A The duality between the DINA and the DINO model and some technical constructions

We establish the duality between the DINA and the DINO model.

Proposition 1 *Consider a response vector* $\mathbf{R} = (R^1, ..., R^J)$ *following a DINA model with latent attribute* $\boldsymbol{\alpha}$ *and* $\mathbf{R}' = (R'^1, ..., R'^J)$ *following the DINO model with latent attribute* $\boldsymbol{\alpha}'$*. Their slipping and guessing parameters are denoted by* s_j*,* g_j*,* s'_j*, and* g'_j*, respectively. If* $1 - s_j = g'_j$*,* $g_j = 1 - s'_j$*, and* $\alpha_j = 1 - \alpha'_j$*, then* $\mathbf{R}$ *and* $\mathbf{R}'$ *are identically distributed.*

The above proposition is straightforward to verify through the ideal response indicators in (2) and (5). Thus, we omit the detailed proof. The above proposition suggests that the DINA and the DINO model are mathematically the same but with different parameterizations. Therefore, all the theoretical results we developed for the DINA model can be directly translated to the DINO model based on the above proposition. Therefore, the rest of the technical proofs are all for the DINA model. In the rest of this subsection, we present some technical construction for the subsequent proof.

T-matrix for the DINA model. For notational convenience, we will write

$$c = 1 - s$$

that is the correct response probability for capable students ("c" for correct). Then,

$$\mathbf{c} = \mathbf{1} - \mathbf{s}$$

is the corresponding parameter vector.

The T-matrix serves as a connection between the observed response distribution and the model structure. We first specify each row vector of the T-matrix for a general conjunctive diagnostic model.

For each item j, we have

$$P(R^j = 1|Q, \mathbf{p}, \boldsymbol{\theta}) = \sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} c_{j,\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} P(R^j = 1|Q, \boldsymbol{\alpha}, \boldsymbol{\theta}), \tag{16}$$

We create a row vector $B_{\boldsymbol{\theta},Q}(j)$ of length 2^K containing the probabilities $c_{j,\boldsymbol{\alpha}}$ for all $\boldsymbol{\alpha}$'s and arrange those elements in an appropriate order, then we write (16) in the form of a matrix product

$$\sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} c_{j,\boldsymbol{\alpha}} = B_{\boldsymbol{\theta},Q}(j)\mathbf{p},$$

where $\mathbf{p}$ is the column vector containing the probabilities $p_{\boldsymbol{\alpha}}$. For each pair of items, we may establish that the probability of responding positively to both items j_1 and j_2 is

$$P(R^{j_1} = 1, R^{j_2} = 1|Q, \mathbf{p}, \boldsymbol{\theta}) = \sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} c_{j_1,\boldsymbol{\alpha}} c_{j_2,\boldsymbol{\alpha}} = B_{\boldsymbol{\theta},Q,}(j_1, j_2)\mathbf{p}.$$

where $B_{\boldsymbol{\theta},Q}(j_1, j_2)$ is defined as a row vector containing the probabilities $c_{j_1,\boldsymbol{\alpha}} c_{j_2,\boldsymbol{\alpha}}$ for each $\boldsymbol{\alpha}$. Note that each element of $B_{\boldsymbol{\theta},Q}(j_1, j_2)$ is the product of the corresponding elements of

$B_{\boldsymbol{\theta},Q}(j_1)$ and $B_{\boldsymbol{\theta},Q}(j_2)$. With a completely analogous construction, for items $j_1, \cdots, j_l$, we can write the probability of responding positively to all items as

$$P(R^{j_1} = 1, \ldots, R^{j_l} = 1 | Q, \mathbf{p}, \boldsymbol{\theta}) = B_{\boldsymbol{\theta},Q}(j_1, \ldots, j_l)\mathbf{p},$$

Note that $B_{\boldsymbol{\theta},Q}(j_1, \ldots, j_l)$ is the element-by-element product of $B_{\boldsymbol{\theta},Q}(j_1), \ldots, B_{\boldsymbol{\theta},Q}(j_l)$.

The T-matrix for the DINA model has 2^K columns and 2^J rows. Each of the first $2^J - 1$ row vectors of the T-matrix is one of the vectors $B_{\boldsymbol{\theta},Q}(j_1, ..., j_l)$. The last row of the T-matrix is taken as $\mathbf{1}^\top$. The T-matrix can be written as

$$T_{\mathbf{c},\mathbf{g}}(Q) = \begin{pmatrix} B_{\boldsymbol{\theta},Q}(1) \\ \vdots \\ B_{\boldsymbol{\theta},Q}(J) \\ B_{\boldsymbol{\theta},Q}(1,2) \\ \vdots \\ B_{\boldsymbol{\theta},Q}(1, ..., J) \\ \mathbf{1}^\top \end{pmatrix}. \quad (17)$$

Response $\boldsymbol{\gamma}$-vector. We further define $\boldsymbol{\gamma}$ to be the vector containing the probabilities of the empirical distribution corresponding to those in $T_{\boldsymbol{\theta}}(Q)\mathbf{p}$, e.g., the first element of $\boldsymbol{\gamma}$ is $\frac{1}{N}\sum_{i=1}^N I(R_i^1 = 1)$ and the $(J+1)$-th element is $\frac{1}{N}\sum_{i=1}^N I(R_i^1 = 1 \textit{ and } R_i^2 = 1)$, i.e.,

$$\boldsymbol{\gamma} = \begin{pmatrix} \frac{1}{N}\sum_{i=1}^N I(R_i^1 = 1) \\ \vdots \\ \frac{1}{N}\sum_{i=1}^N I(R_i^J = 1) \\ \frac{1}{N}\sum_{i=1}^N I(R_i^1 = 1 \textit{ and } R_i^2 = 1) \\ \vdots \\ \frac{1}{N}\sum_{i=1}^N I(R_i^1 = 1,\ R_i^2 = 1, \cdots,\ \textit{and } R_i^J = 1) \\ 1 \end{pmatrix}. \quad (18)$$

An objective function. Under the true Q-matrix Q, let $(\boldsymbol{\theta}, \mathbf{p})$ be the true model parameters. By the the law of large number, we have that

$$\boldsymbol{\gamma} = \begin{pmatrix} \frac{1}{N}\sum_{i=1}^N I(R_i^1 = 1) \\ \vdots \\ \frac{1}{N}\sum_{i=1}^N I(R_i^J = 1) \\ \frac{1}{N}\sum_{i=1}^N I(R_i^1 = 1 \textit{ and } R_i^2 = 1) \\ \vdots \end{pmatrix} \to \begin{pmatrix} P(R_i^1 = 1 | Q, \boldsymbol{\theta}, \mathbf{p}) \\ \vdots \\ P(R_i^J = 1 | Q, \boldsymbol{\theta}, \mathbf{p}) \\ P(R_i^1 = 1 \textit{ and } R_i^2 = 1 | Q, \boldsymbol{\theta}, \mathbf{p}) \\ \vdots \end{pmatrix} = T_{\boldsymbol{\theta}}(Q)\mathbf{p}$$

almost surely as $N \to \infty$. For each Q, we define

$$S(Q) = \inf_{\mathbf{c},\mathbf{g},\mathbf{p}} |T_{\mathbf{c},\mathbf{g}}(Q)\mathbf{p} - \boldsymbol{\gamma}|^2, \quad (19)$$

where the minimization is subject to the natural constraints that $c_j, g_j, p_{\boldsymbol{\alpha}} \in (0,1)$ and $\sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} = 1$. Here $|\cdot|$ means the Euclidian norm. Thanks to the law of large numbers, $S(Q) \to 0$ as $N \to \infty$. The estimator

$$\tilde{Q} = \text{argmin}_Q S(Q)$$

is consistent meaning that

$$P(\tilde{Q} \sim Q) \to 1$$

if and only if the vector $T_{\mathbf{c},\mathbf{g}}(Q)\mathbf{p} \neq T_{\mathbf{c}',\mathbf{g}'}(Q')\mathbf{p}'$ for $Q' \nsim Q$ and all possible $\mathbf{c}'$, $\mathbf{g}'$ and $\mathbf{p}'$.

B Proof of Theorems

The following proposition provides a connection between the likelihood function and the T-matrix, which makes it possible to the T-matrix to show the model identifiability.

Proposition 2 *Under the DINA and DINO models, for two sets of parameters* $(\hat{\mathbf{c}}, \hat{\mathbf{g}}, \hat{\mathbf{p}})$ *and* $(\bar{\mathbf{c}}, \bar{\mathbf{g}}, \bar{\mathbf{p}})$*,*

$$L(\hat{\mathbf{c}}, \hat{\mathbf{g}}, \hat{\mathbf{p}}, Q) = L(\bar{\mathbf{c}}, \bar{\mathbf{g}}, \bar{\mathbf{p}}, Q)$$

for all $\mathbf{R}$ *if and only if the following equation holds:*

$$T_{\hat{\mathbf{c}},\hat{\mathbf{g}}}(Q)\hat{\mathbf{p}} = T_{\bar{\mathbf{c}},\bar{\mathbf{g}}}(Q)\bar{\mathbf{p}}. \tag{20}$$

The following proposition provides a relationship between T-matrices of different model parameters.

Proposition 3 *There exists an invertible matrix* $D_{\mathbf{g}^*}$ *depending only on* $\mathbf{g}^* = (g_1^*, ..., g_J^*)$*, such that*

$$D_{\mathbf{g}^*} T_{\mathbf{c},\mathbf{g}}(Q) = T_{\mathbf{c}-\mathbf{g}^*,\mathbf{g}-\mathbf{g}^*}(Q).$$

Thus, (20) is equivalent to $T_{\bar{\mathbf{c}}-\mathbf{g}^*,\bar{\mathbf{g}}-\mathbf{g}^*}(Q)\bar{\mathbf{p}} = T_{\hat{\mathbf{c}}-\mathbf{g}^*,\hat{\mathbf{g}}-\mathbf{g}^*}(Q)\hat{\mathbf{p}}$ for some $\mathbf{g}^*$. This is a very important technique that will be used repeatedly in the subsequent development. We now cite a proposition.

Proposition 4 (Proposition 6.6 in Liu et al. (2013)) *For the DINA model, under Condition A1-3,* $T_{\mathbf{c},\mathbf{g}}(Q)\mathbf{p}$ *is not in the column space of* $T_{\mathbf{c}',\mathbf{g}}(Q')$ *for all* $\mathbf{c}'$*, that is,* $T_{\mathbf{c},\mathbf{g}}(Q)\mathbf{p} \neq T_{\mathbf{c}',\mathbf{g}}(Q')\mathbf{p}'$ *for all* $\mathbf{c}'$ *and* $\mathbf{p}'$*. In addition,* $T_{\mathbf{c},\mathbf{g}}(Q)$ *is of full column rank.*

The following proposition provides the first step result.

Proposition 5 *Under the DINA and DINO models, with* Q*,* $\mathbf{s}$*, and* $\mathbf{g}$ *being known, the population proportion parameter* $\mathbf{p}$ *is identifiable if and only if* Q *is complete.*

Proof of Proposition 5. When Q is complete, the matrix $T_{\mathbf{c},\mathbf{g}}(Q)$ has full column rank from Proposition 4. Thus, $\mathbf{p}$ is identifiable by Proposition 2.

Consider the case where the Q is incomplete. Without loss of generality, we assume $\mathbf{e}_1 = (1, 0, \cdots, 0)$ is not in the set of row vectors of Q. Then in the T-matrix $T_{\mathbf{c},\mathbf{g}}(Q)$, the columns corresponding to attribute profiles $\mathbf{0}$ and $\mathbf{e}_1$ are the same. Therefore, by

Proposition 2, we can always find two different set of estimates of $p_{\mathbf{0}}$ and $p_{\mathbf{e}_1}$ such that equation (20) holds and therefore $\mathbf{p} = (p_{\boldsymbol{\alpha}}, \boldsymbol{\alpha} \in \{0,1\}^K)$ is nonidentifiable. ■

Proof of Theorem 2. The identifiability of the Q-matrix for the DINO model is an application of Theorem 1 and Proposition 1. In what follows, we focus on the identifiability of the model parameters $\mathbf{c}$ and $\mathbf{p}$ under the DINA model.

We only need to show that when $\mathbf{g}$ is known, for two sets of parameters $(\hat{\mathbf{c}}, \mathbf{g}, \hat{\mathbf{p}})$ and $(\bar{\mathbf{c}}, \mathbf{g}, \bar{\mathbf{p}})$, $L(\hat{\mathbf{c}}, \mathbf{g}, \hat{\mathbf{p}}, Q) = L(\bar{\mathbf{c}}, \mathbf{g}, \bar{\mathbf{p}}, Q)$ holds if and only if A4 satisfied. By Propositions 2 and 3, two sets of parameters $(\hat{\mathbf{c}}, \mathbf{g}, \hat{\mathbf{p}})$ and $(\bar{\mathbf{c}}, \mathbf{g}, \bar{\mathbf{p}})$ yield identical likelihood if and only if

$$T_{\hat{\mathbf{c}}-\mathbf{g},\mathbf{0}}(Q)\hat{\mathbf{p}} = D_{\mathbf{g}} T_{\hat{\mathbf{c}},\mathbf{g}}(Q)\hat{\mathbf{p}} = D_{\mathbf{g}} T_{\bar{\mathbf{c}},\mathbf{g}}(Q)\bar{\mathbf{p}} = T_{\bar{\mathbf{c}}-\mathbf{g},\mathbf{0}}(Q)\bar{\mathbf{p}}. \tag{21}$$

Thus under the assumption that $c_j > g_j$, we only need to consider that $\mathbf{g} = \mathbf{0}$.

Sufficiency of A4. For notational convenience, we write $B_Q(j_1, ..., j_l) = B_{\mathbf{c},\mathbf{g},Q}(j_1, ..., j_l)$ when $\mathbf{c} = \mathbf{1}$ and $\mathbf{g} = \mathbf{0}$. For each item $j \in 1, \cdots, J$, condition A4 implies that there exist items $j_1, ..., j_l$ (different from j) such that

$$B_Q(j, j_1, ..., j_l) = B_Q(j_1, ..., j_l),$$

that is, the attributes required by item j are a subset of the attributes required by items $j_1, ..., j_l$.

Let a and a_* be the row vectors in $D_{\mathbf{g}}$ corresponding to item combinations $j_1, ..., j_l$ and $j, j_1, ..., j_l$; see (21) for the definition of $D_{\mathbf{g}}$. If $(\hat{\mathbf{c}}, \hat{\mathbf{p}})$ and $(\bar{\mathbf{c}}, \bar{\mathbf{p}})$ satisfy by (21), then

$$\frac{{a_*}^\top T_{\hat{\mathbf{c}},\mathbf{g}}(Q)\hat{\mathbf{p}}}{a^\top T_{\hat{\mathbf{c}},\mathbf{g}}(Q)\hat{\mathbf{p}}} = \frac{{a_*}^\top T_{\bar{\mathbf{c}},\mathbf{g}}(Q)\bar{\mathbf{p}}}{a^\top T_{\bar{\mathbf{c}},\mathbf{g}}(Q)\bar{\mathbf{p}}}.$$

On the other hand, we have that

$$\begin{aligned}
\frac{{a_*}^\top T_{\hat{\mathbf{c}},\mathbf{g}}(Q)\hat{\mathbf{p}}}{a^\top T_{\hat{\mathbf{c}},\mathbf{g}}(Q)\hat{\mathbf{p}}} &= \frac{B_{\hat{\mathbf{c}}-\mathbf{g},\mathbf{0};Q}(j, j_1, ..., j_l)\hat{\mathbf{p}}}{B_{\hat{\mathbf{c}}-\mathbf{g},\mathbf{0};Q}(j_1, ..., j_l)\hat{\mathbf{p}}} = \hat{c}_j - g_j, \\
\frac{{a_*}^\top T_{\bar{\mathbf{c}},\mathbf{g}}(Q)\bar{\mathbf{p}}}{a^\top T_{\bar{\mathbf{c}},\mathbf{g}}(Q)\bar{\mathbf{p}}} &= \frac{B_{\bar{\mathbf{c}}-\mathbf{g},\mathbf{0};Q}(j, j_1, ..., j_l)\bar{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\mathbf{g},\mathbf{0};Q}(j_1, ..., j_l)\bar{\mathbf{p}}} = \bar{c}_j - g_j.
\end{aligned}$$

Therefore, $\hat{c}_j = \bar{c}_j$ for all $j = 1, \cdots, J$, which gives the identifiability of the slipping parameter. According to Proposition 5, the completeness of the Q-matrix ensures that the identifiability of $\mathbf{p}$, therefore we have the sufficiency of A4.

Necessity of A4. We reach the conclusion by contradiction. (21) suggests that it is sufficient to show the necessity for $\mathbf{g} = \mathbf{0}$. Without loss of generality, suppose that the first attribute only appears once in the first column of the Q-matrix, i.e., the Q-matrix takes the following form:

$$Q = \begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & \mathcal{I}_{K-1} \\ \mathbf{0} & Q_1 \end{pmatrix}. \tag{22}$$

We construct $\bar{\mathbf{c}}$ and $\bar{\mathbf{p}}$ different from $\hat{\mathbf{c}}$ and $\hat{\mathbf{p}}$ such that $T_{\hat{\mathbf{c}},\mathbf{0}}(Q)\hat{\mathbf{p}} = T_{\bar{\mathbf{c}},\mathbf{0}}(Q)\bar{\mathbf{p}}$. We write $\hat{\mathbf{c}} = (\hat{c}_1, \cdots, \hat{c}_J)$ and $\hat{\mathbf{p}} = \{\hat{p}_{(b,a)} : b \in \{0,1\}, a \in \{0,1\}^{K-1}\}$. For some x close to 1, define

$$\bar{\mathbf{c}} = (\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_J) = (x\hat{c}_1, \hat{c}_2, \cdots, \hat{c}_J)$$

and

$$\bar{\mathbf{p}} = \{\bar{p}_{(b,a)} : \bar{p}_{(1,a)} = \hat{p}_{(1,a)}/x \text{ and } \bar{p}_{(0,a)} = \hat{p}_{(0,a)} + \hat{p}_{(1,a)}(1 - 1/x), \text{ for all } a \in \{0,1\}^{K-1}\}.$$

Notice that the parameters related to the first item have been changed. Consider the rows in the T-matrix related to the first item. Keeping in mind that $\mathbf{g} = \mathbf{0}$, we have that

$$\hat{c}_1 \sum_{a \in \{0,1\}^{K-1}} \hat{p}_{(1,a)} + g_1 \sum_{a \in \{0,1\}^{K-1}} \hat{p}_{(0,a)} = \bar{c}_1 \sum_{a \in \{0,1\}^{K-1}} \bar{p}_{(1,a)} + g_1 \sum_{a \in \{0,1\}^{K-1}} \bar{p}_{(0,a)}. \tag{23}$$

This corresponds to $P(R^1 = 1)$. Similar identities can be established for $P(R^1 = R^{j_1} = ... = R^{j_l} = 1)$. Therefore, we have constructed $(\bar{\mathbf{c}}, \bar{\mathbf{p}}) \neq (\hat{\mathbf{c}}, \hat{\mathbf{p}})$ such that $T_{\bar{\mathbf{c}},\mathbf{0}}(Q)\bar{\mathbf{p}} = T_{\hat{\mathbf{c}},\mathbf{0}}(Q)\hat{\mathbf{p}}$. Thus, $\mathbf{c}$ and $\mathbf{p}$ are not identifiable if A4 does not hold. ■

Proof of Theorem 3. Consider the true Q and a candidate $Q' \nsim Q$. According to the discussion at the end of Section A, it is sufficient to show that it is impossible to have two sets of parameters $(\hat{\mathbf{c}}, \hat{\mathbf{g}}, \hat{\mathbf{p}})$ and $(\bar{\mathbf{c}}, \bar{\mathbf{g}}, \bar{\mathbf{p}})$ such that $\hat{c}_j > \hat{g}_j$, $\bar{c}_j > \bar{g}_j$, $\hat{p}_{\boldsymbol{\alpha}} > 0$, $\bar{p}_{\boldsymbol{\alpha}} > 0$, and

$$T_{\hat{\mathbf{c}},\hat{\mathbf{g}}}(Q)\hat{\mathbf{p}} = T_{\bar{\mathbf{c}},\bar{\mathbf{g}}}(Q')\bar{\mathbf{p}}. \tag{24}$$

We prove this first assuming that there exist two such sets of parameters and then reach a contradiction. The true matrix Q is arranged as in (8) such that the first $2K$ rows form two identity matrices. We try to reach a contradiction under the following two cases.

Case 1: either $Q'_{1:K}$ or $Q'_{K+1:2K}$ is incomplete. We only focus on the case when $Q'_{1:K}$ is not $\mathcal{I}_K$. We borrow an intermediate result in the proof of Proposition 6.4 in Liu et al. (2013): we can identify an item $1 \le h \le K$ and an item set $\mathcal{H} \subset \{1, \cdots, K\}$ $(h \notin \mathcal{H})$ such that under Q', $\mathcal{H}$ requires all attributes required by item h, that is, if someone is capable of solving all problems in $\mathcal{H}$ then he/she is able to solve problem h. We say someone "is able to" or "can" solve a problem or a set of problems if his/her ideal responses to the set of problems are all one.

For items $K+1, \cdots, 2K$, since $Q_{K+1:2K} = \mathcal{I}_K$, there exists an item set $\mathcal{B} \subset \{K + 1, ..., 2K\}$ such that under Q it requires the same attributes as $\mathcal{H}$, that is, if a person is capable of solving all items in $\mathcal{B}$ if and only if they can solving all problems in $\mathcal{H}$. Since $Q_{1:K} = \mathcal{I}_K$, under Q, the attributes required by $\mathcal{H}$ and $\mathcal{B}$ are different from those of item h. Define

$$\tilde{\mathbf{g}} = (\bar{g}_1, \cdots, \bar{g}_K, \hat{g}_{K+1}, \cdots, \hat{g}_J).$$

Assumption (24) and Proposition 3 suggests $T_{\hat{\mathbf{c}}-\tilde{\mathbf{g}},\hat{\mathbf{g}}-\tilde{\mathbf{g}}}(Q)\hat{\mathbf{p}} = T_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}}}(Q')\bar{\mathbf{p}}$.

Under Q' if h requires strictly fewer attributes than $\mathcal{H}$, there are three types of attributes profiles: unable to answer h (denoted by $0_h 0_{\mathcal{H}}$), unable to answer $\mathcal{H}$ but able to answer h

(denoted by $0_{\mathcal{H}}1_h$), and able to answer $\mathcal{H}$ (denoted by $1_{\mathcal{H}}$). We have

$$\begin{array}{rcccl} & 0_h 0_{\mathcal{H}} & 0_{\mathcal{H}} 1_h & 1_{\mathcal{H}} & \\ B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q'}(\mathcal{H}) = (& 0 & 0 & \prod_{j\in\mathcal{H}}(\bar{c}_j - \bar{g}_j) &), \\ B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q'}(h) = (& 0 & (\bar{c}_h - \bar{g}_h) & (\bar{c}_h - \bar{g}_h) &), \\ B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q'}(\mathcal{H},h) = (& 0 & 0 & (\bar{c}_h - \bar{g}_h)\prod_{j\in\mathcal{H}}(\bar{c}_j - \bar{g}_j) &), \end{array}$$

If h and $\mathcal{H}$ require the same attributes, $0_{\mathcal{H}}1_h$ case does not exist and the above equations do not have the $0_{\mathcal{H}}1_h$ column. Under both situations, we have

$$\bar{c}_h - \bar{g}_h = \frac{B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q'}(\mathcal{H},h)\bar{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q'}(\mathcal{H})\bar{\mathbf{p}}} = \frac{B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q'}(\mathcal{H},h,K+1,\cdots,2K)\bar{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q'}(\mathcal{H},K+1,\cdots,2K)\bar{\mathbf{p}}}. \tag{25}$$

Under Q, we have

$$\begin{array}{rccl} & \boldsymbol{\alpha} \neq \mathbf{1} & \boldsymbol{\alpha} = \mathbf{1} & \\ B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q}(K+1,\cdots,2K) = (& 0 & \prod_{j=K+1}^{2K}(\hat{c}_j - \hat{g}_j) &), \\ B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q}(\mathcal{H},K+1,\cdots,2K) = (& 0 & \prod_{j\in\mathcal{H}}(\hat{c}_j - \bar{g}_j)\prod_{j=K+1}^{2K}(\hat{c}_j - \hat{g}_j) &), \\ B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q}(\mathcal{H},h,K+1,\cdots,2K) = (& 0 & (\hat{c}_h - \bar{g}_h)\prod_{j\in\mathcal{H}}(\hat{c}_j - \bar{g}_j)\prod_{j=K+1}^{2K}(\hat{c}_j - \hat{g}_j) &). \end{array}$$

This gives

$$\hat{c}_h - \bar{g}_h = \frac{B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q}(\mathcal{H},h,K+1,\cdots,2K)\hat{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q}(\mathcal{H},K+1,\cdots,2K)\hat{\mathbf{p}}}. \tag{26}$$

$T_{\hat{\mathbf{c}}-\tilde{\mathbf{g}},\hat{\mathbf{g}}-\tilde{\mathbf{g}}}(Q)\hat{\mathbf{p}} = T_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}}}(Q')\bar{\mathbf{p}}$ allows to equate the right-hand sides of (25) and (26) which yields

$$\hat{c}_h = \bar{c}_h. \tag{27}$$

Now under Q', with a similarly argument, we have

$$\bar{c}_h - \bar{g}_h = \frac{B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q'}(\mathcal{H},h,\mathcal{B})\bar{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q'}(\mathcal{H},\mathcal{B})\bar{\mathbf{p}}}. \tag{28}$$

Under Q, consider three types of attributes profiles: unable to answer $\mathcal{H}$ (denoted by $0_{\mathcal{H}}$), able to answer $\mathcal{H}$ but unable to answer h (denoted by $0_h1_{\mathcal{H}}$), and able to answer both $\mathcal{H}$ and h (denoted by $1_{\mathcal{H}}1_h$). We have

$$\begin{array}{rccc} & 0_{\mathcal{H}} & 0_h 1_{\mathcal{H}} & 1_{\mathcal{H}} 1_h \\ B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q}(\mathcal{H},\mathcal{B}) = (& 0 & \prod_{j\in\mathcal{H}}(\hat{c}_j - \bar{g}_j)\prod_{j\in\mathcal{B}}(\hat{c}_j - \hat{g}_j) & \prod_{j\in\mathcal{H}}(\hat{c}_j - \bar{g}_j)\prod_{j\in\mathcal{B}}(\hat{c}_j - \hat{g}_j)), \\ B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q}(\mathcal{H},h,\mathcal{B}) = (& 0 & (\hat{g}_h - \bar{g}_h)\prod_{j\in\mathcal{H}}(\hat{c}_j - \bar{g}_j)\prod_{j\in\mathcal{B}}(\hat{c}_j - \hat{g}_j) & (\hat{c}_h - \bar{g}_h)\prod_{j\in\mathcal{H}}(\hat{c}_j - \bar{g}_j)\prod_{j\in\mathcal{B}}(\hat{c}_j - \hat{g}_j)). \end{array}$$

Since $\hat{g}_h - \bar{g}_h < \hat{c}_h - \bar{g}_h$ and $p_{\boldsymbol{\alpha}} > 0$ for all $\boldsymbol{\alpha}$, we have that

$$\hat{c}_h - \bar{g}_h \neq \frac{B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q}(\mathcal{H},h,\mathcal{B})\hat{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q}(\mathcal{H},\mathcal{B})\hat{\mathbf{p}}} = \frac{B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q'}(\mathcal{H},h,\mathcal{B})\bar{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q'}(\mathcal{H},\mathcal{B})\bar{\mathbf{p}}}. \tag{29}$$

$T_{\hat{\mathbf{c}}-\tilde{\mathbf{g}},\hat{\mathbf{g}}-\tilde{\mathbf{g}}}(Q)\hat{\mathbf{p}} = T_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}}}(Q')\bar{\mathbf{p}}$ allows use to equate the right-hand sides of (28) and (29), which yields $\hat{c}_h > \bar{c}_h$. This contradicts (27).

Thus, under this case, we have that $T_{\hat{\mathbf{c}}-\tilde{\mathbf{g}},\hat{\mathbf{g}}-\tilde{\mathbf{g}}}(Q)\hat{\mathbf{p}} \neq T_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}}}(Q')\bar{\mathbf{p}}$ if $\hat{c}_j > \hat{g}_j$, $\bar{c}_j > \bar{g}_j$, $\hat{p}_{\boldsymbol{\alpha}} > 0$, $\bar{p}_{\boldsymbol{\alpha}} > 0$. Furthermore, if the conditions in the theorem are satisfied and $Q'_{1:K}$ or $Q'_{(K+1):2K}$ is incomplete, then we cannot find parameters $\bar{\mathbf{c}}$, $\bar{\mathbf{g}}$, and $\bar{\mathbf{p}}$ that yields the same response distribution as Q and thus Q can be differentiated from Q' by the maximum likelihood.

Case 2: both $Q'_{1:K}$ and $Q'_{K+1:2K}$ are complete, but $Q \nsim Q'$. In this case, we can always arrange the columns of Q' such that either $Q'_{1:K} = \mathcal{I}_K$. Redefine

$$\tilde{\mathbf{g}} = (\bar{c}_1, \cdots, \bar{c}_K, \hat{c}_{K+1}, \cdots, \hat{c}_{2K}, 0, \cdots, 0)$$

and assumption (24) suggests that $T_{\hat{\mathbf{c}}-\tilde{\mathbf{g}},\hat{\mathbf{g}}-\tilde{\mathbf{g}}}(Q)\hat{\mathbf{p}} = T_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}}}(Q')\bar{\mathbf{p}}$.

The row vectors of T-matrices corresponding to items 1,..., $2K$ are

$$B_{\hat{\mathbf{c}}-\tilde{\mathbf{g}},\hat{\mathbf{g}}-\tilde{\mathbf{g}},Q}(1,\cdots,2K) = \left(\prod_{k=1}^{K}(\hat{g}_k - \bar{c}_k)\prod_{k=K+1}^{2K}(\hat{g}_k - \hat{c}_k), \mathbf{0}^\top\right)$$

and

$$B_{\hat{\mathbf{c}}-\tilde{\mathbf{g}},\hat{\mathbf{g}}-\tilde{\mathbf{g}},Q'}(1,\cdots,2K) = \left(\prod_{k=1}^{K}(\bar{g}_k - \bar{c}_k)\prod_{k=K+1}^{2K}(\bar{g}_k - \hat{c}_k), \mathbf{0}^\top\right)$$

where only the element corresponding to zero attribute is non-zero. Therefore, for any $j \geq 2K+1$, we have

$$\hat{g}_j = \frac{B_{\hat{\mathbf{c}}-\tilde{\mathbf{g}},\hat{\mathbf{g}}-\tilde{\mathbf{g}},Q}(1,\cdots,2K,\, j)\hat{\mathbf{p}}}{B_{\hat{\mathbf{c}}-\tilde{\mathbf{g}},\hat{\mathbf{g}}-\tilde{\mathbf{g}},Q}(1,\cdots,2K)\hat{\mathbf{p}}} = \frac{B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q'}(1,\cdots,2K,\, j)\bar{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q'}(1,\cdots,2K)\bar{\mathbf{p}}} = \bar{g}_j.$$

Once again, we redefine $\tilde{\mathbf{g}} = (\bar{g}_1, \cdots, \bar{g}_K, 0, \cdots, 0, \hat{g}_{2K+1}, \cdots, \hat{g}_J)$. By Condition A5, we have for $K+1 \leq j \leq 2K$

$$\begin{aligned}\hat{c}_j &= \frac{B_{\hat{\mathbf{c}}-\tilde{\mathbf{g}},\hat{\mathbf{g}}-\tilde{\mathbf{g}},Q}(1,\cdots,K,\, j,\, (2K+1),\cdots,J)\hat{\mathbf{p}}}{B_{\hat{\mathbf{c}}-\tilde{\mathbf{g}},\hat{\mathbf{g}}-\tilde{\mathbf{g}},Q}(1,\cdots,K,(2K+1),\cdots,J)\hat{\mathbf{p}}} \\ &= \frac{B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q'}(1,\cdots,K,\, j,\, (2K+1),\cdots,J)\bar{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q'}(1,\cdots,K,(2K+1),\cdots,J)\bar{\mathbf{p}}} = \bar{c}_j.\end{aligned}$$

Similarly take $\tilde{\mathbf{g}} = (0, \cdots, 0, \bar{g}_{K+1}, \cdots, \bar{g}_{2K}, \hat{g}_{2K+1}, \cdots, \hat{g}_J)$. We have $\hat{c}_j = \bar{c}_j$ for $1 \leq j \leq K$.

Now take $\tilde{\mathbf{g}} = (\bar{c}_1, \cdots, \bar{c}_K, 0, \cdots, 0)$, we have for $K+1 \leq j \leq 2K$

$$\hat{g}_j = \frac{B_{\hat{\mathbf{c}}-\tilde{\mathbf{g}},\hat{\mathbf{g}}-\tilde{\mathbf{g}},Q}(1,\cdots,K,\, j)\hat{\mathbf{p}}}{B_{\hat{\mathbf{c}}-\tilde{\mathbf{g}},\hat{\mathbf{g}}-\tilde{\mathbf{g}},Q}(1,\cdots,K)\hat{\mathbf{p}}} = \frac{B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q'}(1,\cdots,K,\, j)\bar{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\tilde{\mathbf{g}},\bar{\mathbf{g}}-\tilde{\mathbf{g}},Q'}(1,\cdots,K)\bar{\mathbf{p}}} = \bar{g}_j.$$

Similarly, for $\hat{g}_j = \bar{g}_j$ for $j = 1, ..., K$. Thus, we have $\hat{g}_j = \bar{g}_j$ for $j = 1, ..., J$. Therefore, assumption (24) becomes

$$T_{\hat{\mathbf{c}},\hat{\mathbf{g}}}(Q)\hat{\mathbf{p}} = T_{\bar{\mathbf{c}},\hat{\mathbf{g}}}(Q')\bar{\mathbf{p}}. \tag{30}$$

This contradicts Proposition 4. Thus, we have reached the conclusion that

$$T_{\hat{\mathbf{c}},\hat{\mathbf{g}}}(Q)\hat{\mathbf{p}} \neq T_{\bar{\mathbf{c}},\bar{\mathbf{g}}}(Q')\bar{\mathbf{p}}.$$

for all $\hat{c}_j > \hat{g}_j$, $\bar{c}_j > \bar{g}_j$, $\hat{p}_{\boldsymbol{\alpha}} > 0$, $\bar{p}_{\boldsymbol{\alpha}} > 0$ and $Q' \nsim Q$. Thus, by maximizing the profiled likelihood, Q can be consistently estimated. ■

Proof of Theorem 4. Suppose there are two sets of parameters $(\hat{\mathbf{c}}, \hat{\mathbf{g}}, \hat{\mathbf{p}})$ and $(\bar{\mathbf{c}}, \bar{\mathbf{g}}, \bar{\mathbf{p}})$ such that $L(\hat{\mathbf{c}}, \hat{\mathbf{g}}, \hat{\mathbf{p}}) = L(\bar{\mathbf{c}}, \bar{\mathbf{g}}, \bar{\mathbf{p}})$, equivalently, $T_{\hat{\mathbf{c}},\hat{\mathbf{g}}}(Q)\hat{\mathbf{p}} = T_{\bar{\mathbf{c}},\bar{\mathbf{g}}}(Q)\bar{\mathbf{p}}$. We show that $(\hat{\mathbf{c}}, \hat{\mathbf{g}}, \hat{\mathbf{p}}) = (\bar{\mathbf{c}}, \bar{\mathbf{g}}, \bar{\mathbf{p}})$ if $\hat{c}_j > \hat{g}_j$, $\hat{p}_{\boldsymbol{\alpha}} > 0$, $\bar{c}_j > \bar{g}_j$, and $\bar{p}_{\boldsymbol{\alpha}} > 0$. Condition A5 allows us to consider the following three cases.

Case 1. There exit at least three items with Q-matrix row vector $\mathbf{e}_1$. Without loss of generality, we write the Q-matrix as (with reordering of the rows)

$$Q = \begin{pmatrix} 1 & \mathbf{0}^\top \\ 1 & \mathbf{0}^\top \\ 1 & \mathbf{0}^\top \\ \mathbf{0} & \mathcal{I}_{K-1} \\ \mathbf{0} & Q' \end{pmatrix}. \tag{31}$$

In what follows, we show that $\hat{c}_j = \bar{c}_j$ and $\hat{g}_j = \bar{g}_j$ for $j = 1, 2, 3$. By Proposition 3, $T_{\hat{\mathbf{c}},\hat{\mathbf{g}}}(Q)\hat{\mathbf{p}} = T_{\bar{\mathbf{c}},\bar{\mathbf{g}}}(Q)\bar{\mathbf{p}}$ suggests that $T_{\hat{\mathbf{c}}-\hat{\mathbf{g}},\mathbf{0}}(Q)\hat{\mathbf{p}} = T_{\bar{\mathbf{c}}-\hat{\mathbf{g}},\bar{\mathbf{g}}-\hat{\mathbf{g}}}(Q)\bar{\mathbf{p}}$. Together with the fact that

$$\frac{B_{\hat{\mathbf{c}}-\hat{\mathbf{g}},\mathbf{0};\mathbf{Q}}(1,2,3)\hat{\mathbf{p}}}{B_{\hat{\mathbf{c}}-\hat{\mathbf{g}},\mathbf{0};\mathbf{Q}}(1,2)\hat{\mathbf{p}}} = \frac{B_{\hat{\mathbf{c}}-\hat{\mathbf{g}},\mathbf{0};\mathbf{Q}}(1,3)\hat{\mathbf{p}}}{B_{\hat{\mathbf{c}}-\hat{\mathbf{g}},\mathbf{0};\mathbf{Q}}(1)\hat{\mathbf{p}}} = \hat{c}_3 - \hat{g}_3, \tag{32}$$

we have that

$$\frac{B_{\bar{\mathbf{c}}-\hat{\mathbf{g}},\bar{\mathbf{g}}-\hat{\mathbf{g}};Q}(1,3)\bar{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\hat{\mathbf{g}},\bar{\mathbf{g}}-\hat{\mathbf{g}};Q}(1)\bar{\mathbf{p}}} = \frac{B_{\bar{\mathbf{c}}-\hat{\mathbf{g}},\bar{\mathbf{g}}-\hat{\mathbf{g}};Q}(1,2,3)\bar{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\hat{\mathbf{g}},\bar{\mathbf{g}}-\hat{\mathbf{g}};Q}(1,2)\bar{\mathbf{p}}}. \tag{33}$$

Expanding the above identity, we have

$$\begin{aligned} &\frac{(\bar{g}_1 - \hat{g}_1)(\bar{g}_3 - \hat{g}_3)\sum_{a\in\{0,1\}^{K-1}} \bar{p}_{(0,a)} + (\bar{c}_1 - \hat{g}_1)(\bar{c}_3 - \hat{g}_3)\sum_{a\in\{0,1\}^{K-1}} \bar{p}_{(1,a)}}{(\bar{g}_1 - \hat{g}_1)\sum_{a\in\{0,1\}^{K-1}} \bar{p}_{(0,a)} + (\bar{c}_1 - \hat{g}_1)\sum_{a\in\{0,1\}^{K-1}} \bar{p}_{(1,a)}} \\ = &\frac{\prod_{j=1}^3(\bar{g}_j - \hat{g}_j)\sum_{a\in\{0,1\}^{K-1}} \bar{p}_{(0,a)} + \prod_{j=1}^3(\bar{c}_j - \hat{g}_j)\sum_{a\in\{0,1\}^{K-1}} \bar{p}_{(1,a)}}{(\bar{g}_1 - \hat{g}_1)(\bar{g}_2 - \hat{g}_2)\sum_{a\in\{0,1\}^{K-1}} \bar{p}_{(0,a)} + (\bar{c}_1 - \hat{g}_1)(\bar{c}_2 - \hat{g}_2)\sum_{a\in\{0,1\}^{K-1}} \bar{p}_{(1,a)}}, \end{aligned} \tag{34}$$

which can be simplified to $(\bar{g}_1 - \hat{g}_1)(\bar{c}_1 - \hat{g}_1)(\bar{c}_2 - \bar{g}_2)(\bar{c}_3 - \bar{g}_3) = 0$. Then under the constraint that $\bar{c}_j > \bar{g}_j$, we have $\bar{g}_1 = \hat{g}_1$ or $\bar{c}_1 = \hat{g}_1$. A similar argument yields

$$\begin{cases} \bar{g}_2 = \hat{g}_2 \text{ or } \bar{c}_2 = \hat{g}_2 \\ \bar{g}_3 = \hat{g}_3 \text{ or } \bar{c}_3 = \hat{g}_3 \end{cases} \text{ and } \begin{cases} \hat{g}_1 = \bar{g}_1 \text{ or } \hat{c}_1 = \bar{g}_1 \\ \hat{g}_2 = \bar{g}_2 \text{ or } \hat{c}_2 = \bar{g}_2 \\ \hat{g}_3 = \bar{g}_3 \text{ or } \hat{c}_3 = \bar{g}_3 \end{cases}.$$

For $j = 1$, 2, or 3, if $\hat{g}_j \neq \bar{g}_j$ we have $\hat{c}_j = \bar{g}_j$ and $\bar{c}_j = \hat{g}_j$. This contradict the condition that $\hat{c}_j > \hat{g}_j$ and $\bar{c}_j > \bar{g}_j$. Thus we have $\hat{g}_j = \bar{g}_j$ for $j = 1, 2, 3$. Repeating the proof of Theorem 2, we have $\hat{c}_j = \bar{c}_j$ for $i = 1, 2, 3$.

Case 2. There exit two items with row vector $\mathbf{e}_1$. Without loss of generality, we write the Q-matrix as

$$Q = \begin{pmatrix} 1 & \mathbf{0}^\top \\ 1 & \mathbf{0}^\top \\ 1 & \mathbf{v}^\top \\ \mathbf{0} & \mathcal{I}_{K-1} \\ \mathbf{0} & Q' \end{pmatrix}, \quad Q_{1:4} = \begin{pmatrix} 1 & 0 & \mathbf{0}^\top \\ 1 & 0 & \mathbf{0}^\top \\ 1 & 1 & {\mathbf{v}_*}^\top \\ 0 & 1 & \mathbf{0}^\top \end{pmatrix}, \tag{35}$$

where $\mathbf{v}$ is a non-zero vector. Without loss of generality we assume $\mathbf{v}^\top = (1, {\mathbf{v}_*}^\top)$. Consider the sub-matrix containing the first four items. i.e., $Q_{1:4}$ in (35). Similar to the proof of Case 1, for $(\hat{\mathbf{c}}, \hat{\mathbf{g}}, \hat{\mathbf{p}})$ and $(\bar{\mathbf{c}}, \bar{\mathbf{g}}, \bar{\mathbf{p}})$ such that $T_{\hat{\mathbf{c}},\hat{\mathbf{g}}}(Q)\hat{\mathbf{p}} = T_{\bar{\mathbf{c}},\bar{\mathbf{g}}}(Q)\bar{\mathbf{p}}$, we will show

$$\begin{cases} \hat{c}_j = \bar{c}_j & j = 1, 2, 4 \\ \hat{g}_j = \bar{g}_j & j = 1, 2, 3 \end{cases}. \tag{36}$$

A similar argument as in Case 1 yields

$$\frac{B_{\hat{\mathbf{c}}-\hat{\mathbf{g}},\mathbf{0};\mathbf{Q}}(1,3)\hat{\mathbf{p}}}{B_{\hat{\mathbf{c}}-\hat{\mathbf{g}},\mathbf{0};\mathbf{Q}}(3)\hat{\mathbf{p}}} = \hat{c}_1 - \hat{g}_1 = \frac{B_{\hat{\mathbf{c}}-\hat{\mathbf{g}},\mathbf{0};\mathbf{Q}}(1,4,3)\hat{\mathbf{p}}}{B_{\hat{\mathbf{c}}-\hat{\mathbf{g}},\mathbf{0};\mathbf{Q}}(4,3)\hat{\mathbf{p}}}.$$

Together with the fact that $T_{\bar{\mathbf{c}}-\hat{\mathbf{g}},\bar{\mathbf{g}}-\hat{\mathbf{g}}}(Q)\bar{\mathbf{p}} = T_{\hat{\mathbf{c}}-\hat{\mathbf{g}},\mathbf{0}}(Q)\hat{\mathbf{p}}$, we have

$$\frac{B_{\bar{\mathbf{c}}-\hat{\mathbf{g}},\bar{\mathbf{g}}-\hat{\mathbf{g}};Q}(1,3)\bar{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\hat{\mathbf{g}},\bar{\mathbf{g}}-\hat{\mathbf{g}};Q}(3)\bar{\mathbf{p}}} = \frac{B_{\bar{\mathbf{c}}-\hat{\mathbf{g}},\bar{\mathbf{g}}-\hat{\mathbf{g}};Q}(1,4,3)\bar{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\hat{\mathbf{g}},\bar{\mathbf{g}}-\hat{\mathbf{g}};Q}(4,3)\bar{\mathbf{p}}}.$$

This implies

$$\begin{aligned} & \frac{\tilde{g}_1\tilde{g}_4\tilde{g}_3\bar{\mathbf{p}}_{0,0} + \tilde{c}_1\tilde{g}_4\tilde{g}_3\bar{\mathbf{p}}_{1,0} + \tilde{g}_1\tilde{c}_4\tilde{g}_3\bar{\mathbf{p}}_{0,1} + \tilde{c}_1\tilde{c}_4\tilde{c}_3\bar{\mathbf{p}}_{1,1}}{\tilde{g}_4\tilde{g}_3\bar{\mathbf{p}}_{0,0} + \tilde{g}_4\tilde{g}_3\bar{\mathbf{p}}_{1,0} + \tilde{c}_4\tilde{g}_3\bar{\mathbf{p}}_{0,1} + \tilde{c}_4\tilde{c}_3\bar{\mathbf{p}}_{1,1}} \\ = & \frac{\tilde{g}_1\tilde{g}_3\bar{\mathbf{p}}_{0,0} + \tilde{c}_1\tilde{g}_3\bar{\mathbf{p}}_{1,0} + \tilde{g}_1\tilde{g}_3\bar{\mathbf{p}}_{0,1} + \tilde{c}_1\tilde{c}_3\bar{\mathbf{p}}_{1,1}}{\tilde{g}_3\bar{\mathbf{p}}_{0,0} + \tilde{g}_3\bar{\mathbf{p}}_{1,0} + \tilde{g}_3\bar{\mathbf{p}}_{0,1} + \tilde{c}_3\bar{\mathbf{p}}_{1,1}}, \end{aligned} \tag{37}$$

where $\tilde{g}_j = \bar{g}_j - \hat{g}_j$ for $j = 1, 3, 4$, $\tilde{c}_j = \bar{c}_j - \hat{g}_j$ for $j = 1, 4$,

$$\tilde{c}_3 = \frac{(\bar{c}_3 - \hat{g}_3)\sum_{\mathbf{v}_* \preceq a \in \{0,1\}^{K-2}} \bar{p}_{(1,1,a)} + (\bar{g}_3 - \hat{g}_3)\sum_{\mathbf{v}_* \npreceq a \in \{0,1\}^{K-2}} \bar{p}_{(1,1,a)}}{\sum_{a \in \{0,1\}^{K-2}} \bar{p}_{(1,1,a)}},$$

and $\bar{\mathbf{p}}_{i,j} = \sum_{a \in \{0,1\}^{K-2}} \bar{p}_{(i,j,a)}$ for $i, j \in \{0, 1\}$. Here $\mathbf{v}_* \preceq a$ means that each element of $\mathbf{v}_*$ is less than or equals to the corresponding element of a, and $\mathbf{v}_* \npreceq a$ means that $\mathbf{v}_* \preceq a$ does not hold.

Simplifying (37), we obtain $\bar{\mathbf{p}}_{0,0}\bar{\mathbf{p}}_{1,1}\tilde{g}_3\tilde{c}_3(\tilde{g}_1 - \tilde{c}_1) = \bar{\mathbf{p}}_{1,0}\bar{\mathbf{p}}_{0,1}\tilde{g}_3\tilde{g}_3(\tilde{g}_1 - \tilde{c}_1)$. Since $\tilde{g}_1 - \tilde{c}_1 \neq 0$, we have

$$\tilde{g}_3 = 0 \quad \text{or} \quad \bar{\mathbf{p}}_{0,0}\bar{\mathbf{p}}_{1,1}\tilde{c}_3 = \bar{\mathbf{p}}_{1,0}\bar{\mathbf{p}}_{0,1}\tilde{g}_3. \tag{38}$$

We show that $\tilde{g}_3$ has to be zero. Otherwise, we have

$$\bar{\mathbf{p}}_{0,0}\bar{\mathbf{p}}_{1,1}(\bar{c}_3^* - \hat{g}_3) = \bar{\mathbf{p}}_{1,0}\bar{\mathbf{p}}_{0,1}(\bar{g}_3 - \hat{g}_3), \tag{39}$$

where

$$\bar{c}_3^* = \tilde{c}_3 + \hat{g}_3 = \frac{\bar{c}_3 \sum_{v_* \preceq a \in \{0,1\}^{K-2}} \bar{p}_{(1,1,a)} + \bar{g}_3 \sum_{v_* \npreceq a \in \{0,1\}^{K-2}} \bar{p}_{(1,1,a)}}{\sum_{a \in \{0,1\}^{K-2}} \bar{p}_{(1,1,a)}}.$$

A similar argument gives that

$$\hat{\mathbf{p}}_{0,0}\hat{\mathbf{p}}_{1,1}(\hat{c}_3^* - \bar{g}_3) = \hat{\mathbf{p}}_{1,0}\hat{\mathbf{p}}_{0,1}(\hat{g}_3 - \bar{g}_3), \tag{40}$$

where

$$\hat{c}_3^* = \frac{\hat{c}_3 \sum_{v_* \preceq a \in \{0,1\}^{K-2}} \hat{p}_{(1,1,a)} + \hat{g}_3 \sum_{v_* \npreceq a \in \{0,1\}^{K-2}} \hat{p}_{(1,1,a)}}{\sum_{a \in \{0,1\}^{K-2}} \hat{p}_{(1,1,a)}}.$$

Equations (39) and (40) imply that $\hat{c}_3^* > \hat{g}_3 > \bar{c}_3^* > \bar{g}_3$ or $\bar{c}_3^* > \bar{g}_3 > \hat{c}_3^* > \hat{g}_3$, which conflicts with the equation that $B_{\hat{\mathbf{c}},\hat{\mathbf{g}};Q}(3)\hat{\mathbf{p}} = B_{\bar{\mathbf{c}},\bar{\mathbf{g}};Q}(3)\bar{\mathbf{p}}$, i.e.,

$$\hat{g}_3(\hat{\mathbf{p}}_{0,0} + \hat{\mathbf{p}}_{1,0} + \hat{\mathbf{p}}_{0,1}) + \hat{c}_3^*\hat{\mathbf{p}}_{1,1} = \bar{g}_3(\bar{\mathbf{p}}_{0,0} + \bar{\mathbf{p}}_{1,0} + \bar{\mathbf{p}}_{0,1}) + \bar{c}_3^*\bar{\mathbf{p}}_{1,1}.$$

To see this, notice that $\hat{\mathbf{p}}_{0,0} + \hat{\mathbf{p}}_{1,0} + \hat{\mathbf{p}}_{0,1} = 1 - \hat{\mathbf{p}}_{1,1}$, $\bar{\mathbf{p}}_{0,0} + \bar{\mathbf{p}}_{1,0} + \bar{\mathbf{p}}_{0,1} = 1 - \bar{\mathbf{p}}_{1,1}$, and $\hat{\mathbf{p}}_{1,1}, \bar{\mathbf{p}}_{1,1} \in (0,1)$. By simple algebra, the above identity cannot be achieved if either $\hat{c}_3^* > \hat{g}_3 > \bar{c}_3^* > \bar{g}_3$ or $\bar{c}_3^* > \bar{g}_3 > \hat{c}_3^* > \hat{g}_3$ is true. Therefore, we have $\tilde{g}_3 = \bar{g}_3 - \hat{g}_3 = 0$. Let $\underline{\mathbf{g}} = (0, 0, \hat{g}_3, 0, \cdots, 0)$. $T_{\bar{\mathbf{c}}-\underline{\mathbf{g}},\bar{\mathbf{g}}-\underline{\mathbf{g}}}(Q)\bar{\mathbf{p}} = T_{\hat{\mathbf{c}}-\underline{\mathbf{g}},\hat{\mathbf{g}}-\underline{\mathbf{g}}}(Q)\hat{\mathbf{p}}$ yields

$$\begin{aligned}
\bar{c}_1 &= \frac{B_{\bar{\mathbf{c}}-\underline{\mathbf{g}},\bar{\mathbf{g}}-\underline{\mathbf{g}};Q}(1,4,3)\bar{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\underline{\mathbf{g}},\bar{\mathbf{g}}-\underline{\mathbf{g}};Q}(4,3)\bar{\mathbf{p}}} = \frac{B_{\hat{\mathbf{c}}-\underline{\mathbf{g}},\hat{\mathbf{g}}-\underline{\mathbf{g}};Q}(1,4,3)\hat{\mathbf{p}}}{B_{\hat{\mathbf{c}}-\underline{\mathbf{g}},\hat{\mathbf{g}}-\underline{\mathbf{g}};Q}(4,3)\hat{\mathbf{p}}} = \hat{c}_1, \\
\bar{c}_2 &= \frac{B_{\bar{\mathbf{c}}-\underline{\mathbf{g}},\bar{\mathbf{g}}-\underline{\mathbf{g}};Q}(2,4,3)\bar{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\underline{\mathbf{g}},\bar{\mathbf{g}}-\underline{\mathbf{g}};Q}(4,3)\bar{\mathbf{p}}} = \frac{B_{\hat{\mathbf{c}}-\underline{\mathbf{g}},\hat{\mathbf{g}}-\underline{\mathbf{g}};Q}(2,4,3)\hat{\mathbf{p}}}{B_{\hat{\mathbf{c}}-\underline{\mathbf{g}},\hat{\mathbf{g}}-\underline{\mathbf{g}};Q}(4,3)\hat{\mathbf{p}}} = \hat{c}_2, \\
\bar{c}_4 &= \frac{B_{\bar{\mathbf{c}}-\underline{\mathbf{g}},\bar{\mathbf{g}}-\underline{\mathbf{g}};Q}(1,4,3)\bar{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\underline{\mathbf{g}},\bar{\mathbf{g}}-\underline{\mathbf{g}};Q}(1,3)\bar{\mathbf{p}}} = \frac{B_{\hat{\mathbf{c}}-\underline{\mathbf{g}},\hat{\mathbf{g}}-\underline{\mathbf{g}};Q}(1,4,3)\hat{\mathbf{p}}}{B_{\hat{\mathbf{c}}-\underline{\mathbf{g}},\hat{\mathbf{g}}-\underline{\mathbf{g}};Q}(1,3)\hat{\mathbf{p}}} = \hat{c}_4.
\end{aligned}$$

Consider items 1 and 2. Let $\underline{\mathbf{c}} = (\hat{c}_1, \hat{c}_2, 0, \cdots, 0)$. $T_{\hat{\mathbf{c}},\hat{\mathbf{g}}}(Q)\hat{\mathbf{p}} = T_{\bar{\mathbf{c}},\bar{\mathbf{g}}}(Q)\bar{\mathbf{p}}$ yields

$$\begin{aligned}
\bar{g}_1 &= \frac{B_{\bar{\mathbf{c}}-\underline{\mathbf{c}},\bar{\mathbf{g}}-\underline{\mathbf{c}};Q}(1,2)\bar{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\underline{\mathbf{c}},\bar{\mathbf{g}}-\underline{\mathbf{c}};Q}(2)\bar{\mathbf{p}}} = \frac{B_{\hat{\mathbf{c}}-\underline{\mathbf{c}},\hat{\mathbf{g}}-\underline{\mathbf{c}};Q}(1,2)\hat{\mathbf{p}}}{B_{\hat{\mathbf{c}}-\underline{\mathbf{c}},\hat{\mathbf{g}}-\underline{\mathbf{c}};Q}(2)\hat{\mathbf{p}}} = \hat{g}_1, \\
\bar{g}_2 &= \frac{B_{\bar{\mathbf{c}}-\underline{\mathbf{c}},\bar{\mathbf{g}}-\underline{\mathbf{c}};Q}(1,2)\bar{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\underline{\mathbf{c}},\bar{\mathbf{g}}-\underline{\mathbf{c}};Q}(1)\bar{\mathbf{p}}} = \frac{B_{\hat{\mathbf{c}}-\underline{\mathbf{c}},\hat{\mathbf{g}}-\underline{\mathbf{c}};Q}(1,2)\hat{\mathbf{p}}}{B_{\hat{\mathbf{c}}-\underline{\mathbf{c}},\hat{\mathbf{g}}-\underline{\mathbf{c}};Q}(1)\hat{\mathbf{p}}} = \hat{g}_2.
\end{aligned}$$

Therefore, (36) is true.

Now combining the results in Cases 1 and 2, we have that for the Q-matrix taking the form of (8), the following holds:

$$\begin{cases} \hat{c}_j = \bar{c}_j & j = 1, \cdots, 2K \\ \hat{g}_j = \bar{g}_j & j = 1, \cdots, J \end{cases}. \tag{41}$$

Let $\mathbf{g}^* = (\hat{c}_1, \cdots, \hat{c}_K, \hat{g}_{K+1}, \cdots, \hat{g}_J)$. For each $j \in \{(2K+1), \cdots, J\}$, let $\mathcal{A}_j$ be the set of items $\{(K+1), \cdots, J\} \setminus \{j\}$, i.e., the set of all items from $K+1$ to J except the jth one. For the sub-matrix $Q_{K+1:J}$, condition A5 implies that each attribute appears at least twice.

Therefore, we have

$$\hat{c}_j - \hat{g}_j = \frac{B_{\hat{\mathbf{c}}-\mathbf{g}^*,\hat{\mathbf{g}}-\mathbf{g}^*;Q}(\mathcal{A}_j, j)\hat{\mathbf{p}}}{B_{\hat{\mathbf{c}}-\mathbf{g}^*,\hat{\mathbf{g}}-\mathbf{g}^*;Q}(\mathcal{A}_j)\hat{\mathbf{p}}} = \frac{B_{\bar{\mathbf{c}}-\mathbf{g}^*,\bar{\mathbf{g}}-\mathbf{g}^*;Q}(\mathcal{A}_j, j)\bar{\mathbf{p}}}{B_{\bar{\mathbf{c}}-\mathbf{g}^*,\bar{\mathbf{g}}-\mathbf{g}^*;Q}(\mathcal{A}_j)\bar{\mathbf{p}}} = \bar{c}_j - \hat{g}_j.$$

This gives $\hat{c}_j = \bar{c}_j$ for $j = 2K+1, \cdots, J$. Together with (41), $\hat{c}_j = \bar{c}_j$ for all $j = 1, \cdots, J$. This further yields $\hat{\mathbf{p}} = \bar{\mathbf{p}}$ due to the full column rank of the matrix $T_{\hat{\mathbf{c}},\hat{\mathbf{g}}}(Q)$.

Therefore, for two sets of parameters $(\hat{\mathbf{c}}, \hat{\mathbf{g}}, \hat{\mathbf{p}})$ and $(\bar{\mathbf{c}}, \bar{\mathbf{g}}, \bar{\mathbf{p}})$ such that $T_{\hat{\mathbf{c}},\hat{\mathbf{g}}}(Q)\hat{\mathbf{p}} = T_{\bar{\mathbf{c}},\bar{\mathbf{g}}}(Q)\bar{\mathbf{p}}$, we have $(\hat{\mathbf{c}}, \hat{\mathbf{g}}, \hat{\mathbf{p}}) = (\bar{\mathbf{c}}, \bar{\mathbf{g}}, \bar{\mathbf{p}})$. This finishes the proof of Theorem 4. ■

C Proof of Propositions

Proof of Proposition 2. Notice that the column vector $T_{\mathbf{c},\mathbf{g}}(Q)\mathbf{p}$ contains the probabilities $P(R_{j_1} = 1, ..., R_{j_l} = 1)$ for all possible distinct combinations j_1,...,j_l. Thus, $T_{\mathbf{c},\mathbf{g}}(Q)\mathbf{p}$ completely characterizes the distribution of $\mathbf{R}$. Two sets of parameters $T_{\hat{\mathbf{c}},\hat{\mathbf{g}}}(Q)\hat{\mathbf{p}} = T_{\bar{\mathbf{c}},\bar{\mathbf{g}}}(Q)\bar{\mathbf{p}}$ if and only if they correspond to the same distribution of $\mathbf{R}$. This concludes the proof. ■

Proof of the Proposition 3. In what follows, we construct a D matrix satisfying the condition in the proposition. We show that there exists a matrix D only depending on g^* so that $DT_{\mathbf{c},\mathbf{g}}(Q) = T_{\mathbf{c}-\mathbf{g}^*,\mathbf{g}-\mathbf{g}^*}(Q)$. Note that each row of $DT_{\mathbf{c},\mathbf{g}}(Q)$ is just a row linear transform of $T_{\mathbf{c},\mathbf{g}}(Q)$. Then, it is sufficient to show that each row vector of $T_{\mathbf{c}-\mathbf{g}^*,\mathbf{g}-\mathbf{g}^*}(Q)$ is a linear transform of rows of $T_{\mathbf{c},\mathbf{g}}(Q)$ with coefficients only depending on g^*. We prove this by induction.

First, note that

$$B_{\mathbf{c}-\mathbf{g}^*,\mathbf{g}-\mathbf{g}^*;Q}(j) = B_{\mathbf{c},\mathbf{g};Q}(j) - g_j^*\mathbf{1}^\top$$

where $\mathbf{1}^\top$ is a row vector with all elements being 1. Then all row vectors of $T_{\mathbf{c}-\mathbf{g}^*,\mathbf{g}-\mathbf{g}^*}(Q)$ of the form $B_{\mathbf{c}-\mathbf{g}^*,\mathbf{g}-\mathbf{g}^*,Q}(j)$ are inside the row space of $T_{\mathbf{c},\mathbf{g}}(Q)$ with coefficients only depending on g^*. Suppose that all the vectors of the form

$$B_{\mathbf{c}-\mathbf{g}^*,\mathbf{g}-\mathbf{g}^*;Q}(j_1, ..., j_l)$$

for all $1 \leq l \leq \iota$ can be written linear combinations of the row vectors of $T_{\mathbf{c},\mathbf{g}}(Q)$ with coefficients only depending on g^*. Then, we consider

$$B_{\mathbf{c},\mathbf{g};Q}(j_1, ..., j_{\iota+1}) = \Upsilon_{h=1}^{\iota+1}\left(B_{\mathbf{c}-\mathbf{g}^*,\mathbf{g}-\mathbf{g}^*;Q}(j_h) + g_{j_h}^*\mathbf{1}^\top\right),$$

where "Υ" refers to element by element multiplication. The left hand side is just a row vector of $T_{\mathbf{c},\mathbf{g}}(Q)$. We expand the right hand side of the above display. Note that the last term is precisely

$$B_{\mathbf{c}-\mathbf{g}^*,\mathbf{g}-\mathbf{g}^*;Q}(j_1, ..., j_{\iota+1}) = \Upsilon_{h=1}^{\iota+1}B_{\mathbf{c}-\mathbf{g}^*,\mathbf{g}-\mathbf{g}^*;Q}(j_h).$$

The rest terms are all of the form $B_{\mathbf{c}-\mathbf{g}^*,\mathbf{g}-\mathbf{g}^*;Q}(j_1, ..., j_l)$ for $1 \leq l \leq \iota$ multiplied by coefficients only depending on g^*. Therefore, according to the induction assumption, we have that $B_{\mathbf{c}-\mathbf{g}^*,\mathbf{g}-\mathbf{g}^*;Q}(j_1, ..., j_{\iota+1})$ can be written as linear combinations of rows of $T_{\mathbf{c},\mathbf{g}}(Q)$ with coefficients only depending on g^*. ■