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Absorbing boundary conditions for numerical simulation of waves

(artificial boundaries/numerical approximation)

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ABSTRACT In practical calculations, it is often essential to introduce artificial boundaries to limit the area of computation. Here we develop a systematic method for obtaining a hierarchy of local boundary conditions at these artificial boundaries. These boundary conditions not only guarantee stable difference approximations, but also minimize the (unphysical) artificial reflections that occur at the boundaries.

When calculating solutions to partial differential equations it is often essential to introduce artificial boundaries to limit the area of computation. Important areas of application that use artificial boundaries are local weather prediction (see refs. 1 and 2), geophysical calculations involving acoustic and elastic waves (see refs. 3 and 4), and a variety of other problems in fluid dynamics. One always needs some boundary conditions at these boundaries to guarantee a unique and well-posed solution to the differential equation. In turn, this is a necessary condition for *stable* difference approximation. Of course these artificial boundaries are only a computational necessity and have no physical significance. Thus, it is highly desirable to design boundary conditions for these artificial boundaries that minimize the amplitudes of reflected waves.

Our objective is to design a hierarchy of boundary conditions at these artificial boundaries with the following properties:

(*i*) These boundary conditions together with the associated differential equation guarantee a well-posed mixed initial boundary value problem.

(*ii*) The amplitudes of the reflection coefficients of these boundary conditions are as small as possible.

(*iii*) These boundary conditions are local.

The importance of the conditions in (*i*) and (*ii*) has been discussed above. The condition in (*iii*) is essential for a reasonable control of the operation count for the numerical approximation.

By using the recently developed theory of reflection of singularities (see refs. 5–7), one can develop perfectly absorbing boundary conditions (with reflection coefficients identically zero) for general variable coefficient systems of hyperbolic differential equations. Unfortunately, these boundary conditions are necessarily nonlocal in *both space and time* and thus are not useful for practical calculations. Nevertheless, it is possible to approximate these perfectly absorbing nonlocal boundary conditions by hierarchies of local boundary conditions so that (*i*), (*ii*), and (*iii*) above are satisfied. Numerical experiments support these conclusions. We have developed (see ref. 8) a complete analysis of the above ideas together with a report on test calculations for the wave equation and linearized shallow water equations. Here we shall only report on the methods and the nature of the resulting boundary conditions for some special cases.

The wave equation in a half-space

We design absorbing boundary conditions on the wall $x = 0$ for solutions w of the wave equation

$$\Box w = \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = 0$$

in the quarter space $t,x \geq 0$. Plane wave solutions traveling to the left are given by

$$w = e^{i(\sqrt{\zeta^2-\omega^2}x + \zeta t + \omega y)}, \quad \zeta^2 - \omega^2 > 0, \quad \zeta > 0.$$

Here ζ has the interpretation of frequency and $\omega/\zeta = \sin\theta$, where θ is the angle of incidence of the wave upon the boundary $x = 0$. If (ω,ζ) is held fixed, the first order boundary condition that produces a zero reflection coefficient is given by

$$\left(\frac{d}{dx} - i\sqrt{\zeta^2 - \omega^2}\right) w|_{x=0} = 0.$$

By superposition, more general wave packets traveling to the left are represented by

$$w(x,y,t) = \int\int_{\sqrt{\zeta^2-\omega^2}>0} e^{i(\sqrt{\zeta^2-\omega^2}x + \zeta t + \omega y)}\hat{w}(0,\zeta,\omega)d\zeta\, d\omega$$

where $\hat{w}$ denotes the Fourier transform in (t,y). It is easily seen that the boundary condition that produces zero reflection coefficient for these wave packets is given by

$$\left(\frac{\partial}{\partial x} - \sqrt{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}}\right) w|_{x=0} = 0 \qquad [1]$$

where $\sqrt{\partial^2/\partial t^2 - \partial^2/\partial y^2}$ is defined by

$$\sqrt{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}} = \int\int e^{i(\zeta t + \omega y)} i\sqrt{\zeta^2 - \omega^2} \times \hat{w}(0,\zeta,w)d\zeta\, d\omega.$$

For this special case, the *boundary condition in* [**1**] is *the theoretical nonlocal boundary condition* alluded to above.

To develop a hierarchy of local highly absorbing approximations we first observe that under Fourier transform

$$i\zeta \leftrightarrow \frac{\partial}{\partial t} \text{ and } i\omega \leftrightarrow \frac{\partial}{\partial y}. \qquad [2]$$

We then write $i\sqrt{\zeta^2 - \omega^2}$ in the form $i\zeta\sqrt{1 - (\omega^2/\zeta^2)}$, and with $x = \omega/\zeta$ we then approximate $\sqrt{1+x}$ using

$$\text{(1st)}\quad \sqrt{1+x} = 1 + 0(|x|)$$

$$\text{(2nd)}\quad \sqrt{1+x} = 1 + \frac{1}{2}x + 0(|x|^2)$$

$$\text{(3rd)}\quad \sqrt{1+x} = 1 + \frac{x}{2 + (x/2)} + 0(|x|^3). \qquad [3]$$

After clearing the denominator by multiplying by appropriate powers of $i\zeta$ and using the correspondence in [**2**], we obtain the following *hierarchy* of *highly absorbing* boundary conditions for the wave equation on the wall $x = 0$:

$$\text{(1st approx.)}\quad w_x - w_t|_{x=0} = 0$$

$$\text{(2nd approx.)}\quad w_{xt} - w_{tt} + \frac{1}{2}w_{yy}|_{x=0} = 0$$

$$\text{(3rd approx.)}\quad w_{xtt} - w_{ttt} - \frac{1}{4}w_{xyy} + \frac{3}{4}w_{tyy}|_{x=0} = 0. \qquad [4]$$

It is verified in ref. 8 that each of these boundary conditions satisfies the three conditions discussed in the introduction. For a wave with a 45° angle of incidence the (1st), (2nd), and (3rd) approximations reflect waves with amplitudes, respectively, 17%, 3%, and 0.5% of the amplitude of the incident wave. We remark here that another obvious approximation for $\sqrt{1+x}$ is given by the Taylor expansion,

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + 0(|x|^3).$$

If we follow the same procedure, we obtain the absorbing boundary condition

$$w_{xttt} - w_{tttt} + \frac{1}{2}w_{yytt} + \frac{1}{8}w_{yyyy}|_{x=0} = 0. \qquad [5]$$

However, by applying the normal mode analysis developed in ref. 9, we have observed that the *boundary condition* in [5] is *strongly ill-posed for the wave equation* and therefore is *useless* for *practical calculations*.

As previously mentioned, more sophisticated versions of the above arguments using the theory of pseudo-differential operators (see ref. 6) apply to yield highly absorbing boundary conditions for general variable coefficient hyperbolic systems. To illustrate these facts, we list these boundary conditions in two important special cases.

The wave equation in polar coordinates

We design highly absorbing boundary conditions on the circle $r = a$ for solutions of the wave equation inside the region, $r \leq a$. The operator we study has the form

$$L = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial^2\theta} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial t^2}$$

and the approximations analogous to those in [4] above are calculated to be given by

$$\text{(1st approx.)} \quad \frac{\partial}{\partial r} + \frac{\partial}{\partial t} + \frac{1}{2a}w|_{r=a} = 0$$

$$\text{(2nd approx.)} \quad \frac{\partial^3}{\partial r\partial t^2} + \frac{\partial^3}{\partial t^3} - \frac{1}{2a^2}\frac{\partial^3}{\partial t\partial\theta^2} + \frac{1}{2a}\frac{\partial^2}{\partial t^2} + \frac{1}{2a^3}\frac{\partial^2}{\partial\theta^2}w|_{r=a} = 0.$$

We remark that with $r \to \infty$, outgoing waves have the form

$$\frac{f(r-t)}{r^{1/2}}[a(\theta) + 0(1/r)].$$

so that the first approximation could be deduced directly from the outgoing condition. We do not know of a derivation of the 2nd approximation from similar *ad hoc* physical reasoning.

The linearized shallow water equations

Here we illustrate our theory for the linearized shallow water equations in the quarter space $x,t \geq 0$. This 3×3 matrix system has the form

$$\frac{\partial u}{\partial t} = \begin{pmatrix} a & 0 & c \\ 0 & a & 0 \\ c & 0 & a \end{pmatrix}\frac{\partial u}{\partial x} + \begin{pmatrix} b & 0 & 0 \\ 0 & b & c \\ 0 & c & b \end{pmatrix}\frac{\partial u}{\partial y} + \begin{pmatrix} 0 & f & 0 \\ -f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}u$$

together with the physical restrictions, $0 < a^2 + b^2 < c^2$, $c > 0$. For simplicity we assume all of the above matrices are constants. To aid in defining our boundary conditions, we set $w = \mathcal{U}^{-1}u$ where $\mathcal{U}$ is the unitary matrix given by

$$\mathcal{U} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

The boundary conditions depend upon whether we have linearized about an *inflowing state* with $a < 0$ or about an *outflowing state* with $a > 0$.

The inflow case: Analogous to [4] we have for

$$w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix},$$

$$\text{(1st approx.)} \quad \begin{matrix} w_1|_{x=0} = 0 \\ w_2|_{x=0} = 0 \end{matrix}$$

$$\text{(2nd approx.)} \quad \left[\begin{array}{l} \frac{\partial w_1}{\partial t} + \frac{\sqrt{2}}{4}(a+c)\frac{\partial w_2}{\partial y}\Big|_{x=0} = 0 \\ \frac{\partial w_2}{\partial t} + \frac{\sqrt{2}}{2}\frac{b}{c}(a - \sqrt{2}(a+c))\frac{\partial w_2}{\partial y} \\ \qquad - \frac{a+c}{\sqrt{2}}\frac{\partial w_3}{\partial y} + \frac{(a+c)f}{c\sqrt{2}}w_3|_{x=0} = 0. \end{array}\right.$$

Numerical experiments indicate that in this situation the 2nd approximation is roughly 4 times as effective as the 1st approximation. Similar absorbing boundary conditions also can be developed in the outflow case (see ref. 8).

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