Supporting Information: Avoiding or Restricting Defectors in Public Goods Games?

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1 Payoff formulas

First, we derive the the payoffs $\Pi_{ij}(k)$ for the five strategies AVOID (or RESTRICT), C, FREE, D, and FAKE (denoted 1, 2, 3, 4, 5, respectively, as in the main text). Recall that $\Pi_{ij}(k)$ denotes the payoff of a strategist of type i (resp., type j) when the random sampling consists of k players of type i and $N-k$ players of type j.

Denote $\Pi(k) = \{\Pi_{ij}(k)\}_{i,j=1,i\neq j}^{5}$, where, abusing notation, k is the number of AVOID (or RESTRICT) players if they are present in the pair; otherwise, the number of C players if C is present in the pair. Except for $\Pi_{31}(0) = \Pi_{51}(0) = 0$, we have

$$\Pi(k) = \begin{array}{c} \\ 1 \\ C \\ FREE \\ D \\ FAKE \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & C & FREE & D & FAKE \end{array} \\ \begin{pmatrix} - & rc-c-\frac{\epsilon_P}{k} & rc-c-\frac{\epsilon_P}{k} & \Pi_{14}(k) & (\frac{rk}{N}-1)c+\frac{N\delta-\epsilon_P}{k}-\delta \\ rc-c & - & \frac{rk}{N}c-c & \frac{rk}{N}c-c & \frac{rk}{N}c-c \\ rc-c & \frac{rk}{N}c & - & 0 & 0 \\ \Pi_{41}(k) & \frac{rk}{N}c & 0 & - & 0 \\ \frac{rk}{N}c-\delta & \frac{rk}{N}c & 0 & 0 & - \end{pmatrix} \end{array} \tag{1}$$

where

- for RESTRICT, $\Pi_{14}(k) = \frac{rkc}{k+\psi(N-k)} - c - \frac{\epsilon_P+\epsilon_R}{k}\ \forall 1 \leq k \leq N$ and $\Pi_{41}(k) = \frac{rkc\psi}{k+\psi(N-k)}\ \forall 1 \leq k \leq N-1$ and $\Pi_{41}(0) = 0$;
- for AVOID, $\Pi_{14}(N) = rc - c - \frac{\epsilon_P}{N}$ and $\Pi_{14}(k) = 0\ \forall 1 \leq k \leq N-1$, and $\Pi_{41}(k) = 0\ \forall 0 \leq k \leq N-1$.

We now derive the average payoffs $Pij(x)$ and $Pji(x)$ defined in the main text. For simplicity, consider $c = 1$. We have

$$P_{12}(x) = P_{13} = \sum_{k=0}^{N-1} H(k, N-1, x-1, Z-1)\,(r - 1 - \frac{\epsilon}{k+1}) = r - 1 - \frac{\binom{Z}{N} - \binom{Z-x}{N}}{x\binom{Z-1}{N-1}}\epsilon_P$$

$$P_{21}(x) = \sum_{k=0}^{N-1} H(k, N-1, x, Z-1)\,(r-1) = r - 1$$

$$P_{31}(x) = \sum_{k=1}^{N-1} H(k, N-1, x, Z-1)\,(r-1) = (r-1)\left(1 - \frac{\binom{Z-1-x}{N-1}}{\binom{Z-1}{N-1}}\right)$$

$$P_{15}(x) = \sum_{k=0}^{N-1} H(k, N-1, x-1, Z-1)\,(\frac{r(k+1)}{N} + \frac{N\delta - \epsilon_P}{k+1} - \delta - 1) =$$

$$= \frac{r}{N}\left(1 + (x-1)\frac{N-1}{Z-1}\right) + \frac{\binom{Z}{N} - \binom{Z-x}{N}}{x\binom{Z-1}{N-1}}(N\delta - \epsilon_P) - \delta - 1$$

$$P_{51}(x) = \sum_{k=1}^{N-1} H(k, N-1, x-1, Z-1)\,(\frac{rk}{N} - \delta) = \frac{r(N-1)}{N(Z-1)}x - \delta\left(1 - \frac{\binom{Z-1-x}{N-1}}{\binom{Z-1}{N-1}}\right)$$

$$P_{23}(x) = P_{24} = P_{25} = \frac{r}{N}\left(1 + (x-1)\frac{N-1}{Z-1}\right) - 1$$

$$P_{32}(x) = P_{42} = P_{52} = \frac{r(N-1)}{N(Z-1)}x$$

$$P_{34}(x) = P_{43} = P_{35} = P_{53} = P_{45} = P_{54} = 0$$

For AVOID: $P_{41}(x) = 0$ and $P_{14}(x) = \frac{\binom{x-1}{N-1}}{\binom{Z-1}{N-1}}(r - 1 - \frac{\epsilon_P}{N})$

For RESTRICT: P_{14} and P_{41} are hard to compute analytically, we follow the sum formulas in our numerical simulations.

2 Some simplifications of the analytical results and proofs

2.1 Some simplifications

Here, using the well-known inequalities[4]

$$\log N + \gamma < F_N = \sum_{k=1}^{N} \frac{1}{k} \leq \log N + 1$$

where $\gamma = 0.577215$, we provide some simplifications of the conditions obtained in the main text. First of all, regarding the conditions for risk-dominance of AVOID against D, FREE and FAKE:

$$\begin{aligned} \epsilon_P &\leq \frac{c(r-1)}{\log N + \gamma} \\ \delta &\geq \frac{N-r}{NF_{N-1}} c + \frac{F_N}{NF_{N-1}} \epsilon_P. \end{aligned} \tag{2}$$

They can be simplified to

$$\begin{aligned} \epsilon_P &\leq c(r-1)/F_N \\ \delta &\geq \frac{(N^2 - rN)c + \epsilon_P}{N^2 \left(\log(N-1) + 1\right)} + \frac{\epsilon_P}{N}. \end{aligned} \tag{3}$$

Now, the necessary condition for RESTRICT to be risk-dominant against D, which is

$$\epsilon_P + \epsilon_R \leq \frac{N(r-1)}{F_N} c, \tag{4}$$

can be simplified to

$$\epsilon_P + \epsilon_R < \frac{N(r-1)}{\log N + \gamma} c. \tag{5}$$

Furthermore, the necessary condition for RESTRICT to be favored to AVOID

$$(r-1)c \geq \frac{F_N}{N-1} \epsilon_R + \frac{F_{N-1}}{N-1} \epsilon_P \tag{6}$$

can be simplified to

$$(r-1)c \geq \frac{\epsilon_R(\log N + \gamma) + \epsilon_P(\log(N-1) + \gamma)}{N-1} \tag{7}$$

2.2 Some proofs

2.2.1 Ratio of fixation probabilities

It has been shown that[5]

$$\frac{\rho_{j,i}}{\rho_{i,j}} = \prod_{k=1}^{N-1} \frac{T^-(k)}{T^+(k)} = \prod_{k=1}^{N-1} \frac{1 + e^{\beta[P_{ij}(k) - P_{ji}(k)]}}{1 + e^{-\beta[P_{ij}(k) - P_{ji}(k)]}} = e^{\beta \sum_{k=1}^{N-1}(P_{ij}(k) - P_{ji}(k))}$$

Hence, considering two different strategies j and j', the inequality

$$\frac{\rho_{j,i}}{\rho_{i,j}} \geq \frac{\rho_{j',i}}{\rho_{i,j'}}$$

holds if and only if

$$\sum_{k=1}^{N-1} (\pi_{ij}(k) - P_{ji}(k)) \geq \sum_{k=1}^{N-1} (P_{ij'}(k) - P_{j'i}(k))$$

This can be further simplified, in large population limit, to[1]

$$\sum_{k=1}^{N} P_{ij}(k) - \sum_{k=0}^{N-1} P_{ji}(k) \geq \sum_{k=1}^{N} P_{ij'}(k) - \sum_{k=0}^{N-1} P_{j'i}(k)$$

2.2.2 Decrease of F_N/N and $F_N/(N-1)$

We prove that $F_N/N > F_{N+1}/(N+1)$ and that $F_N/(N-1) > F_{N+1}/N$. Indeed, we have

$$\begin{aligned}(N+1)F_N - NF_{N+1} &= N(F_N - F_{N+1}) + F_N = F_N - \frac{N}{N+1} \\ &= \sum_{k=1}^{N} (\frac{1}{k} - \frac{1}{N+1}) > 0\end{aligned} \tag{8}$$

Moreover,

$$NF_N - (N-1)F_{N+1} = N(F_N - F_{N+1}) + F_{N+1} > N(F_N - F_{N+1}) + F_N > 0 \tag{9}$$

Furthermore, since $\lim_{N\to+\infty} F_N = \log N + \gamma$[4], we have

$$\lim_{N\to+\infty} \left(\frac{F_N}{N-1}\epsilon_R + \frac{F_{N-1}}{N-1}\epsilon_P \right) = 0.$$

2.2.3 Properties of the function in Equation (10) in the main text

Consider the following formula

$$\sum_{k=1}^{N-1} \frac{k(1-\psi)}{\psi N + k(1-\psi)} rc - F_N \epsilon_R - F_{N-1}\epsilon_P - (N-1)c. \tag{10}$$

It is clear that it is a decreasing function of ϵ_R and ϵ_P since F_N and F_{N-1} are positive. It increases with r for a similar reason. Moreover, it decreases with ψ since

$$\frac{k(1-\psi)}{\psi N + k(1-\psi)} = \frac{k}{k + N\frac{\psi}{1-\psi}} = \frac{k}{k + N\left(\frac{1}{1-\psi} - 1\right)} \tag{11}$$

is a decreasing function of $\psi \in (0,1)$ for all $1 \le k \le N-1$.

Furthermore, when r tends to infinity, fixing other parameters, ψ (and hence also its threshold below which RESTRICT is better than AVOID, ψ^{AVOID}) tends to 1 since

$$\lim_{r \to +\infty} \frac{F_N \epsilon_R - F_{N-1}\epsilon_P - (N-1)c}{rc} = 0$$

and

$$\sum_{k=1}^{N-1} \frac{k(1-\psi)}{\psi N + k(1-\psi)} = 0 \text{ at } \psi = 1.$$

3 Performance of AVOID and RESTRICT depending on the arrangement cost

In Fig. S1 we show the frequencies of the five strategies in case of AVOID and RESTRICT for varying the cost of arranging commitment ϵ_P. In general, the smaller this cost, the higher the frequency of AVOID and RESTRICT. For small cost of arranging commitment, both AVOID and RESTRICT are highly frequent, dominating their population. When the cost is sufficiently large, in case of AVOID the commitment free-riders FREE takes over. This observation is similar to the pairwise case[2]. But in case of in case of RESTRICT the non-committers D take over. Note that AVOID players do not have to pay this cost when playing with D because no game is played between these strategies (see the models in the main text), while RESTRICT

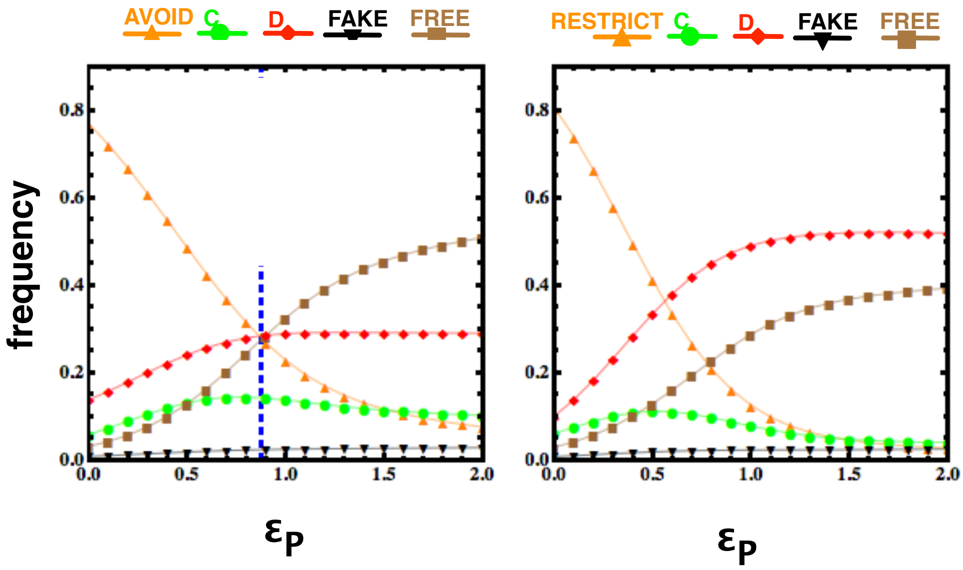

Figure S1: Frequency of each strategy in case of AVOID (left) and RESTRICT (right) for varying ϵ_P. For small cost of arranging commitment, both AVOID and RESTRICT are dominant, while commitment free-riders FREE takes over when the cost is high in the first case, and the non-committers take over in the second case. The blue line is the analytical threshold (derived in the main text of ϵ_P) for which AVOID is risk-dominant against all defectors and free-riders. Clearly, analytical results complies with numerical ones. Parameters: In the right panel, $\epsilon_R = 1.0$; In both cases, $N = 5$, $Z = 100$, $r = 3$, $\delta = 2$; $\beta = 0.1$;

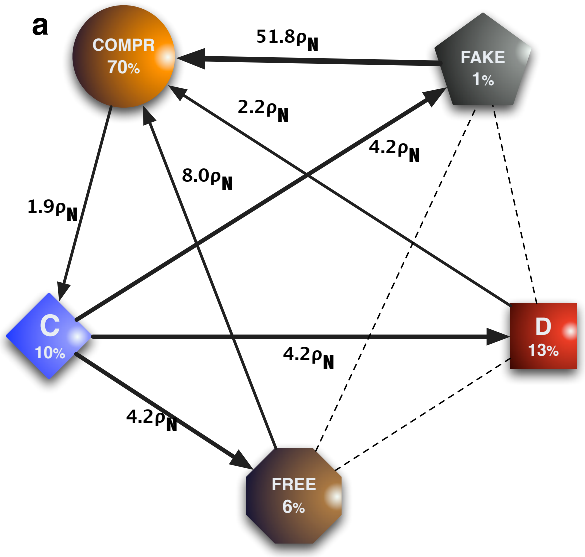

Figure S2: Transition probabilities and stationary distributions in case of RESTRICT. For a given cost of restriction ϵ_R, the better the effect of restriction on non-commiters D, the better RESTRICT. Note the arrow from D to RESTRICT for small ψ **(panel a,** $\psi = 0.25$) which disappears when ψ is large **(panel b,** $\psi = 0.5$). Parameters: $N = 5$, $Z = 100$, $r = 3$; $\epsilon_R = 0.5$; $\beta = 0.1$;

players have to (and also the cost of restriction ϵ_R) when playing with D. We therefore see additionally that in case of AVOID when ϵ_P is sufficiently large, D does not increase in terms of frequency while it does so in case of RESTRICT.

4 Contour plots for AVOID with varying N

For varying N, AVOID is abundant whenever a sufficient compensation is associated with the commitment deal, see Figure S3. Hence, ϵ_P is the essential parameter deciding whether the commitment strategy is successful. Furthermore, when the cost is small the frequency of AVOID decreases with group size; but when the cost is sufficiently large this frequency increases. It is like when we have a good law-enforcing system which reduces the cost of arranging commitment: then AVOID can lead to better cooperation; but once that cost cannot be reduced sufficiently, then interacting in larger groups is actually better for AVOID because the cost is shared between more AVOID players.

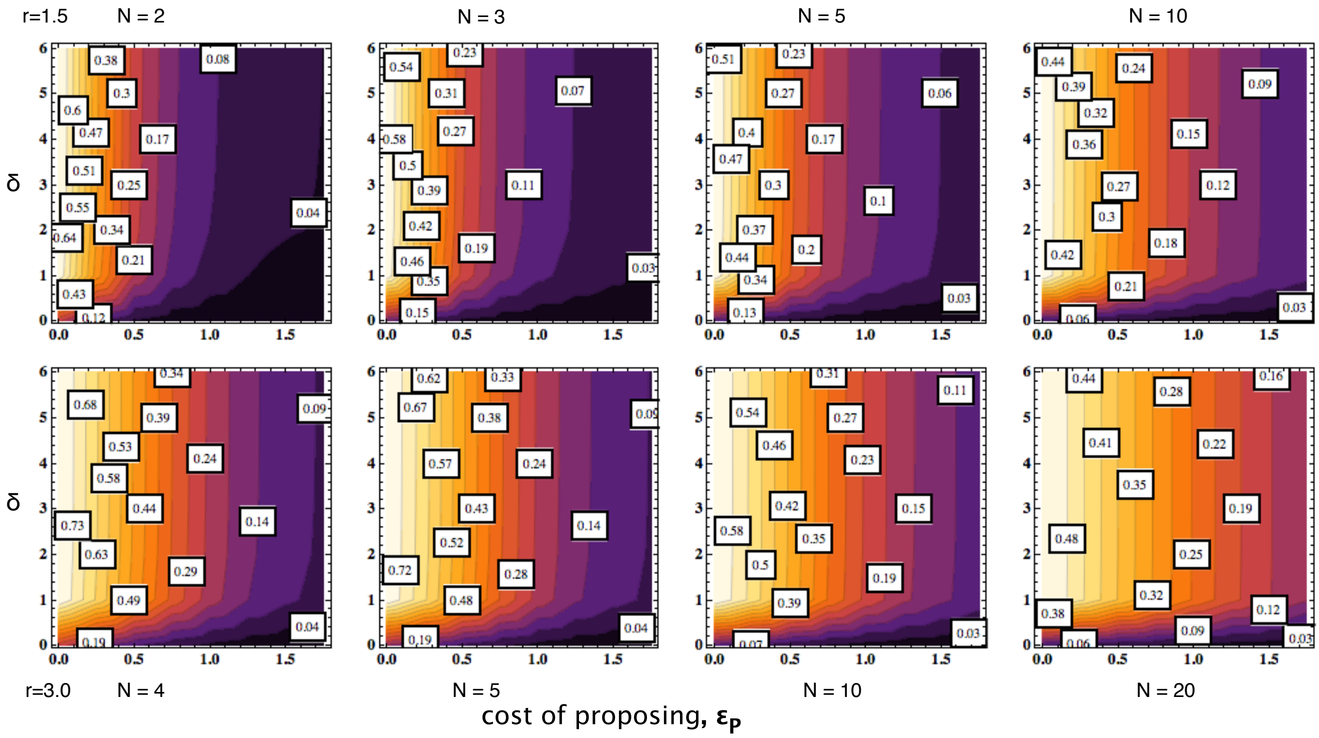

Figure S3: Contour plot of the frequency of AVOID as a function of ϵ_P and δ, for different group sizes N. Parameters: $Z = 100$, $\beta = 0.1$. In general, for small enough cost of arranging the commitment, AVOID is abundant whenever a sufficient compensation is associated with the commitment deal. That is, ϵ_P is the essential parameter for the commitment strategy. Nonetheless, for small ϵ_P the frequency of AVOID decreases with N, while for larger ϵ_p, it increases.

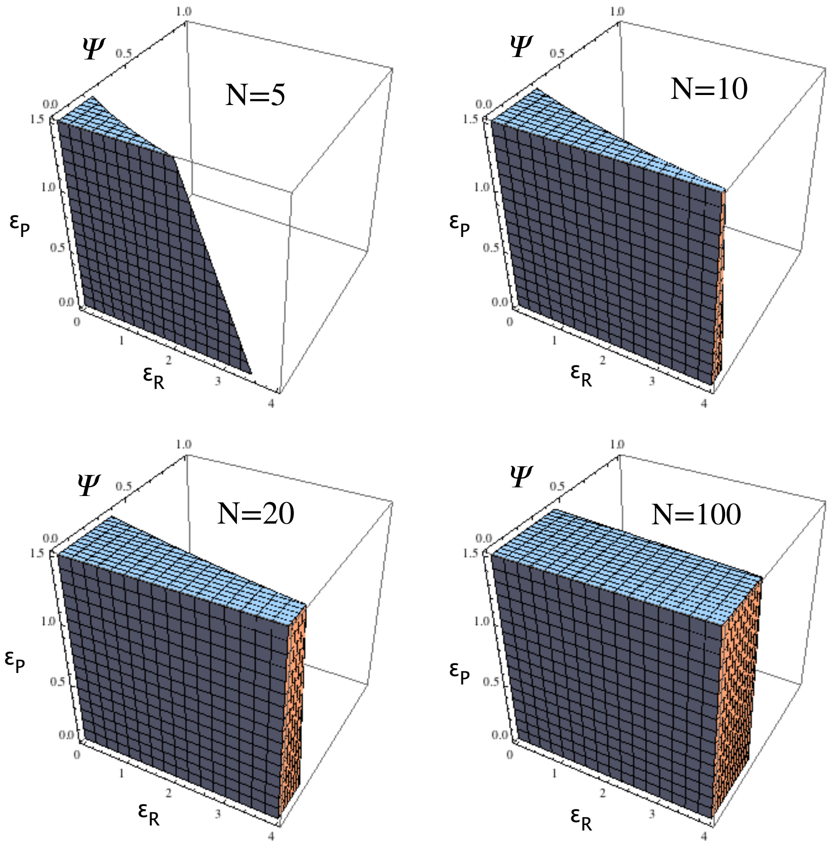

Figure S4: Range of parameters ψ, ϵ_R and ϵ_P, generated from the analytical formula in Eq. (10) in the main text, in which RESTRICT is better than AVOID, for different values of N. In general, the larger N, the larger the parameter space in which RESTRICT is advantageous to AVOID in dealing with non-committers D. Parameters: $Z = 100$, $\epsilon_P = 0.25$, $r = 3$.

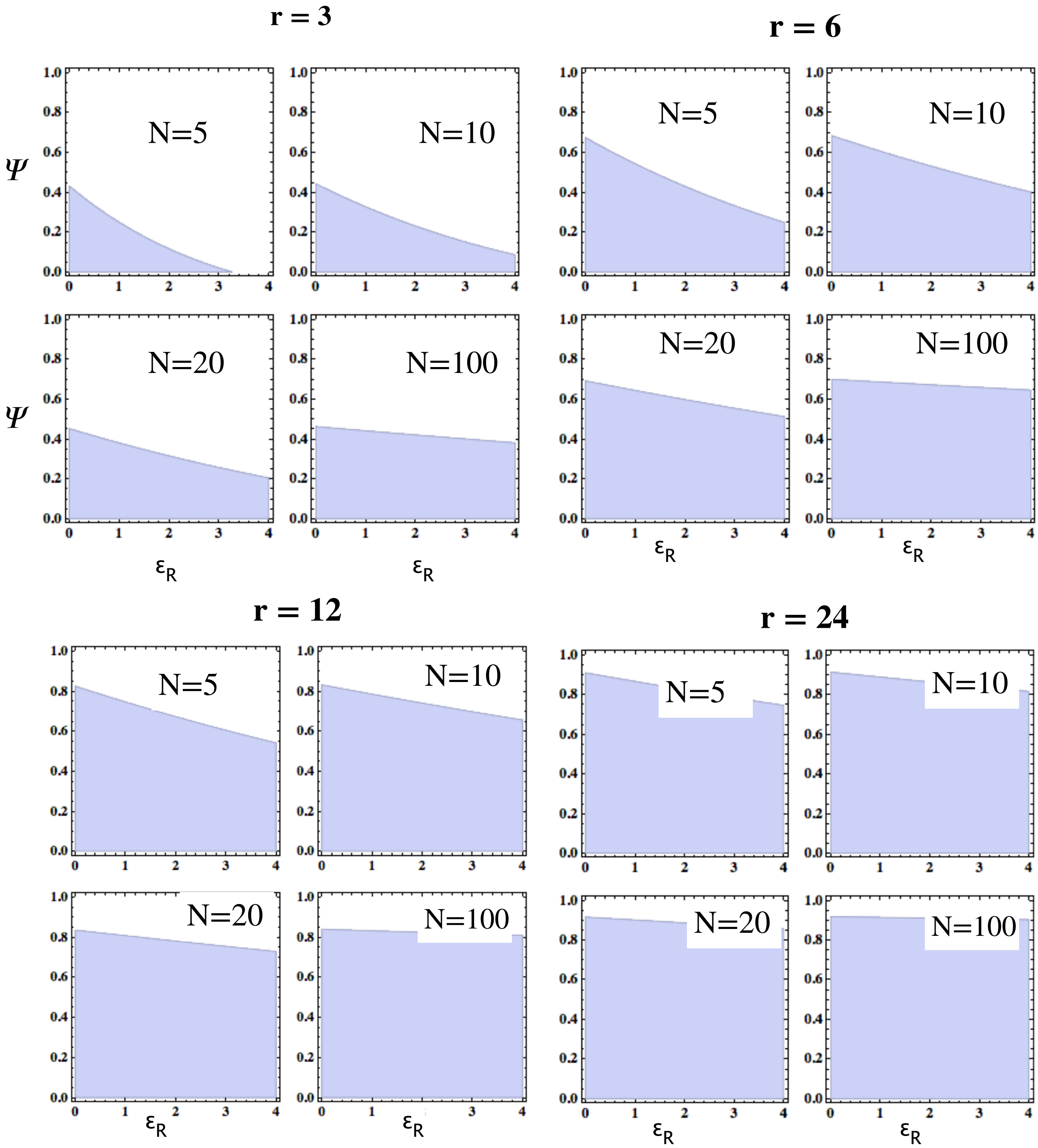

Figure S5: Range of parameters ψ, ϵ_R and ϵ_P, generated from the analytical formula in Eq. ... in the main text, in which RESTRICT is better than AVOID, for different values of N and r. In general, the larger r and N, the larger the parameter space in which RESTRICT is advantageous to AVOID in dealing with non-committers D. Parameters: $Z = 100$, $\epsilon_P = 0.25$.

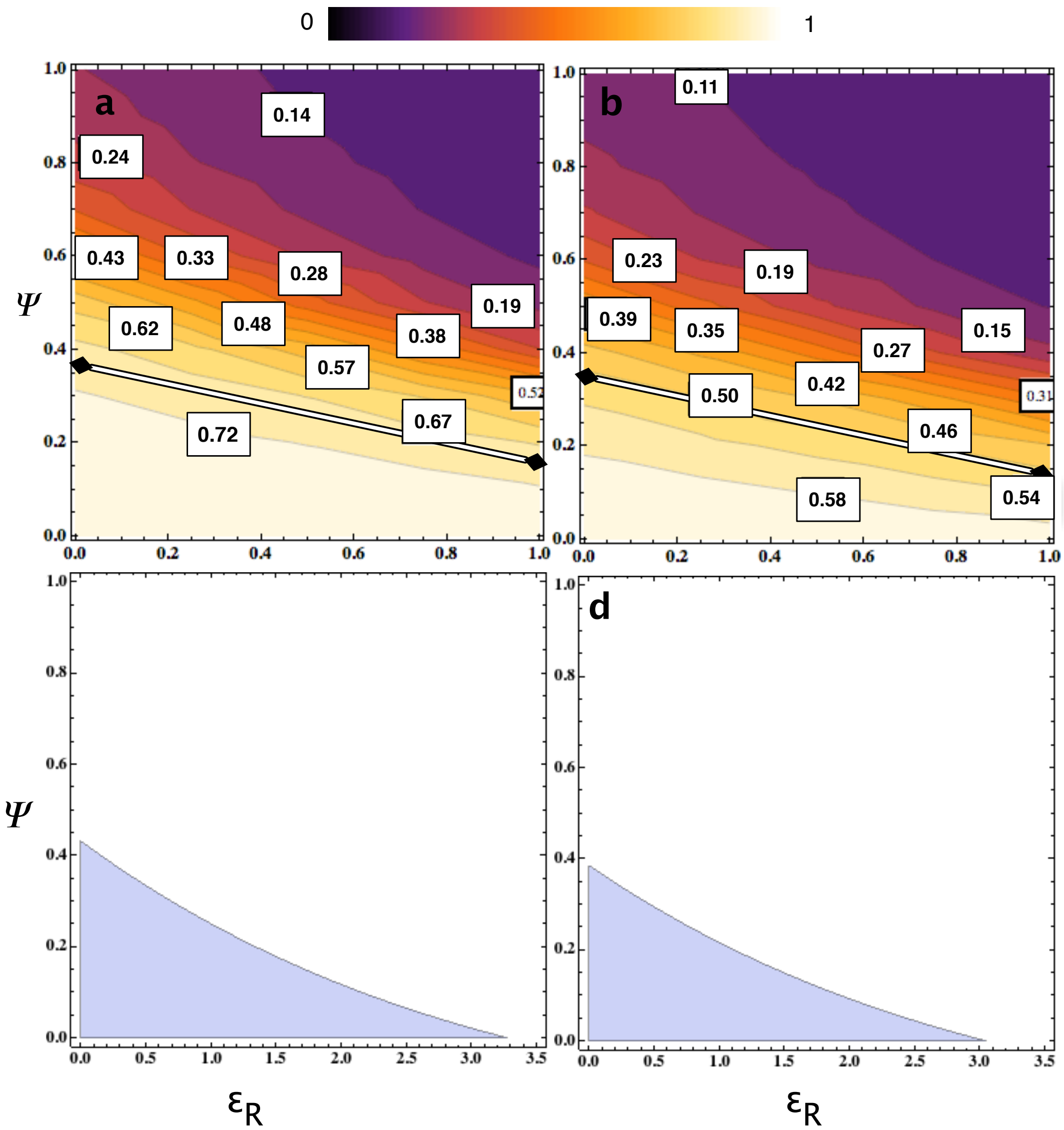

Figure S6: Frequency of RESTRICT as a function of ϵ_R and ψ, with (a) $\epsilon_P = 0.25$ and (b) $\epsilon_P = 0.5$. For a large range of cost for restricting the access of non-committers, ϵ_R, and the restriction, ψ, RESTRICT is better than AVOID. See the area below the double-stroke curves, which corresponds to the frequency of AVOID (0.64 in panel a and 0.49 in panel b). In general, the larger ϵ_R, the smaller ψ required for RESTRICT to be advantageous to AVOID. This clearly complies with **analytical results generated by Eq. (10) in the main text, as shown in the panels (c)** $\epsilon_P = 0.25$ **and (d)** $\epsilon_P = 0.5$. Interestingly, ψ is the decisive parameter on the frequency of RESTRICT. Parameters: $N = 5$, $Z = 100$, $r = 3$; $\delta = 2$; $\beta = 0.1$.

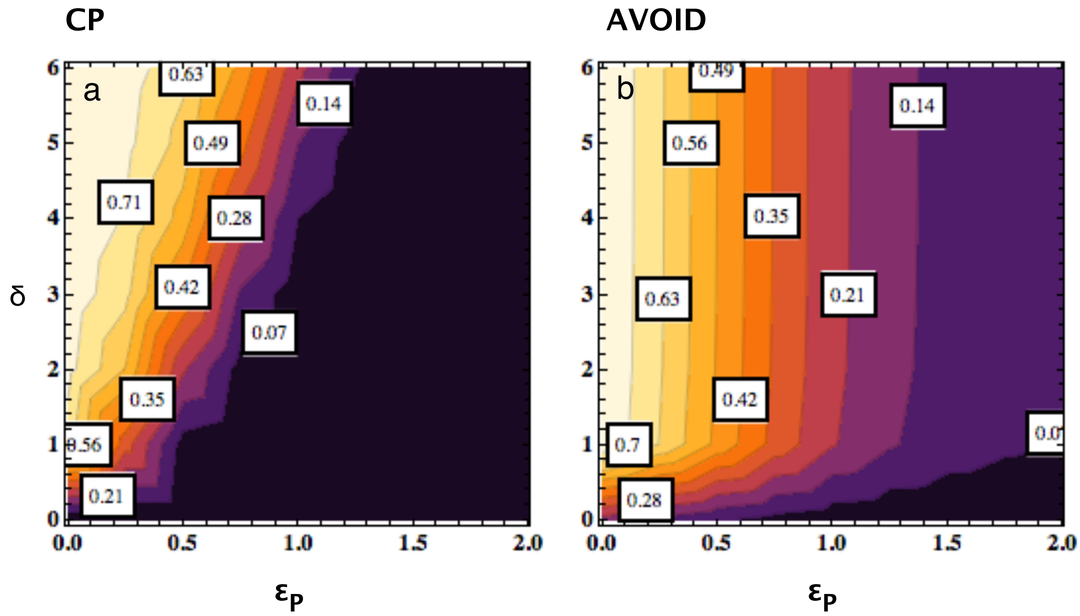

Figure S7: Costly peer punishment (CP) versus AVOID. (a) Fraction of CP in a population with C and D; (b) fraction of AVOID in a population with C, D, FREE and FAKE. Parameters: $N = 5$, $Z = 100$, $r = 3$; $\delta = 2$; $\epsilon_P = 0.25$; $\beta = 0.1$.

5 RESTRICT vs. AVOID for varying N and r

We generate analytical results using Eq. (10) in the main text, describing the parameter space where RESTRICT is better than AVOID in dealing with non-committers (hence, becomes more frequent in the population with the other four non-proposing strategies). In general, the larger N, the larger the parameter space in which RESTRICT is advantageous to AVOID in dealing with non-committers, see Figure S4.

In Fig. S5 we show similar results for varying the public goods producing factor r. The results show that the larger r, the larger parameter space where RESTRICT is advantageous to AVOID. It complies with the Eq (10) in the main text, the left hand size of which is clearly an increasing function of r.

In Fig. S6 we also show that these analytical results corroborate with the the numerical simulations.

6 Simple Punishment vs. AVOID

A costly peer punishment strategy, CP, in the PGG game, contributes to the public good. After the PGG was played, the punisher can impose a fine δ upon each non-contributor (defector) D, at a personal cost ϵ_P (see more details in reference[3]).

Figure S7 shows that, differently from AVOID where ϵ is the crucial parameter as long as δ is sufficiently large, the frequency of CP always increases with δ. We observe that AVOID is more frequent than CP most of the time.

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