Appendix for ”Flexible combination of multiple diagnostic biomarkers to improve diagnostic accuracy” by Xu, Rong, Fang and Wang

Proof of Proposition 1. Since $L_\delta(u) = L_{01}(u) + \delta^{-1}(\delta - u)I(0 \le u \le \delta)$, we have

$$\begin{aligned} E\Big(w(Y,\mathbf{Z})L_\delta(Y(g(\mathbf{X}) - c(\mathbf{Z}))\Big) &= E\Big(w(Y,\mathbf{Z})L_{01}(Y(g(\mathbf{X}) - c(\mathbf{Z}))\Big) \\ &\quad + E\Big(w(Y,\mathbf{Z})\frac{\delta - Y(g(\mathbf{X}) - c(\mathbf{Z}))}{\delta} I(0 \le Y(g(\mathbf{X}) - c(\mathbf{Z})) \le \delta)\Big). \end{aligned} \tag{1}$$

Note that $E\big(w(Y,\mathbf{Z})\frac{\delta - Y(g(\mathbf{X}) - c(\mathbf{Z}))}{\delta} I(0 \le Y(g(\mathbf{X}) - c(\mathbf{Z})) \le \delta)\big)$ is decreasing in δ, and approaches 0 when $\delta \to 0$. Furthermore, for any given $\epsilon > 0$,

$$\begin{aligned} &E\big(w(Y,\mathbf{Z})L_{01}(Y(g(\mathbf{X}) - c(\mathbf{Z})))\big) - E\big(w(Y,\mathbf{Z})L_{01}(Y(g^*(\mathbf{X}) - c^*(\mathbf{Z}))\big) \\ =& \int_{\mathcal{D}_{g,c,0} \cap \mathcal{D}^c_{g^*,c^*,0}} \frac{\pi_\mathbf{z} - p_\mathbf{z}(\mathbf{x})}{\pi_\mathbf{z}(1-\pi_\mathbf{z})} f(\tilde{\mathbf{x}})d\tilde{\mathbf{x}} + \int_{\mathcal{D}^c_{g,c,0} \cap \mathcal{D}_{g^*,c^*,0}} \frac{p_\mathbf{z}(\mathbf{x}) - \pi_\mathbf{z}}{\pi_\mathbf{z}(1-\pi_\mathbf{z})} f(\tilde{\mathbf{x}})d\tilde{\mathbf{x}} \\ \ge& \int_{\mathcal{D}_{g,c,\epsilon} \cap \mathcal{D}^c_{g^*,c^*,\epsilon}} \frac{\pi_\mathbf{z} - p_\mathbf{z}(\mathbf{x})}{\pi_\mathbf{z}(1-\pi_\mathbf{z})} f(\tilde{\mathbf{x}})d\tilde{\mathbf{x}} + \int_{\mathcal{D}^c_{g,c,\epsilon} \cap \mathcal{D}_{g^*,c^*,\epsilon}} \frac{p_\mathbf{z}(\mathbf{x}) - \pi_\mathbf{z}}{\pi_\mathbf{z}(1-\pi_\mathbf{z})} f(\tilde{\mathbf{x}})d\tilde{\mathbf{x}}, \end{aligned} \tag{2}$$

where $f(\tilde{\mathbf{x}})$ denotes the density function of $\tilde{\mathbf{x}}$. By (**??**), we have

$$\begin{cases} \pi_\mathbf{z} - p_\mathbf{z}(\mathbf{x}) > \epsilon, & \text{if } \tilde{\mathbf{x}} \in \mathcal{D}_{g,c,\epsilon} \cap \mathcal{D}^c_{g^*,c^*,\epsilon}; \\ p_\mathbf{z}(\mathbf{x}) - \pi > \epsilon, & \text{if } \tilde{\mathbf{x}} \in \mathcal{D}^c_{g,c,\epsilon} \cap \mathcal{D}_{g^*,c^*,\epsilon}. \end{cases}$$

Therefore,

$$E\big(w(Y,\mathbf{Z})L_{01}(Y(g^*_\delta(\mathbf{X}) - c^*_\delta(\mathbf{Z})))\big) - E\big(w(Y,\mathbf{Z})L_{01}(Y(g^*(\mathbf{X}) - c^*(\mathbf{Z})) > \epsilon Pr\left(\mathcal{D}_{g^*_\delta, c^*_\delta, \epsilon} \triangle \mathcal{D}_{g^*,c^*,\epsilon}\right).$$

By the fact that $E\Big(w(Y,\mathbf{Z})L_\delta(Y(g^*_\delta(\mathbf{X}) - c^*_\delta(\mathbf{Z}))\Big) \le E\Big(w(Y,\mathbf{Z})L_\delta(Y(g^*(\mathbf{X}) - c^*(\mathbf{Z}))\Big)$, we have

$$\begin{aligned} &E\big(w(Y,\mathbf{Z})L_{01}(Y(g^*_\delta(\mathbf{X}) - c^*_\delta(\mathbf{Z})))\big) - E\big(w(Y,\mathbf{Z})L_{01}(Y(g^*(\mathbf{X}) - c^*(\mathbf{Z})))\big) \\ &\le E\Big(w(Y,\mathbf{Z})\frac{\delta - Y(g^*(\mathbf{X}) - c^*(\mathbf{Z}))}{\delta} I(0 \le Y(g^*(\mathbf{X}) - c^*(\mathbf{Z})) \le \delta)\Big). \end{aligned} \tag{3}$$

It follows immediately that $Pr\left(\mathcal{D}_{g^*_\delta, c^*_\delta, \epsilon} \triangle \mathcal{D}_{g^*,c^*,\epsilon}\right) \to 0$ as $\delta \to 0$. □