Arbitrary, direct and deterministic manipulation of vector beams via electrically-tuned q-plates - Supplementary information

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We have seen that one can manipulate the inhomogeneous state of polarization of a vector-vortex (VV) beam, identified through a hybrid Poincaré sphere (HPS) representation in a opposite topological charge OAM eingenstate basis, by acting with a q-plate with arbitrary retardation δ. The q-plate action consists of a rotation around an axis lying in the xy plane, whose orientation depends on the QP specific geometry defined by α_0. Consider an input state of polarization, corresponding to a point $\vec{r}_0 = (x_0, y_0, z_0)$ lying on the unit sphere represented in cartesian coordinates (i.e. the generalized Stokes parameters). Arbitrary manipulation implies capability to reach an arbitrary location $\vec{r}_1 = (x_1, y_1, z_1)$ on the same sphere. We show how it is possible to accomplish this task by exploiting two sequential rotations around orthogonal axes lying in the xy plane. Without loss of generality, we can choose such axes to be precisely x and y. The pair of rotations is hence represented by the following matrices

$$R_x[\alpha] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \tag{1}$$

and

$$R_y[\beta] = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \tag{2}$$

The strategy we want to adopt is the following: consider the orbit traced on the sphere by the two points $\vec{r}_0$ and $\vec{r}_1$ rotating respectively around the x and y axis. As first step, suppose such trajectories intersect in some point $\vec{t}$. Then it is enough to solve the two systems

$$R_x[\alpha] \cdot \vec{r}_0 = \vec{t}, \qquad R_y[\beta] \cdot \vec{t} = \vec{r}_1 \tag{3}$$

to determine the rotation angles α and β required to end up on $\vec{r}_1$, upon starting from $\vec{r}_0$. To determine the intersection point $\vec{t}$, we observe that the two rotation trajectories belong to orthogonal planes, represented by

$$\begin{aligned} P_0 &= \{(x, y, z) \,|\, x = x_0 \, and \, y, z \in \mathbb{R}\} \\ P_1 &= \{(x, y, z) \,|\, y = y_1 \, and \, x, z \in \mathbb{R}\} \end{aligned} \tag{4}$$

P_0 is the plane orthogonal to x and passing through $\vec{r}_0$, while P_1 is orthogonal to y and containing $\vec{r}_1$. Since such planes are not parallel, they intersect on a straight line, parametrized by (x_0, y_1, z) with $z \in \mathbb{R}$. Thus, the coordinates of the orbit intersection points are identified by the constraint $x_0^2 + y_1^2 + z^2 = 1$ and correspond to the two values

$$\bar{z} = \pm\sqrt{1 - x_0^2 - y_1^2} \tag{5}$$

We recognize that equation (5) cannot be solved for all possible x_0 and y_1 values, since $x_0^2 + y_1^2 > 1$ implies $\bar{z}$ being an imaginary number. This condition corresponds to cases in which the P_i planes intersect outside the unit sphere, a clear example being the choice $\vec{r}_0 = (1, 0, 0)$ and $\vec{r}_1 = (0, 1, 0)$. However, for our purposes it is enough to demonstrate that, in such cases, one can simply switch R_x and R_y roles (i.e. rotating first around the y axis and then around x) to accomplish the desidered transformation. Inverting the rotation order consists in P_0, P_1 becoming the planes passing through y_0 and x_1 respectively. Therefore, the intersection equation (5) becomes

$$\bar{z} = \pm\sqrt{1 - x_1^2 - y_0^2} \tag{6}$$

As a last step, we show that for any starting and ending point choice, at least one of the rotation's order allows one to connect $\vec{r}_1$ and $\vec{r}_0$. Indeed, suppose that the $R_x R_y$ strategy cannot be accomplished, meaning that the coordinates of the two points satisfy $x_0^2 + y_1^2 > 1$. Then, by exploiting the normalization conditions $x_i^2 + y_i^2 + z_i^2 = 1$ with $i = 0, 1$, one obtains

$$1 - y_0^2 - z_0^2 + 1 - x_1^2 - z_1^2 > 1 \tag{7}$$

and hence

$$x_1^2 + y_0^2 < 1 - (z_0^2 + z_1^2) < 1, \tag{8}$$

implying that the orbit intersection point can be found by solving (6). Therefore, if $R_x R_y$ is not the right order choice, then $R_y R_x$ is. We point out that there are also several cases in which both choices are capable of accomplishing the required transformation.

To conclude, we showed that a device represented by two q-plates with the same topological charge and initial α_0 orientations giving rise to orthogonal rotation axes can accomplish an arbitrary VV manipulation in the HPS, the only additional requirement being the possibility to exchange the rotation order. Since α_0 can be modified by simply rotating the q-plates, the exchange can be easily accomplished by switching the q-plates initial orientations. The only exception is the case $q = 1$, whose geometry is rotationally invariant. In this situation three q-plates are required, combined for instance in the following sequence $R_x R_y R_x$, in order to be allowed to chose in each case between acting with the first or the last R_x. Another possible solution consist in exploiting rotation around three orthogonal axes, such as $R_x R_y R_z$. Indeed, R_z action corresponds to a phase retarding between right and left circular polarization components, hence it can easily accomplished by means of a polarization rotator [1].

[1] Q. Zhan and J. R. Leger, Applied optics **41**, 4630 (2002).