Mathematical Proofs of Biological Auctions with Multiple Rewards

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Proof of Lemma 1. Recall that we assume an infinite population size and that s_1 is strictly greater than s_2. We denote by x the frequency of the s_2 strategists. Hence, $1-x$ denotes the frequency of the s_1 strategists. The individuals compete for two rewards with value v_1 and v_2. Since the participants in each auction are chosen randomly from the population, the probability that k out of the n participants in an auction are s_1 strategists is $\binom{n}{k}x^{n-k}(1-x)^k$.

Following Chatterjee et al. (2012), we prove the expected payoff of an s_1 strategist as follows. Let i be the number of individuals with strategy s_1 participating in the same auction apart from the s_1 strategist under consideration. The probability for $i=0$ is $\binom{n-1}{0}x^{n-1}$ in which case the s_1 strategist surely wins reward v_1. If $0 < i \leq n-1$, the expected reward of the s_1 strategist is $\frac{v_1+v_2}{i+1}$. Since the payment of each individual is equivalent to its strategy, the expected payoff of an s_1 strategist is given by

$$\begin{aligned}
p(s_1) &= \sum_{i=1}^{n-1} \binom{n-1}{i} x^{n-1-i}(1-x)^i \frac{v_1+v_2}{i+1} + v_1 x^{n-1} - s_1 \\
&= \sum_{i=0}^{n-1} \binom{n-1}{i} x^{n-1-i}(1-x)^i \frac{v_1+v_2}{i+1} - v_2 x^{n-1} - s_1 \\
&= \frac{v_1+v_2}{n} \sum_{i=0}^{n-1} \frac{n\cdot(n-1)!}{(i+1)\cdot i!\cdot(n-(i+1))!} \cdot x^{n-(i+1)} \cdot (1-x)^i - v_2 x^{n-1} - s_1 \\
&= \frac{v_1+v_2}{n\cdot(1-x)} \sum_{i=0}^{n-1} \binom{n}{i+1} \cdot x^{n-(i+1)} \cdot (1-x)^{i+1} - v_2 x^{n-1} - s_1 \\
&= \frac{v_1+v_2}{n\cdot(1-x)} \underbrace{\sum_{i=1}^{n} \binom{n}{i} \cdot x^{n-i} \cdot (1-x)^i}_{1-x^n \text{ since } \sum_{k=0}^{n}\binom{n}{k}x^{n-k}(1-x)^k=1} - v_2 x^{n-1} - s_1 \\
&= \frac{v_1+v_2}{n\cdot(1-x)} \cdot (1-x^n) - v_2 x^{n-1} - s_1
\end{aligned} \tag{S1}$$

Next, we calculate the expected payoff of a s_2 strategist. The s_2 strategist wins the first reward v_1 only in the case there are no s_1 strategists chosen for the same auction. In this case the s_2 strategists wins v_1 with probability $1/n$ or wins v_2 with probability $(1-1/n)/(n-1)$. In the case of a single s_1 bidder participating in the auction, the s_2 strategist can only win v_2 with probability $1/(n-1)$. Accounting for the probabilities of both cases and the payment of s_2 which is independent of the outcome, we derive the following expected payoff of an s_2

strategist:

$$p(s_2) = \binom{n}{0} x^{n-1}(1-x)^0 \left[\frac{v_1}{n} + \left(1-\frac{1}{n}\right)\frac{v_2}{n-1}\right] + \binom{n}{1} x^{n-2}(1-x)\frac{v_2}{n-1} - s_2$$
$$= x^{n-1}\left[\frac{v_1+v_2}{n}\right] + x^{n-2}(1-x)v_2 - s_2 = \frac{x^{n-1}(v_1+v_2)}{n} + x^{n-2}(1-x)v_2 - s_2 \ . \quad \text{(S2)}$$

□

Proof of Theorem 1. Using the results of Lemma 1 we can analyze situations in which a strategy s is beaten by another strategy s'. We distinguish two situations (i.e., s' has a higher expected payoff than s):

– $s' < s$: Since we want $p(s')$ to be larger as $p(s)$, the following inequality has to hold:

$$\frac{(v_1+v_2)x^{n-1}}{n} + v_2 x^{n-2}(1-x) - s' > \frac{(v_1+v_2)(1-x^n)}{n(1-x)} - v_2 x^{n-1} - s$$

and can be rewritten as:

$$s - s' > \frac{(v_1+v_2)(1-x^{n-1})}{n(1-x)} - v_2 x^{n-2} \quad \text{(S3)}$$

– $s' > s$: Again we want $p(s')$ to be larger as $p(s)$ which requires that:

$$\frac{(v_1+v_2)(1-x^n)}{n(1-x)} - v_2 x^{n-1} - s' > \frac{(v_1+v_2)x^{n-1}}{n} + v_2 x^{n-2}(1-x) - s$$

and can be rewritten as:

$$s' - s < \frac{(v_1+v_2)(1-x^{n-1})}{n(1-x)} - v_2 x^{n-2} \quad \text{(S4)}$$

Next, we assume that the frequency of the invaders with strategy s' is $\epsilon \to 0$ and the frequency of the s strategists is $(1-\epsilon)$. Similar as in Chatterjee et al. (2012), higher order terms of ϵ are ignored. The mutant strategy s' can invade a strategy s if $p(s') > p(s)$. In the case of $s' < s$, we replace x by ϵ and obtain:

$$s - s' > \frac{(v_1+v_2)(1-\epsilon^{n-1})}{n(1-\epsilon)} - v_2\epsilon^{n-2} \approx \frac{v_1+v_2}{n} \ . \quad \text{(S5)}$$

In the case of $s' > s$, we replace x by $1-\epsilon$ and obtain:

$$s'-s < \frac{(v_1+v_2)(1-(1-\epsilon)^{n-1})}{n\epsilon} - v_2(1-\epsilon)^{n-2} \approx \frac{(n-1)(v_1+v_2)}{n} - v_2 = v_1 - \frac{v_1+v_2}{n} \ . \quad \text{(S6)}$$

Theorem 1 follows from Eq. (S5) and Eq. (S6). □

Proof of Theorem 2. We prove the mixed equilibria for two rewards with $n = 2$ and $n = 3$ as follows. For both values of n we differentiate Equation (3) with respect to s and set it to zero. In the case of $n = 2$, we get

$$\frac{\delta E(s, I)}{\delta s} = (v_1 - v_2) \cdot p(s) - 1 = 0 \ .$$

and hence obtain $p(s) = \frac{1}{v_1 - v_2}$. In the case of $n = 3$, we get

$$\frac{\delta E(s, I)}{\delta s} = 2p(s)\left[P(s)(v_1 - 2v_2) + v_2\right] - 1 = 0,$$

rewriting and integrating w.r.t. s gives $(v_1 - 2v_2)P(s)^2 + 2v_2P(s) = s$. We solve this quadratic equation for $P(s)$ and obtain the two possible solutions: $\frac{-v_2 \pm \sqrt{v_2^2 + s(v_1 - 2v_2)}}{v_1 - 2v_2}$. Since only in the plus case the solution for the probability density function provides positive values for $0 \leq s \leq v_1$, we can disregard the minus case. We obtain $p(s) = \frac{1}{2\sqrt{v_2^2 + s(v_1 - 2v_2)}}$ for $n = 3$ and hence we have proven that both probability density functions given in Theorem 2 are indeed mixed equilibria. Next we check if the derived probability density functions are also ESS.

We let I be a mixed strategy and J be some pure strategy. Similar as in Chatterjee et al. (2012), we denote by $E(X, (Y^i, Z^j))$ the expected payoff of the strategy X playing against i individuals with strategy Y and j individuals with strategy Z, where $X, Y, Z \in I, J$, and $i + j = n - 1$. A mixed strategy I is an ESS iff one of the following two conditions holds for all strategies J different from I (Maynard Smith and Price, 1973; Maynard Smith, 1974, 1982; Haigh and Cannings, 1989):

1. $E(I, (I^{n-1}, J^0)) > E(J, (I^{n-1}, J^0))$; or
2. $E(I, (I^{n-1}, J^0)) = E(J, (I^{n-1}, J^0))$ and $E(I, (I^{n-2}, J^1)) > E(J, (I^{n-2}, J^1))$.

In the case of $n = 2$, we need to calculate the following four payoffs where I is the mixed equilibrium strategy in Equation (4) and J is a pure strategy with value s in $[0, v_1 - v_2]$:

- $E(I,(I^{n-1},J^0)) = E(I,(I^1,J^0))$: Since both players use the same strategy their expected reward is $(v_1+v_2)/2$. The expected payment of a strategy uniformly distributed in $[0,v_1-v_2]$ is $\int_0^{v_1-v_2} x \cdot p(x)\ dx = (v_1-v_2)/2$ and hence the expected payoff is v_2.

- $E(J,(I^{n-1},J^0)) = E(J,(I^1,J^0))$: The expected payoff of J against I is $(v_1-v_2)\cdot \int_0^s p(x)\ dx + v_2 - s = v_2$.

- $E(I,(I^{n-2},J^1)) = E(I,(I^0,J^1))$: The probability of I to win v_1 is given by $1-\int_0^s p(x)\ dx$. Therefore, the expected reward is $(v_1-v_2)[1-s/(v_1-v_2)]+v_2 = v_1 - s$. The expected payment is $\int_0^{v_1-v_2} x\cdot p(x)\ dx = (v_1-v_2)/2$ and hence the payoff is $(v_1+v_2)/2 - s$.

- $E(J,(I^{n-2},J^1)) = E(J,(I^0,J^1))$: The expected payoff in this case is $(v_1+v_2)/2 - s$.

We observe that the first ESS condition is not satisfied because $E(I,(I^1,J^0)) = E(J,(I^1,J^0)) = v_2$. Since the second ESS condition does also not hold because $E(I,(I^0,J^1)) = E(J,(I^0,J^1)) = (v_1+v_2)/2 - s$, the mixed equilibrium given in Equation (4) is not an ESS.

In the case of $n=3$, we calculate the following expected payoffs to prove that Equation (5) is an ESS. Suppose strategy I is given by Equation (5) and J is any other pure strategy with value $s \geq 0$:

- $E(I,(I^{n-1},J^0)) = E(I,(I^2,J^0))$: Since all three participants in the auction use the same strategy, their expected payoff has to be identical. Their expected reward is $\frac{v_1+v_2}{3}$ and their expected payment is given by $\int_0^{v_1} x\ p(x)\mathrm{d}x$. Evaluating both terms gives that:

$$E(I,(I^2,J^0)) = \frac{v_1+v_2}{3} - \frac{\sqrt{v_1^2-2v_1v_2+v_2^2}\cdot(v_1^2-2v_1v_2-2v_2^2)+2v_2^3}{3(v_1-2v_2)^2} = 0\ .$$

- $E(J,(I^{n-1},J^0)) = E(J,(I^2,J^0))$: The J-strategist wins reward v_1 iff both I-strategists bid less than s, which happens with probability $(\int_0^s p(x)\mathrm{d}x)^2$. The J-strategist wins reward v_2 iff one of the two I-strategists bids more than s. The probability of this happening is $2(\int_0^s p(x)\mathrm{d}x)(1-\int_0^s p(x)\mathrm{d}x)$. Thus, the expected payoff $E(J,(I^{n-1},J^0))$ is equal to $v_1(\int_0^s p(x)\mathrm{d}x)^2 + 2v_2(\int_0^s p(x)\mathrm{d}x)(1-\int_0^s p(x)\mathrm{d}x) - s$, which simplifies to $(v_1-2v_2)P(s)^2 + 2v_2P(s) - s$. However, from the proof of the mixed equilibrium we know that this term evaluates to zero if s is in the support of I. We follow that $E(J,(I^{n-1},J^0))$ is zero if $s \leq v_1$ and is negative if $s > v_1$.

– $E(I,(I^{n-2},J^1)) = E(I,(I^1,J^1))$: Again we denote by $P(s) = \int_0^s p(x)\mathrm{d}x$. Hence, the expected payoff is given by

$$E(I,(I^1,J^1)) = \frac{v_1+v_2}{2}[1-P(s)]^2 + \frac{2v_1}{2}P(s)[1-P(s)] + \frac{v_2}{2}P(s)^2 - \int_0^{v_1} x\ p(x)\mathrm{d}x\ .$$

– $E(J,(I^{n-2},J^1)) = E(J,(I^1,J^1))$: Similarly as above we obtain for the expected payoff:

$$E(J,(I^1,J^1)) = \frac{v_1+v_2}{2}P(s) + \frac{v_2}{2}[1-P(s)] - s\ .$$

Since $E(I,(I^2,J^0)) = E(J,(I^2,J^0)) = 0$, the first ESS condition does not hold and we focus on the second condition: $E(I,(I^1,J^1)) > E(J,(I^1,J^1))$ which evaluates to:

$$\frac{v_1+v_2-3s}{6} > \frac{v_1\left(\sqrt{v_1 s + v_2^2 - 2v_2 s} - 2s\right) - 2v_2^2 + 4v_2 s}{2(v_1-2v_2)}$$
$$\frac{4v_2^2 + v_1(v_1+3s) - v_2(v_1+6s) - 3v_1\sqrt{v_2^2 + v_1 s - 2v_2 s}}{6(v_1-2v_2)} > 0.$$

To check whether this condition is met, we take the first and second derivative of $f(s)$ to find the minimum of this function:

$$f'(s) = \frac{3v_1 - 6v_2 - \frac{3v_1(v_1-2v_2)}{2\sqrt{v_1 s + v_2^2 - 2v_2 s}}}{6(v_1-2v_2)}$$
$$f''(s) = \frac{v_1(v_1-2v_2)}{8\sqrt{(v_1 s + v_2^2 - 2v_2 s)^3}}\ .$$

We observe that $f''(s)$ is positive iff $v_2 < v_1/2$ and is negative iff $v_2 > v_1/2$. Hence, $f(s)$ at s^* is a minimum iff $v_2 < v_1/2$ and a maximum iff $v_2 > v_1/2$. We obtain $s^* = \frac{v_1+2v_2}{4}$ via setting $f'(s^*) = 0$. Plugging s^* into our original function we get that $f(s^*) = \frac{v_1-2v_2}{24}$ which is positive iff $v_1 > 2v_2$. Therefore, the mixed equilibrium given in Equation (5) is an ESS iff $v_1 > 2v_2$ holds which completes the proof of Theorem 2. □

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