R_\theta)^j_k x^k\right]\partial'_j=(R_\theta)^j_i\partial'_j+(\partial_i R_\theta)^j_k x^k\partial'_j. \qquad \text{[S33]}$$

Due to the second term of the last expression, the Lagrangian now explicitly depends on the coordinate. This makes the Lagrangian after the local rotation completely useless. Especially, we cannot use the Fourier transformation (despite the fact that the translation is not actually broken), and hence we cannot discuss the behavior of couplings in the limit of the small momentum transfer in this frame.

Magnetic Translation. Finally we discuss the magnetic translation. We want to remove $\vec{u}$ without derivatives in the Lagrangian,

$$L_{\text{el+int}}=i\psi^\dagger\partial_t\psi-\frac{\left|\left(\vec{\nabla}-ie\vec{A}\right)\psi\right|^2}{2m}-\psi^\dagger\psi V\left(\vec{x}-\vec{u}\right). \qquad \text{[S34]}$$

If we just change the integration variable to $\vec{x}'=\vec{x}-\vec{u}(\vec{x},t)$, then $\vec{u}$ without derivatives appears in the vector potential,

$$\vec{A}=B\begin{pmatrix}-y\\0\\0\end{pmatrix}=B\begin{pmatrix}-y'-u_y\\0\\0\end{pmatrix}. \qquad \text{[S35]}$$

To absorb this new $\vec{u}$ dependence, one can further perform a local gauge transformation, $\psi'=e^{-ieBx'u_y}\psi$. When u_y is a constant, this procedure successfully removes all u_y s from the Lagrangian. However, for a generic $u_y(\vec{x},t)$, we have

$$\vec{\nabla}'\psi'=e^{-ieBx'u_y}\left(\vec{\nabla}'\psi-ieB\hat{x}u_y\psi-ieBx'\psi\vec{\nabla}'u_y\right). \qquad \text{[S36]}$$

Again the last term introduces an undesirable coordinate dependence to the Lagrangian.

Landau Levels on a Torus

Here we summarize the wave function of Landau levels (following ref. 3) that simultaneously diagonalize Hamiltonian and lattice translations,

$$H=\frac{\left(p_x+eBy\right)^2+p_y^2}{2m},\quad T_x=e^{ip_x a_x},\quad T_y=e^{i\left(p_y+eBx\right)a_y}. \qquad \text{[S37]}$$

We assume a rectangular lattice with primitive lattice vectors $\vec{a}_x=a_x\hat{x}$ and $\vec{a}_y=a_y\hat{y}$ and a flux quantum per unit cell $eBa_xa_y=2\pi$. We work in a torus $a_xN_x\times a_yN_y$ $(N_x,N_y\in\mathbb{Z})$ and impose the periodic boundary condition $T_x^{N_x}=T_y^{N_y}=1$. The number of degeneracy is precisely the number of lattice points,

$$\frac{a_xa_yN_xN_y}{2\pi\ell^2}=N_xN_y,\quad \ell\equiv(eB)^{-1/2}. \qquad \text{[S38]}$$

For each $k=(2\pi/a_xN_x)i\quad(i\in\{1,2,\ldots,N_xN_y\})$, the function

$$\psi_{nk}(\vec{x})=\sum_{j\in\mathbb{Z}}\frac{H_n\left(y/\ell+k\ell+(2\pi\ell/a_x)jN_y\right)e^{-(1/2)\left(y/\ell+k\ell+(2\pi\ell/a_x)jN_y\right)^2}}{\sqrt{2^n n!\sqrt{\pi}\ell}}\times\frac{e^{i\left(k+(2\pi/a_x)jN_y\right)x}}{\sqrt{a_xN_x}} \qquad \text{[S39]}$$

represents a simultaneous eigenfunction of the Hamiltonian and the lattice translation T_x with the eigenvalue $(eB/m)(n+1/2)$ and e^{ika_x}, respectively. To make it a simultaneous eigenfunction of T_y as well, we take a superposition

$$\begin{aligned}\Psi_{n\vec{k}}(\vec{x})&\equiv\sum_{m=1}^{N_y}\frac{e^{-ik_ya_ym}}{\sqrt{N_y}}\psi_{n,k_x+(2\pi/a_x)m}(\vec{x})\\&=\sum_{m\in\mathbb{Z}}\frac{H_n\left(y/\ell+k_x\ell+(2\pi\ell/a_x)m\right)e^{-(1/2)(y/\ell+k_x\ell+(2\pi\ell/a_x)m)^2}}{\sqrt{2^n n!\sqrt{\pi}\ell}}\\&\quad\times\frac{e^{-ik_ya_ym}}{\sqrt{N_y}}\frac{e^{i(k_x+(2\pi/a_x)m)x}}{\sqrt{a_xN_x}},\end{aligned} \qquad \text{[S40]}$$

where $k_\alpha=(2\pi/a_\alpha)i_\alpha$ $(i_\alpha\in\{1,2,\ldots,N_\alpha\}$ and $\alpha\in\{x,y\})$. One can check the validity of this wave function by an explicit calculation.

Cancelation of the Induced Mass of NGBs

For completeness, here we check the absence of mass terms of NGBs $[(1/2)\Pi_{ab}(0)\pi^a\pi^b]$ generated by gapless electron bubbles. Because we are interested in the $O(\pi^2)$ term, we have to include the vertex to the same order. Although here we examine only a few examples, their absence is ultimately due to the (broken) symmetry and hence should be very general.

Rotation. Let us start with the example of the spatial rotation discussed in the main text. For a constant θ, we have

$$\begin{aligned}H_{\text{int}}&=\frac{\chi}{2m}\left[\left(k_x\cos\theta+k_y\sin\theta\right)^2-k_x^2\right]\psi_k^\dagger\psi_k\\&=\frac{\chi}{m}\left[\theta k_xk_y+\frac{1}{2}\theta^2\left(k_y^2-k_x^2\right)+O\left(\theta^3\right)\right]\psi_k^\dagger\psi_k\\&=\left[-\theta\partial_{\phi_{\vec{k}}}\epsilon_{\vec{k}}+\frac{1}{2}\theta^2\partial^2_{\phi_{\vec{k}}}\epsilon_{\vec{k}}+O\left(\theta^3\right)\right]\psi_k^\dagger\psi_k,\end{aligned} \qquad \text{[S41]}$$

where $ke^{i\phi_{\vec{k}}}=k_x+ik_y$ and $\epsilon_{\vec{k}}$ is the electron dispersion,

$$\epsilon_{\vec{k}}=\frac{(1+\chi)k_x^2+k_y^2}{2m}-\mu. \qquad \text{[S42]}$$

Note that the vertex with a single NGB field is proportional to $\partial_{\phi_{\vec{k}}}\epsilon_{\vec{k}}$ and the one with two NGB fields is proportional to $\partial^2_{\phi_{\vec{k}}}\epsilon_{\vec{k}}$. This relation is dictated by the broken rotation symmetry.

The boson self-energy Π at $\vec{q}=0$ and $\nu=0$ receives two contributions at the one-loop level,

$$\Pi(0)=\int\frac{\mathrm{d}^2k\mathrm{d}\omega}{(2\pi)^3}\left[\left(\partial_{\phi_{\vec{k}}}\epsilon_{\vec{k}}G\left(\vec{k},\omega\right)\right)^2+\partial^2_{\phi_{\vec{k}}}\epsilon_{\vec{k}}G\left(\vec{k},\omega\right)\right]. \qquad \text{[S43]}$$

The first and second terms represent the diagrams in Fig. S1 *A* and *B*, respectively. To show their cancelation, we use the relation of the electron Green function $G^{-1}(\vec{k},\omega)=\omega-\epsilon_{\vec{k}}$:

$$\vec{\nabla}_{\vec{k}}G\left(\vec{k},\omega\right)=\left[G\left(\vec{k},\omega\right)\right]^2\vec{\nabla}_{\vec{k}}\epsilon_{\vec{k}}. \qquad \text{[S44]}$$

By integration by part, we get

$$\Pi(0)=\int\frac{\mathrm{d}^2k\mathrm{d}\omega}{(2\pi)^3}\left[\partial_{\phi_{\vec{k}}}\epsilon_{\vec{k}}\,\partial_{\phi_{\vec{k}}}G\left(\vec{k},\omega\right)+\partial^2_{\phi_{\vec{k}}}\epsilon_{\vec{k}}\,G\left(\vec{k},\omega\right)\right]=0. \qquad \text{[S45]}$$

Magnetic Translation. Next, for the electron–phonon problem under a magnetic field, we have

$$
\begin{aligned}
H_{\mathrm{int}} &= -\tilde{V}_x\left[\cos\left(k_y a_y - \frac{2\pi}{a_x}u_x\right) - \cos\left(k_y a_y\right)\right] \\
&\quad - \tilde{V}_y\left[\cos\left(k_x a_x + \frac{2\pi}{a_y}u_y\right) - \cos\left(k_x a_x\right)\right] \\
&\simeq \left[\left(\frac{2\pi u_y}{a_y}\right)\partial_{k_x a_x}\epsilon_{\vec{k}} - \left(\frac{2\pi u_x}{a_x}\right)\partial_{k_y a_y}\epsilon_{\vec{k}}\right] \\
&\quad + \frac{1}{2}\left[\left(\frac{2\pi u_y}{a_y}\right)^2\partial^2_{k_x a_x}\epsilon_{\vec{k}} + \left(\frac{2\pi u_x}{a_x}\right)^2\partial^2_{k_y a_y}\epsilon_{\vec{k}}\right]
\end{aligned}
\qquad \text{[S46]}
$$

to the order $O(u^3)$. Again, the coupling to the linear (quadratic) order in u_y is proportional to $\partial_{k_x a_x}\epsilon_{\vec{k}}$ ($\partial^2_{k_x a_x}\epsilon_{\vec{k}}$). Therefore, following exactly the same argument, we get $\Pi_{ij}(0) = 0$ $(i,j \in \{x,y\})$.

Bosonic Self-Energy Correction

Here we discuss the boson self-energy correction for a general $\vec{q}$ and ν. At the leading order in q, the contribution of the diagram in Fig. S1*A* is given by (we drop the band index n for brevity)

$$
\begin{aligned}
\prod_{ab}(\nu,\vec{q}) &= \int \frac{\mathrm{d}^d k \mathrm{d}\omega}{(2\pi)^{d+1}} v^a_{\vec{k},\vec{k}+\vec{q}} v^b_{\vec{k}+\vec{q},\vec{k}} G\left(\vec{k},\omega\right) G\left(\vec{k}+\vec{q},\omega+\nu\right) \\
&= \int \frac{\mathrm{d}^d k}{(2\pi)^d} v^a_{\vec{k},\vec{k}+\vec{q}} v^b_{\vec{k}+\vec{q},\vec{k}} \frac{f\left(\epsilon_{\vec{k}}\right) - f\left(\epsilon_{\vec{k}+\vec{q}}\right)}{\nu + i\delta - \left(\epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}}\right)} \\
&\simeq \int \frac{\mathrm{d}^d k}{(2\pi)^d} \delta\left(\epsilon_{\vec{k}}\right) v^a_{\vec{k},\vec{k}} v^b_{\vec{k},\vec{k}} \frac{\hat{q}\cdot\vec{\nabla}_{\vec{k}}\epsilon_{\vec{k}}}{\nu/q + i\delta - \hat{q}\cdot\vec{\nabla}_{\vec{k}}\epsilon_{n\vec{k}}}.
\end{aligned}
\qquad \text{[S47]}
$$

As discussed in the previous section, the constant term

$$
\prod_{ab}(0) = -\int \frac{\mathrm{d}^d k}{(2\pi)^d} \delta\left(\epsilon_{n\vec{k}}\right) v^a_{\vec{k},\vec{k}} v^b_{\vec{k},\vec{k}} \qquad \text{[S48]}
$$

is exactly canceled by the diamagnetic term (Fig. S1*B*). The imaginary part is therefore given by

$$
\begin{aligned}
\mathrm{Im}\prod_{ab}(\nu,\vec{q}) &= -\pi\frac{\nu}{q}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \delta\left(\epsilon_{\vec{k}}\right) v^a_{\vec{k},\vec{k}} v^b_{\vec{k},\vec{k}} \delta\left(\nu/q - \hat{q}\cdot\vec{\nabla}_{\vec{k}}\epsilon_{n\vec{k}}\right) \\
&\simeq -\pi\frac{\nu}{q}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \delta\left(\epsilon_{\vec{k}}\right) v^a_{\vec{k},\vec{k}} v^b_{\vec{k},\vec{k}} \delta\left(\hat{q}\cdot\vec{\nabla}_{\vec{k}}\epsilon_{\vec{k}}\right).
\end{aligned}
\qquad \text{[S49]}
$$

Electronic Bandwidth Under a Magnetic Field

Here we show a simple numerical result on the bandwidth of the electron band structure under a uniform magnetic field, to support the claim

$$
(\text{band width}) \propto e^{-C\ell^2/a^2}. \qquad \text{[S50]}
$$

Here $\ell = (eB)^{-1/2}$ is the magnetic length and a is the lattice constant of the tight-binding model. In the continuum limit $a \to 0$, the Landau levels are completely flat. For a finite a, the lattice potential produces nonzero dispersions.

By denoting the number of the flux per unit cell by ϕ,

$$
\frac{\ell^2}{a^2} = \frac{1}{eBa^2} = \frac{1}{\phi}. \qquad \text{[S51]}
$$

Thus, Eq. **S50** suggests that

$$
\log(\text{band width}) = (\text{const.}) - C\phi^{-1}. \qquad \text{[S52]}
$$

In Fig. S2, we show the numerical result for the lowest Landau levels in the tight-binding model on the square lattice with the nearest neighbor hopping. The logarithm of the bandwidth is indeed proportional to ϕ^{-1}. This result holds for other Landau levels as well, as long as the van Hove singularity energy is avoided.

1. Coleman S, Wess J, Zumino B (1969) Structure of phenomenological Lagrangians. I. *Phys Rev* 177:2239–2247.
2. Callan CG, Coleman S, Wess J, Zumino B (1969) Structure of phenomenological Lagrangians. II. *Phys Rev* 177:2247–2250.
3. Haldane FDM, Rezayi EH (1985) Periodic Laughlin-Jastrow wave functions for the fractional quantized Hall effect. *Phys Rev B Condens Matter* 31(4):2529–2531.

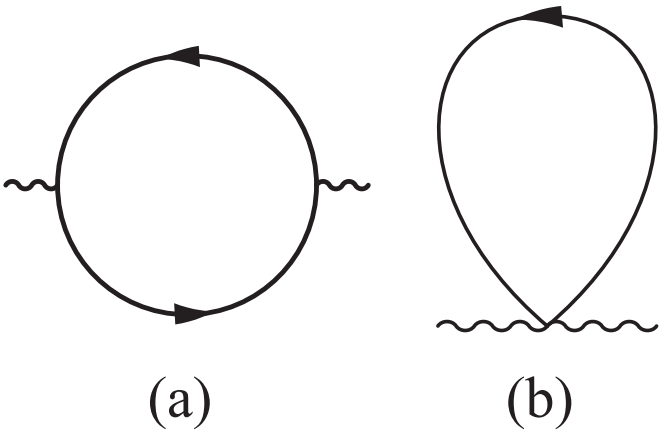

Fig. S1. (*A* and *B*) One-loop diagrams for boson self-energy corrections.

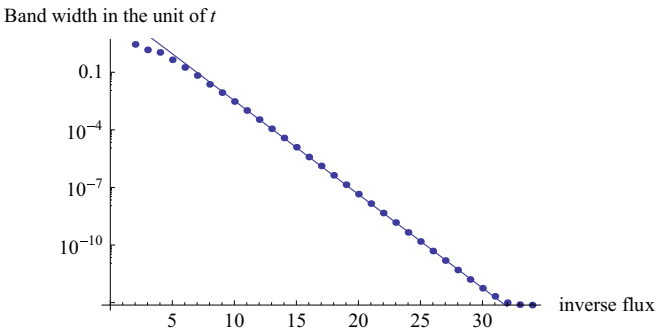

Fig. S2. The bandwidth of the lowest Landau level in the tight-binding model as a function of the inverse flux ϕ^{-1}.