Electronic supplementary material for the paper "A Stefan problem on an evolving surface"

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(ESM1) *Proof that L^p_X is a normed space (slight modification of the proof of Theorem 2.8 in [1]).* It is easy to verify that the expressions in (2.2) define norms if the integrals on the right hand sides are well-defined, which we now check. So let $u \in L^p_X$. Then $\tilde{u} := \phi_{-(\cdot)}u(\cdot) \in L^p(0,T;X_0)$. Define $F\colon [0,T] \times X_0 \to \mathbb{R}$ by $F(t,x) = \|\phi_t x\|_{X(t)}$. By assumption, $t \mapsto F(t,x)$ is measurable for all $x \in X_0$, and if $x_n \to x$ in X_0, then by the reverse triangle inequality,

$$|F(t,x_n) - F(t,x)| \leq \|\phi_t(x_n - x)\|_{X(t)} \leq C_X \|x_n - x\|_{X_0} \to 0,$$

so $x \mapsto F(t,x)$ is continuous. Thus F is a Carathéodory function. Due to the condition $|F(t,x)| \leq C_X \|x\|_{X_0}$, by Remark 3.4.5 of [2], the Nemytskii operator N_F defined by $(N_F x)(t) := F(t,x(t))$ maps $L^p(0,T;X_0) \to L^p(0,T)$, so that

$$\|N_F \tilde{u}\|^p_{L^p(0,T)} = \int_0^T \|u(t)\|^p_{X(t)} < \infty.$$

□

(ESM2) *Proof of Lemma 2.3.* First we show that if $u \in L^p(0,T;X_0)$, then $\phi_{(\cdot)}u(\cdot) \in L^p_X$.

Let $u \in L^p(0,T;X_0)$ be arbitrary. By density, there exists a sequence of simple functions $u_n \in L^p(0,T;X_0)$ with

$$\|u_n - u\|_{L^p(0,T;X_0)} \to 0$$

and thus for almost every t,

$$\|u_n(t) - u(t)\|_{X_0} \to 0$$

for a subsequence, which we relabelled. We have that $\phi_t u_n(t) \to \phi_t u(t)$ in $X(t)$ by continuity; this implies

$$\|\phi_t u_n(t)\|_{X(t)} \to \|\phi_t u(t)\|_{X(t)} \qquad \text{pointwise a.e.} \tag{1}$$

Write $u_n(t) = \sum_{i=1}^{M_n} u_{n,i}\mathbf{1}_{B_i}(t)$ where the $u_{n,i} \in X_0$ and the B_i are measurable, disjoint and partition $[0,T]$. Then

$$\phi_t u_n(t) = \sum_{i=1}^{M_n} \phi_t(u_{n,i})\mathbf{1}_{B_i}(t) \in X(t).$$

Taking norms and exponentiating, we get

$$\|\phi_t u_n(t)\|^p_{X(t)} = \sum_{i=1}^{M_n} \|\phi_t u_{n,i}\|^p_{X(t)} \mathbf{1}^p_{B_i}(t),$$

which is measurable (with respect to t) since, by assumption, the $\|\phi_t u_{n,i}\|_{X(t)}$ are measurable and a finite sum of measurable functions is measurable. Thus, by (1), $\|\phi_t u(t)\|_{X(t)}$, is measurable. Finally,

$$\int_0^T \|\phi_t u(t)\|^p_{X(t)} \leq \int_0^T C_X^p \|u(t)\|^p_{X_0} = C_X^p \|u\|^p_{L^p(0,T;X_0)},$$

so $\phi_{(\cdot)}u(\cdot) \in L^p_X$.

So there is a map from $L^p(0,T;X_0)$ to L^p_X and vice-versa from the definition of L^p_X. The isomorphism between the spaces is $T\colon L^p(0,T;X_0) \to L^p_X$ where

$$Tu = \phi_{(\cdot)}u(\cdot), \quad \text{and} \quad T^{-1}v = \phi_{-(\cdot)}v(\cdot).$$

It is easy to check that T is linear and bijective. The equivalence of norms follows by the bounds on $\phi_{-t}\colon X(t) \to X_0$

$$\frac{1}{C_X}\|u(t)\|_{X(t)} \leq \|\phi_{-t}u(t)\|_{X_0} \leq C_X \|u(t)\|_{X(t)}.$$

□

(ESM3) *(Slight modification of the proof of Lemma 2.13 in [1])* By consideration of the Caratheodory map $F\colon [0,T] \times X_0^* \times X_0 \to \mathbb{R}$ defined by

$$F(t, x^*, x) = \langle \phi_{-t}^* x^*, \phi_t x \rangle_{X^*(t), X(t)}$$

and using Remark 3.4.2 of [2], given $g \in L^p_{X^*}$ and $f \in L^q_X$, we have with $\tilde{g} := \phi_{(\cdot)}^* g(\cdot)$ and $\tilde{f} := \phi_{-(\cdot)} f(\cdot)$ that $t \mapsto \langle \phi_{-t}^* \tilde{g}(t), \phi_t \tilde{f}(t) \rangle_{X^*(t), X(t)} = \langle g(t), f(t) \rangle_{X^*(t), X(t)}$ is measurable, since $t \mapsto \tilde{g}(t)$ and $t \mapsto \tilde{f}(t)$ are measurable.

(ESM4) Such a choice is possible because by definition of supremum, for any $\delta > 0$, there exists an element $\tilde{x}_{i,t} \in X(t)$ of norm 1 such that

$$\sup_{\substack{x \in X(t) \\ \|x\|_{X(t)}=1}} |\langle x_{i,t}^*, x \rangle_{X^*(t), X(t)}| - |\langle x_{i,t}^*, \tilde{x}_{i,t} \rangle_{X^*(t), X(t)}| < \delta. \tag{2}$$

Then setting $x_{i,t} = \tilde{x}_{i,t} \operatorname{sign}(\langle x_{i,t}^*, \tilde{x}_{i,t} \rangle_{X^*(t), X(t)})$ and choosing $\delta = \frac{\epsilon}{2\|h\|_{L^1(0,T)}}$ implies (2.5).

(ESM5) *Proof of Lemma 2.8.* For $q < \infty$, it is easy to check that $\phi_t\colon L^q(\Omega_0) \to L^q(\Omega(t))$ is a linear homeomorphism satisfying the additional boundedness requirements. Therefore, let us discuss $q = \infty$. Let $u \in L^\infty(\Omega(t))$. We have

$$\|u\|_{L^\infty(\Omega(t))} = \operatorname*{ess\,sup}_{x \in \Omega(t)} |u(x)| = \operatorname*{ess\,sup}_{y \in \Omega_0} |u(\Phi_t^0(y))| = \|\tilde{u}\|_{L^\infty(\Omega_0)}$$

because Φ_t^0 is a diffeomorphism so null sets are mapped to null sets. Now an application of Lemma 2.3 yields the result. The above calculation also shows that the norm is preserved for $q = \infty$. □

(ESM6) *Proof of Lemma 2.9.* That the embedding is continuous is obvious. Let w_n be a bounded sequence in $W(H^1, H^{-1})$. Then $\phi_{-(\cdot)} w_n$ is a bounded sequence in $\mathcal{W}(H^1, H^{-1})$, and by $\mathcal{W}(H^1, H^{-1}) \overset{c}{\hookrightarrow} L^2(0,T; L^2(\Omega_0))$, there is a subsequence $\phi_{-(\cdot)} w_{n_k} \to \tilde{w}$ that converges in $L^2(0,T; L^2(\Omega_0))$. Hence $w_{n_k} \to \phi_{(\cdot)} w$ in $L^2_{L^2}$. □

(ESM7) *Proof of Theorem 2.10.* First we prove the following.

1 Lemma. *Let $\{w_n\}$ and w be functions such that $\{\tilde{w}_n\}$ and $\tilde{w}$ are measurable (eg. membership of $L^1_{L^1}$ will suffice). If for almost all $t \in [0,T]$,*

$$\begin{aligned} w_n(t,x) &\to w(t,x) && \text{for almost all } x \in \Omega(t) \\ |w_n(t,x)| &\le C && \text{for almost all } x \in \Omega(t) \text{ for all } n, \end{aligned}$$

then for almost all $t \in [0,T]$, $|w(t,x)| \le C$ a.e. in $\Omega(t)$.

Proof. The first statement implies that there exist null sets N and M_t such that for all $t \in [0,T] \backslash N$, $|w_n(t,x)| \to |w(t,x)|$ for all $x \in \Omega(t) \backslash M_t$. The second statement is that for all $t \in [0,T] \backslash S_n$, $|w_n(t,x)| \le C$ for all $x \in \Omega(t) \backslash R_{n,t}$ where again S_n and $R_{n,t}$ are null sets. Combining these, we find that for all $t \in [0,T] \backslash (N \cup \bigcup S_n)$, $|w(t,x)| \le C$ for all $x \in \Omega(t) \backslash (M_t \cup \bigcup R_{n,t})$. We conclude after recalling that the countable union of null sets is null. □

In this proof, $N, \hat{N}, M_t$ and $\hat{M}_t$ are null sets.

Define $v_n = w_n - w$. Then the first premise means that for all $t \in [0,T] \backslash N$, for all $\epsilon > 0$, there exists K such that if $n \ge K$, $|v_n(t,x)| \le \epsilon$ for all $x \in \Omega(t) \backslash M_t$; if we set $y = \Phi_0^t(x)$, this is $|\tilde{v}_n(t,y)| \le \epsilon$ for all $y \in \Omega_0 \backslash \Phi_0^t(M_t)$. In other words,

$$\tilde{v}_n(t,y) \to 0 \qquad \text{for all } y \in \Omega_0 \backslash \Phi_0^t(M_t). \tag{3}$$

The second premise is for $t \in [0,T] \backslash \hat{N}$, $|v_n(t,x)| \le |w(t,x)| + |g(t,x)| \le 2|g(t,x)|$ for all $x \in \Omega(t) \backslash \hat{M}_t$ (where we used Lemma 1 and chose $\hat{N}$ and $\hat{M}_t$ to take into account the null sets of the lemma), i.e., $|\tilde{v}_n(t,y)| \le 2|\tilde{g}(t,y)|$ for all $y \in \Omega_0 \backslash \hat{\Phi}_0^t(M_t)$, which implies

$$|\tilde{v}_n(t,y)|^q \le 2^q |\tilde{g}(t,y)|^q \qquad \text{for all } y \in \Omega_0 \backslash \Phi_0^t(\hat{M}_t). \tag{4}$$

Thus from (3) and (4), by the dominated convergence theorem, we have

$$\left(\int_{\Omega_0} |\tilde{v}_n(t)|^q \right)^{\frac{1}{q}} \to 0 \qquad \text{for } t \in [0,T] \backslash (N \cup \hat{N}).$$

This is precisely the statement that for all $t \in [0,T]\backslash(N \cup \hat{N})$, $\|\tilde{v}_n(t)\|_{L^q(\Omega_0)} \to 0$. We know $\|\tilde{v}_n(t)\|_{L^q(\Omega_0)} \leq 2\|\tilde{g}(t)\|_{L^q(\Omega_0)}$ and the right hand side is in $L^p(0,T)$ by assumption. So again we use the dominated convergence theorem which tells us that

$$\int_0^T \|\tilde{v}_n(t)\|^p_{L^q(\Omega_0)} \to 0,$$

i.e., $\tilde{v}_n = \tilde{w}_n - \tilde{w} \to 0$ in $L^p(0,T;L^q(\Omega_0))$ or $\tilde{w}_n \to \tilde{w}$ in $L^p(0,T;L^q(\Omega_0))$. □

(ESM8) We have

$$\begin{aligned}\int_0^T \langle \dot{u}_n(t), u_n^+(t)\rangle_{H^{-1}(\Omega(t)),H^1(\Omega(t))} &= \int_0^T \int_{\Omega(t)} \partial^\bullet u_n^+(t) u_n^+(t)\\ &= \int_0^T \frac{1}{2}\frac{d}{dt}\int_{\Omega(t)} u_n^+(t)^2 - \frac{1}{2}\int_0^T\int_{\Omega(t)} u_n^+(t)^2 \nabla_\Omega \cdot \mathbf{w}\\ &= \frac{1}{2}\int_{\Omega(T)} u_n^+(T)^2 - \frac{1}{2}\int_{\Omega_0} u_n^+(0)^2 - \frac{1}{2}\int_0^T\int_{\Omega(t)} u_n^+(t)^2\nabla_\Omega\cdot\mathbf{w}.\end{aligned}$$

(ESM9) To see this, let us show that if $w_n(x) \to w(x)$ pointwise a.e. in Ω, then $g(w_n(x)) \to g(w(x))$ pointwise a.e. in Ω. Fix x such that $w(x) > 0$. Then for every ϵ, there exists an N_ϵ such that if $n \geq N_\epsilon$, we have $|w_n(x) - w(x)| \leq \epsilon$. In other words, $w(x) - \epsilon \leq w_n(x) \leq w(x) + \epsilon$. So if we pick $\epsilon = \epsilon'$ small enough, we find $w_n(x) \geq w(x) - \epsilon' \geq 0$. Therefore it follows that $g(w_n) = 1$ for $n \geq N_{\epsilon'}$. This shows that when x is s.t $w(x) > 0$, $g(w_n) \to g(w) = 1$ pointwise. When x is such that $g(x) < 0$, a similar argument shows again that we get convergence. When x is such that $w(x) = 0$, we cannot control the sign of $w_n(x)$ but $\nabla w = 0$ on the set $w = 0$.

(ESM10) *Proof of Lemma 3.5.* Testing with $\mathcal{E}_\epsilon(u_\epsilon)$ gives

$$\begin{aligned}\langle \partial^\bullet(\mathcal{E}_\epsilon(u_\epsilon(t))), \mathcal{E}_\epsilon(u_\epsilon(t))\rangle_{V^*(t),V(t)} + \int_{\Omega(t)} \nabla_\Omega u_\epsilon(t)\nabla_\Omega(\mathcal{E}_\epsilon(u_\epsilon(t))) + \int_{\Omega(t)} (\mathcal{E}_\epsilon(u_\epsilon(t)))^2 \nabla_\Omega\cdot\mathbf{w}\\ = \int_{\Omega(t)} f(t)\mathcal{E}_\epsilon(u_\epsilon(t))\end{aligned}$$

which, since $\nabla_\Omega u_\epsilon \nabla_\Omega(\mathcal{E}_\epsilon(u_\epsilon)) = (\mathcal{E}_\epsilon)'(u_\epsilon)|\nabla_\Omega u_\epsilon|^2 \geq |\nabla_\Omega u_\epsilon|^2$ gives us

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{E}_\epsilon(u_\epsilon(t))\|^2_{L^2(\Omega(t))} + \int_{\Omega(t)} |\nabla_\Omega u_\epsilon(t)|^2 + \frac{1}{2}\int_{\Omega(t)} (\mathcal{E}_\epsilon(u_\epsilon(t)))^2 \nabla_\Omega\cdot\mathbf{w} \leq \int_{\Omega(t)} f(t)\mathcal{E}_\epsilon(u_\epsilon(t)).$$

Integrating over time and using the previous estimate, we find

$$\frac{1}{2}\|\mathcal{E}_\epsilon(u_\epsilon(T))\|^2_{L^2(\Omega(T))} + \int_0^T\int_{\Omega(t)} |\nabla_\Omega u_\epsilon(t)|^2 \leq \frac{1}{2}(1+M)^2|\Omega_0| + C_1(T,M,\mathbf{w},f).$$

For the time derivative:

$$\begin{aligned}\|\partial^\bullet(\mathcal{E}_\epsilon u_\epsilon)\|_{L^2_{H^{-1}}} &\leq \sup_{\substack{v\in L^2_{H^1},\\ \|v\|=1}} \int_0^T\int_{\Omega(t)} |\nabla_\Omega u_\epsilon(t)\nabla_\Omega v(t) + \mathcal{E}_\epsilon(u_\epsilon(t))v(t)\nabla_\Omega\cdot\mathbf{w}| + \int_0^T\int_{\Omega(t)} |f(t)v(t)|\\ &\leq C_2(T,\Omega,M,\mathbf{w},f).\end{aligned}$$

□

(ESM11) First, we need the following lemma.

2 Lemma. *The map*

$$\lambda \to \int_0^T\int_{\Omega(t)} x(t)\mathcal{U}(u(t)+\lambda v(t)) \quad \forall u,v,x \in L^2_{L^2}$$

is continuous on $[0,1]$.

Proof. Consider for a.a t

$$F_t(\lambda) = \int_{\Omega(t)} x(t,y)\mathcal{U}(u(t,y) + \lambda v(t,y))\,\mathrm{d}y.$$

The integrand is evidently continuous with respect to λ, and $|x(t,y)\mathcal{U}(u(t,y)+\lambda v(t,y))| \leq (|u(t,y)| + |v(t,y)|)|w(t,y)|$ (because $\lambda \leq 1$) and the right hand side is integrable over $\Omega(t)$. Thus F_t is continuous. Now consider

$$\lambda \mapsto \int_0^T F_t(\lambda)\,\mathrm{d}t.$$

We just showed that the integrand $\lambda \mapsto F_t(\lambda)$ is continuous, and $|F_t(\lambda)| \leq \|u(t)\|_{L^2(\Omega(t))}\|w(t)\|_{L^2(\Omega(t))} + \|v(t)\|_{L^2(\Omega(t))}\|w(t)\|_{L^2(\Omega(t))}$, and the right hand side is integrable. Therefore $\int_0^T F_t(\lambda)$ is continuous in λ. ☐

Let us pick $w = \chi - \lambda x$, where $\lambda \in [0,1]$ and $x \in L^2_{L^2}$. Then this becomes

$$\int_0^T \int_{\Omega(t)} \lambda x(u - \mathcal{U}(\chi - \lambda x)) \geq 0.$$

Since λ is positive, we can divide through this expression by λ, and then we can send $\lambda \to 0$ to receive by continuity (see Lemma 2)

$$\int_0^T \int_{\Omega(t)} x(u - \mathcal{U}(\chi)) \geq 0.$$

This then implies that $u = \mathcal{U}(\chi)$.

(ESM12) *Proof of Lemma 3.7.* To see this, for $s \in (0,T]$, consider the function $\chi_{\epsilon,s}(t) = \min\left(1, \epsilon^{-1}(s-t)^+\right)$ which has a weak derivative

$$\chi'_{\epsilon,s}(t) = \begin{cases} 0 & : t \in (0, s-\epsilon) \\ -\frac{1}{\epsilon} & : t \in (s-\epsilon, s) \\ 0 & : t \in (s,T) \end{cases}.$$

Take the test function in (1.3) to be $\chi_{\epsilon,T}\eta$ where $\eta \in W(H^1, L^2)$,

$$-\int_0^T \int_{\Omega(t)} (\dot{\chi}_{\epsilon,T}(t)\eta(t) + \chi_{\epsilon,T}(t)\dot{\eta}(t))e(t) + \int_0^T \int_{\Omega(t)} \chi_{\epsilon,T}(t)\nabla_\Omega u \nabla_\Omega \eta(t) = \int_0^T \int_{\Omega(t)} \chi_{\epsilon,T}(t) f(t)\eta(t) + \int_{\Omega_0} \chi_{\epsilon,T}(0)e_0\eta(0)$$

and this becomes

$$\frac{1}{\epsilon}\int_{T-\epsilon}^T \int_{\Omega(t)} \eta(t)e(t) - \int_0^T \int_{\Omega(t)} \chi_{\epsilon,T}(t)\dot{\eta}(t)e(t) = \int_0^T \int_{\Omega(t)} \chi_{\epsilon,T}(t)\left(f(t)\eta(t) - \nabla_\Omega u(t)\nabla_\Omega \eta(t)\right) + \int_{\Omega_0} \chi_{\epsilon,T}(0)e_0\eta(0).$$

Send $\epsilon \to 0$ and use the Lebesgue differentiation theorem on the left hand side to yield

$$\int_{\Omega(T)} \eta(T)e(T) - \int_0^T \int_{\Omega(t)} \dot{\eta}(t)e(t) = \int_0^T \int_{\Omega(t)} \left(-\nabla_\Omega u(t)\nabla_\Omega \eta(t) + \int_{\Omega(t)} f(t)\eta(t)\right) + \int_{\Omega_0} e_0\eta(0).$$

☐

(ESM13) The regularisations a_ϵ exist for the following reason. Define $\tilde{a} = a \circ \Phi_t^0$, and note that $0 \leq \tilde{a} \leq 1$ and $\tilde{a} \in L^2((0,T)\times\Omega_0)$ (it is measurable since it is piecewise measurable). By density of $C^2([0,T]\times\Omega_0) \subset L^2((0,T)\times\Omega_0)$, there exist $\tilde{a}_\epsilon \in C^2([0,T]\times\Omega_0)$ that satisfy $0 \leq \tilde{a}_\epsilon \leq 1$ a.e. and $\|\tilde{a}_\epsilon - \tilde{a}\|_{L^2((0,T)\times\Omega_0)} \leq C_H\epsilon$. But

$$C_H\|a_\epsilon - a\|_{L^2_{L^2}} \leq \|\tilde{a}_\epsilon - \tilde{a}\|_{L^2(0,T;L^2(\Omega_0))} = \|\tilde{a}_\epsilon - \tilde{a}\|_{L^2((0,T)\times\Omega_0)} \leq C_H\epsilon,$$

where $a_\epsilon := \tilde{a}_\epsilon$.

(ESM14) By changing variables $s = t - \tau$, we rewrite (3.9) as

$$\begin{aligned}\partial^\bullet_\tau \eta_\epsilon(t-s) + (a_\epsilon(x,t-s)+\epsilon)\Delta_\Omega \eta_\epsilon(t-s) &= 0\\ \eta_\epsilon(t) &= \xi.\end{aligned}$$

Noticing $\partial^\bullet_\tau \eta_\epsilon(t-s) = -\partial^\bullet_s \eta_\epsilon(t-s)$ and substituting $\varphi_\epsilon(s) = \eta_\epsilon(t-s)$, this can be written as

$$\begin{aligned}\dot{\varphi}_\epsilon(s) - (a_\epsilon(x,t-s)+\epsilon)\Delta_\Omega \varphi_\epsilon(s) &= 0\\ \varphi_\epsilon(0) &= \xi\end{aligned}$$

which holds for $s \in (0,t)$. This is precisely in the form of the PDE in Lemma 2.13.

(ESM15) We can write

$$\begin{aligned}&\int_{\Omega(t)} (e_1(t)-e_2(t))\eta(t)\\ &\quad - \int_0^t \int_{\Omega(\tau)} (e_1(\tau)-e_2(\tau))(\dot{\eta}(\tau) + (a_\epsilon(x,\tau)+\epsilon)\Delta_\Omega \eta(\tau) - (a_\epsilon(x,\tau) - a(x,\tau)+\epsilon)\Delta_\Omega \eta(\tau))\\ &\quad = \int_0^t \int_{\Omega(\tau)} (f_1(\tau)-f_2(\tau))\eta(\tau)\rangle + \int_{\Omega_0} (e_0^1 - e_0^2)\eta(0).\end{aligned}$$

(ESM16) We can estimate as follows:

$$\begin{aligned}\int_0^t \int_{\Omega(\tau)} |a(x,\tau)-a_\epsilon(x,\tau)||\Delta_\Omega \eta_\epsilon(\tau)| &\le \left(\int_0^t \int_{\Omega(\tau)} \frac{|a(x,\tau)-a_\epsilon(x,\tau)|^2}{\epsilon}\right)^{\frac{1}{2}} \left(\int_0^t \int_{\Omega(\tau)} \epsilon |\Delta_\Omega \eta_\epsilon(\tau)|^2\right)^{\frac{1}{2}}\\ &\le \sqrt{\epsilon}\, \|a - a_\epsilon\|_{L^2_{L^2}} \sqrt{(2+\epsilon)(1+e^{2C_{\mathbf{w}}(2+\epsilon)t})} \left(\int_{\Omega_0} |\nabla_\Omega \xi|^2\right)^{\frac{1}{2}} \to 0\end{aligned}$$

and

$$\begin{aligned}\int_0^t \int_{\Omega(\tau)} |\epsilon \Delta_\Omega \eta_\epsilon| &\le \left(\int_0^t \int_{\Omega(\tau)} \epsilon\right)^{\frac{1}{2}} \left(\int_0^t \int_{\Omega(\tau)} \epsilon (\Delta_\Omega \eta_\epsilon)^2\right)^{\frac{1}{2}}\\ &\le \sqrt{t|\Omega|\epsilon(2+\epsilon)(1+e^{2C_{\mathbf{w}}(2+\epsilon)t})} \left(\int_{\Omega_0} |\nabla_\Omega \xi|^2\right)^{\frac{1}{2}} \to 0.\end{aligned}$$

References

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