Supporting Information

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SI Appendix

Assumptions and Corollaries. Throughout the binary-choice model, we assume that (z_a, z_b), $(y_{a,i}, y_{b,i})$, $\log(fz_a + (1-f)z_b)$, and $\log(fy_{a,i} + (1-f)y_{b,i})$ have finite moments up to order 2 for all $f \in [0,1]$ and i. We assume further that, for all $f \in [0,1]$, $\mathbb{E}[fz_a + (1-f)z_b] = \mathbb{E}[fy_{a,i} + (1-f)y_{b,i}]$ for all i, so that we can compare growth rates of populations in different environments. More specifically, the optimal log geometric average growth rate increases as the portion of idiosyncratic risk increases:

$$\alpha_{\lambda_1}\left(f^*_{\lambda_1}\right) \ge \alpha_{\lambda_2}\left(f^*_{\lambda_2}\right) \quad \text{if} \quad 0 \le \lambda_1 \le \lambda_2 \le 1.$$

To prove this result, for any given f, take the first derivative of $\alpha_\lambda(f)$ with respect to λ:

$$\frac{\partial \alpha_\lambda(f)}{\partial \lambda} = \mathbb{E}_z\left[\frac{z^f - \mathbb{E}\left[y^f\right]}{\lambda z^f + (1-\lambda)\mathbb{E}\left[y^f\right]}\right].$$

Evaluate the first derivative at $\lambda = 0, 1$, and recall that our assumptions imply that $\mathbb{E}_z[z^f] = \mathbb{E}_y[y^f]$:

$$\left.\frac{\partial \alpha_\lambda(f)}{\partial \lambda}\right|_{\lambda=0} = \mathbb{E}_z\left[\frac{z^f - \mathbb{E}_y\left[y^f\right]}{\mathbb{E}_y\left[y^f\right]}\right] = 0,$$
$$\left.\frac{\partial \alpha_\lambda(f)}{\partial \lambda}\right|_{\lambda=1} = \mathbb{E}_z\left[\frac{z^f - \mathbb{E}_y\left[y^f\right]}{z^f}\right] = 1 - \mathbb{E}_y\left[y^f\right]\mathbb{E}_z\left[\frac{1}{z^f}\right] \le 0,$$

where the last step uses Jensen's Inequality. Now, take the second derivative of $\alpha_\lambda(f)$ with respect to λ:

$$\frac{\partial^2 \alpha_\lambda(f)}{\partial \lambda^2} = \mathbb{E}_z\left[\frac{-\left(z^f - \mathbb{E}\left[y^f\right]\right)^2}{\left(\lambda z^f + (1-\lambda)\mathbb{E}\left[y^f\right]\right)^2}\right] \le 0.$$

Therefore, for any given f, $\alpha_\lambda(f)$ is a nonincreasing concave function in the interval $0 \le \lambda \le 1$. Because $\alpha_\lambda(f^*_\lambda)$ is the maximum of $\alpha_\lambda(f)$ over all f, it follows that $\alpha_{\lambda_1}(f^*_{\lambda_1}) \ge \alpha_{\lambda_2}(f^*_{\lambda_2})$ whenever $\lambda_1 \le \lambda_2$, as desired.

Proof of Eqs. 2 and 3. The total number of offspring of type f in generation t is simply the sum of all of the offspring from the type f individuals of the previous generation:

$$\begin{aligned} n^f_t &= \sum_{i=1}^{n^f_{t-1}} x^f_{i,t} = \lambda \sum_{i=1}^{n^f_{t-1}} z^f_{i,t} + (1-\lambda)\sum_{i=1}^{n^f_{t-1}} y^f_{i,t} \\ &= \lambda\left(z_{a,t}\sum_{i=1}^{n^f_{t-1}} I^f_{i,t} + z_{b,t}\sum_{i=1}^{n^f_{t-1}}\left(1 - I^f_{i,t}\right)\right) \\ &\quad + (1-\lambda)\left(\sum_{i=1}^{n^f_{t-1}} I^f_{i,t} y_{a,i,t} + \sum_{i=1}^{n^f_{t-1}}\left(1 - I^f_{i,t}\right) y_{b,i,t}\right), \end{aligned}$$

where we have added time subscripts to the relevant variables to clarify their temporal ordering. As n^f_{t-1} increases without bound, the Law of Large Numbers implies that

$$\begin{aligned} n^f_t &\stackrel{p}{=} n^f_{t-1}\left(\lambda\left(fz_{a,t} + (1-f)z_{b,t}\right) + (1-\lambda)\left(f\mathbb{E}_y[y_a] + (1-f)\mathbb{E}_y[y_b]\right)\right) \\ &= n^f_{t-1}\left(\lambda\left(fz_{a,t} + (1-f)z_{b,t}\right) + (1-\lambda)\mathbb{E}_y\left[y^f\right]\right), \end{aligned}$$

where $\stackrel{p}{=}$ denotes equality in probability. Through backward recursion and assuming that $n^f_0 = 1$ without loss of generality, the population of type f individuals in generation T is given by

$$n^f_T \stackrel{p}{=} \prod_{t=1}^{T}\left(\lambda\left(fz_{a,t} + (1-f)z_{b,t}\right) + (1-\lambda)\mathbb{E}_y\left[y^f\right]\right).$$

Taking the logarithm on both sides and again, using the Law of Large Numbers, we get

$$\begin{aligned} \frac{1}{T}\log n^f_T &\stackrel{p}{=} \frac{1}{T}\sum_{t=1}^{T}\log\left(\lambda\left(fz_{a,t} + (1-f)z_{b,t}\right)\right. \\ &\quad \left. + (1-\lambda)\mathbb{E}_y\left[y^f\right]\right) \stackrel{p}{\to} \mathbb{E}_z\left[\log\left(\lambda z^f + (1-\lambda)\mathbb{E}_y\left[y^f\right]\right)\right], \end{aligned}$$

where $\stackrel{p}{\to}$ denotes convergence in probability, which completes the proof of Eq. **2**. Eq. **3** simply rewrites Eq. **2**.

Examples for Common Distributions of Relative Fecundity. Define $R = \omega^\lambda_a / \omega^\lambda_b$ to be the relative fecundity of two actions. We can characterize the growth-optimal behavior f^* for common distribution of R. Fig. S1 plots f^* for five distributions of R as a function of distribution parameters.

Lognormal distribution. Let R follow the lognormal distribution $\log N(\Delta\alpha, \Delta\beta^2)$. The expectations of R and $1/R$ are

$$\mathbb{E}[R] = \exp\left(\Delta\alpha + \frac{\Delta\beta^2}{2}\right) \text{ and}$$

$$\mathbb{E}[1/R] = \exp\left(-\Delta\alpha + \frac{\Delta\beta^2}{2}\right).$$

Therefore, the optimal behavior f^* is given by

$$f^* = \begin{cases} 1 & \text{if } \dfrac{\Delta\beta^2}{2} < \Delta\alpha \\ \text{between 0 and 1} & \text{if } \dfrac{\Delta\beta^2}{2} \ge |\Delta\alpha| \\ 0 & \text{if } \dfrac{\Delta\beta^2}{2} < -\Delta\alpha. \end{cases}$$

γ-Distribution. Let R follow the γ-distribution $\gamma(\alpha, \beta)$, where $\alpha > 0, \beta > 0$. The expectation of $1/R$ exists only for $\alpha > 1$, and therefore, the parameter space is restricted to $\alpha > 1, \beta > 0$:

$$\mathbb{E}[R] = \frac{\alpha}{\beta} \text{ and}$$
$$\mathbb{E}[1/R] = \frac{\beta}{\alpha - 1}.$$

Therefore, the optimal behavior f^* is given by

$$f^* = \begin{cases} 1 & \text{if } \beta < \alpha - 1 \\ \text{between 0 and 1} & \text{if } \alpha - 1 \le \beta \le \alpha \\ 0 & \text{if } \beta > \alpha. \end{cases}$$

As special cases of γ-distribution, we automatically have the results for the exponential, χ^2, and Erlang distribution.

Pareto distribution. Let R follow the Pareto distribution Pareto(x_m, α), where $x_m > 0, \alpha > 0$. The expectation of R exists only for $\alpha > 1$, and therefore, the parameter space is restricted to $x_m > 0, \alpha > 1$:

$$\mathbb{E}[R] = \frac{\alpha x_m}{\alpha - 1} \text{ and}$$

$$\mathbb{E}[1/R] = \frac{\alpha}{(\alpha+1)x_m}.$$

Therefore, the optimal behavior f^* is given by

$$f^* = \begin{cases} 1 & \text{if } x_m > 1 - \dfrac{1}{\alpha+1} \\ \text{between 0 and 1} & \text{if } 1 - \dfrac{1}{\alpha} \le x_m \le 1 - \dfrac{1}{\alpha+1} \\ 0 & \text{if } x_m < 1 - \dfrac{1}{\alpha}. \end{cases}$$

β′-Distribution. Let R follow the β'-distribution $\beta'(\alpha, \beta)$, where $\alpha > 0, \beta > 0$. The expectation of R exists only for $\beta > 1$, and the expectation of $1/R$ exists only for $\alpha > 1$; therefore, the parameter space is restricted to $\alpha > 1, \beta > 1$:

$$\mathbb{E}[R] = \frac{\alpha}{\beta - 1} \text{ and}$$

$$\mathbb{E}[1/R] = \frac{\beta}{\alpha - 1}.$$

Therefore, the optimal behavior f^* is given by

$$f^* = \begin{cases} 1 & \text{if } \beta < \alpha - 1 \\ \text{between 0 and 1} & \text{if } \alpha - 1 \le \beta \le \alpha + 1 \\ 0 & \text{if } \beta > \alpha + 1. \end{cases}$$

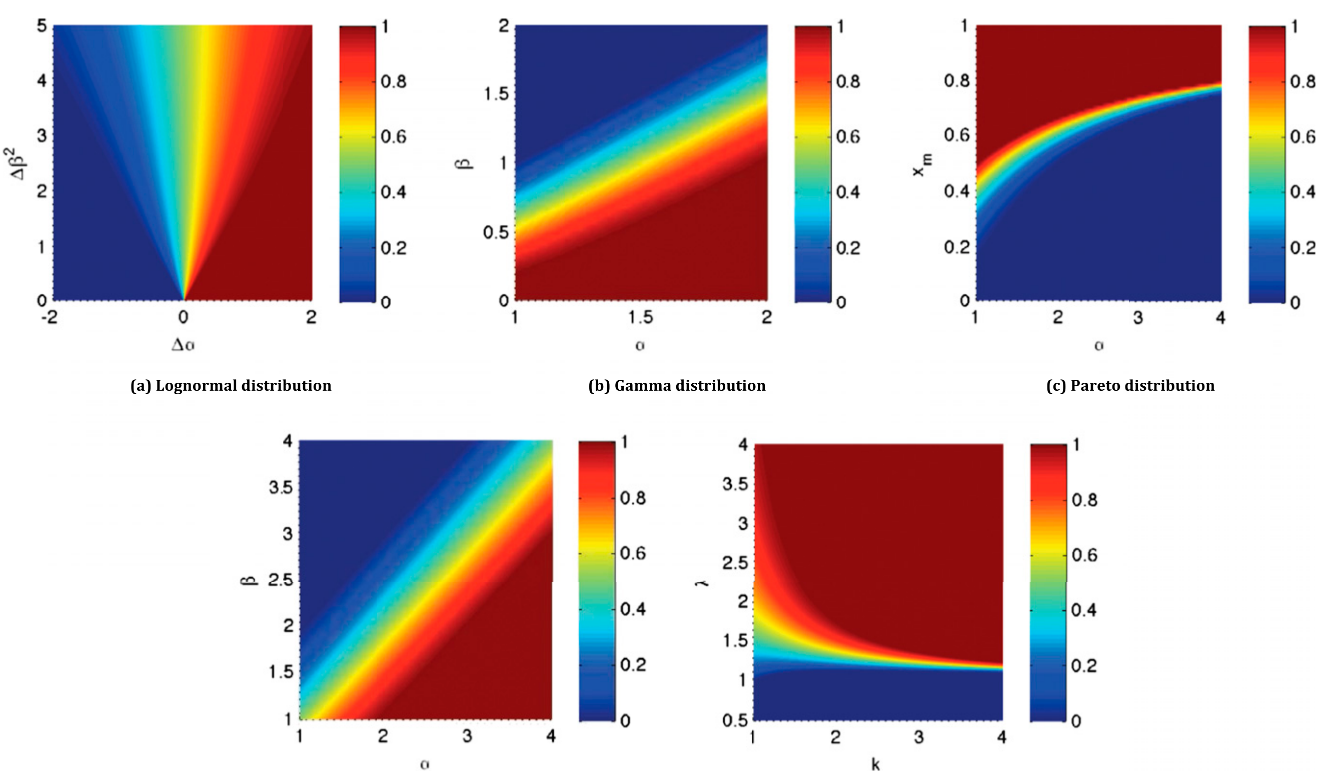
Weibull distribution. Let R follow the Weibull distribution Weibull(k, λ), where $k > 0, \lambda > 0$. The expectation of $1/R$ exists only for $k > 1$, and therefore, the parameter space is restricted to $k > 1, \lambda > 0$:

$$\mathbb{E}[R] = \lambda \Gamma\left(1 + \frac{1}{k}\right) \text{ and}$$

$$\mathbb{E}[1/R] = \frac{1}{\lambda} \Gamma\left(1 - \frac{1}{k}\right),$$

where $\Gamma(.)$ is the γ-function. Therefore, the optimal behavior f^* is given by

$$f^* = \begin{cases} 1 & \text{if } \lambda > \Gamma\left(1 - \dfrac{1}{k}\right) \\ \text{between 0 and 1} & \text{if } \dfrac{1}{\Gamma\left(1 + \dfrac{1}{k}\right)} \le \lambda \le \Gamma\left(1 - \dfrac{1}{k}\right) \\ 0 & \text{if } \lambda < \dfrac{1}{\Gamma\left(1 + \dfrac{1}{k}\right)}. \end{cases}$$



(a) Lognormal distribution (b) Gamma distribution (c) Pareto distribution

(d) Beta prime distribution (e) Weibull distribution

Fig. S1. Optimal behavior f^* for several distributions of relative fecundity $R = \omega_a^{\lambda}/\omega_b^{\lambda}$. In *A–E*, different colors correspond to deterministic ($f^* = 0$ or 1) or randomizing ($0 < f^* < 1$) behavior given the particular parameters of the distribution.