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Web-based Supporting Materials for "A Marginal-Mean ANOVA Approach for Analyzing Multireader Multicase Radiological Imaging Data" by Stephen Hillis

This section contains tables showing the derivation of the analysis formulas for the study design examples given in Section 5. These derivations are based on the "Algorithm for deriving mm-ANOVA formulas" template given in Table 5 and illustrated in Table 6 for the factorial study design. For these derivations, notation for test, reader and case effects is similar to that for the factorial model. Standard nesting notation is used; e.g., subscript $i(j)$ denotes that the factor indexed by i is nested within the factor indexed by j, and MS$[R * C\,(T)]$ is the mean square for reader-by-case interaction nested within test. Mean squares for the mm-ANOVA model are indicated by a tilde symbol "˜" above "MS"; e.g., $\widetilde{\mathrm{MS}}\,(T)$ is the mean square due to test for the mm-ANOVA model, and $\mathrm{MS}\,(T)$ is for the conventional ANOVA model.

Table S1. Mm-ANOVA approach for reader $\times$ case design (only one test)

1. Derive the mm-ANOVA model
   (a) Conventional ANOVA model: $Y_{jk} = \mu + R_j + C_k + (RC)_{jk} + \varepsilon_{jk}$, $j = 1, \ldots, r; k = 1, \ldots c$, with variance components $\sigma_R^2, \sigma_C^2, \sigma_{RC}^2, \sigma_\varepsilon^2$. Define $\sigma^2 \equiv \sigma_{RC}^2 + \sigma_\varepsilon^2$.
   (b) Mm-ANOVA model (note: $\tilde{Y}_j = Y_{j\bullet}$): $\tilde{Y}_j = \mu + R_j + \tilde{\varepsilon}_j$ where $\tilde{\varepsilon}_j = C_\bullet + (RC)_{j\bullet} + \varepsilon_{j\bullet}$
   (c) Mm-ANOVA error variance and covariances expressed in terms of conventional ANOVA variance components: $\sigma_{\tilde{\varepsilon}}^2 = \frac{1}{c}\left(\sigma_R^2 + \sigma_C^2 + \sigma^2\right)$, $\text{Cov}_2 \equiv \text{cov}(\tilde{\varepsilon}_j, \tilde{\varepsilon}_{j'}) = \frac{1}{c}\sigma_C^2, j \neq j'$
   (d) Covariance constraint: $\text{Cov}_2 \geq 0$
2. Derive the mm-ANOVA test statistic and its null distribution – this step is omitted, since we are only interested in obtained a confidence interval.
3. Derive confidence interval for test accuracy
   (a) Mm-ANOVA expected reader-performance parameter: $\theta = E\left(\tilde{Y}_j\right)$
   (b) Corresponding conventional ANOVA parameter: $\theta = E\left(Y_{j\bullet}\right) = \mu$
   (c) Conventional ANOVA estimate: $\hat{\theta} = Y_{\bullet\bullet}$
   (d) $Y_{\bullet\bullet} = \mu + R_\bullet + C_\bullet + (RC)_{\bullet\bullet} + \varepsilon_{\bullet\bullet} \Longrightarrow V = \frac{1}{rc}\left(c\sigma_R^2 + r\sigma_C^2 + \sigma^2\right)$.
   (e) $V = \frac{1}{rc}E\left[\text{MS}\left(R\right) + \text{MS}\left(C\right) - \text{MS}\left(R * C\right)\right]$
   (f) $V = \frac{1}{r}E\left[\widetilde{\text{MS}}\left(R\right) + U\right]$ where $U = \frac{1}{c}\left[\text{MS}\left(C\right) - \text{MS}\left(R * C\right)\right]$
   (g) $E\left(U\right) = \frac{r\sigma_C^2}{c} = r\,\text{Cov}_2 \Rightarrow V = \frac{1}{r}\left\{E\left[\widetilde{\text{MS}}\left(R\right)\right] + r\,\text{Cov}_2\right\}$
   (h) $\hat{V} = \frac{1}{r}\left[\widetilde{\text{MS}}\left(R\right) + r\max\left(\widehat{\text{Cov}}_2, 0\right)\right]$
   (i) $\text{df}_2 = \frac{\left[\widetilde{\text{MS}}(R) + r\max\left(\widehat{\text{Cov}}_2, 0\right)\right]^2}{\frac{\left[\widetilde{\text{MS}}(R)\right]^2}{r-1}}$
   (j) $\hat{\theta} = \tilde{Y}_\bullet$
   (k) CI: $\tilde{Y}_\bullet \pm t_{\alpha/2;\text{df}_2}\sqrt{\frac{1}{r}\left[\widetilde{\text{MS}}\left(R\right) + r\max\left(\widehat{\text{Cov}}_2, 0\right)\right]}$

Table S2. Mm-ANOVA approach for reader-nested-within-test split-plot design

1. Derive the mm-ANOVA model

   (a) Conventional ANOVA model: $Y_{ijk} = \mu + \tau_i + R_{(i)j} + C_k + (\tau C)_{ik} + (RC)_{(i)jk} + \varepsilon_{ijk},\ i = 1, \ldots, t; j = 1, \ldots, r; k = 1, \ldots c,$ with variance components $\sigma^2_{R(T)}, \sigma^2_C, \sigma^2_{TC}, \sigma^2_{RC(T)}, \sigma^2_\varepsilon$ and constraint $\sum_{i=1}^t \tau_i = 0$. Define $\sigma^2 = \sigma^2_{RC(T)} + \sigma^2_\varepsilon$.

   (b) Mm-ANOVA model (note: $\tilde{Y}_{ij} = Y_{ij\bullet}$): $\tilde{Y}_{ij} = \mu + \tau_i + R_{(i)j} + \tilde{\varepsilon}_{ij}$ where $\tilde{\varepsilon}_{ij} = C_\bullet + (\tau C)_{i\bullet} + (RC)_{(i)j\bullet} + \varepsilon_{ij\bullet}$ and $\sum_{i=1}^t \tau_i = 0$

   (c) Mm-ANOVA error variance and covariances expressed in terms of conventional ANOVA variance-components: $\sigma^2_{\tilde{\varepsilon}} = \frac{1}{c}\left(\sigma^2_C + \sigma^2_{TC} + \sigma^2\right)$; $\text{Cov}_2 \equiv \text{cov}(\tilde{\varepsilon}_{ij}, \tilde{\varepsilon}_{ij'}) = \frac{1}{c}\left(\sigma^2_C + \sigma^2_{TC}\right),\quad j \neq j'$; $\text{Cov}_3 \equiv \text{cov}(\tilde{\varepsilon}_{ij}, \tilde{\varepsilon}_{i'j'}) = \frac{1}{c}\sigma^2_C,\ i \neq i'$

   (d) Covariance constraints: $\text{Cov}_2 \geq \text{Cov}_3 \geq 0$

2. Derive the mm-ANOVA test statistic and its null distribution

   (a) Mm-ANOVA model hypothesis of equal test accuracies: $\text{H}_0 : \theta_1 = \cdots = \theta_t$ where $\theta_i = E\left(\tilde{Y}_{i\bullet}\right)$

   (b) Conventional ANOVA hypothesis: $\theta_i = E\left(Y_{i\bullet\bullet}\right) = \mu + \tau_i \implies \text{H}_0 : \tau_1 = \cdots = \tau_t = 0$

   (c) Conventional ANOVA expected mean squares

Mean square	Expected mean square
$\text{MS}(T)$	$\frac{rc}{(t-1)}\sum_{i=1}^t \tau_i^2 + c\sigma^2_{R(T)} + r\sigma^2_{TC} + \sigma^2$
$\text{MS}[R(T)]$	$c\sigma^2_{R(T)} + \sigma^2$
$\text{MS}(C)$	$tr\sigma^2_C + r\sigma^2_{TC} + \sigma^2$
$\text{MS}(T * C)$	$r\sigma^2_{TC} + \sigma^2$
$\text{MS}[R * C(T)]$	$\sigma^2 \equiv \sigma^2_{RC(T)} + \sigma^2_\varepsilon$

   (d) Conventional ANOVA test statistic: $F = \frac{\text{MS}(T)}{\text{MS}[R(T)] + \text{MS}(T*C) - \text{MS}[R*C(T)]}$

   (e) $\widetilde{\text{MS}}\,(T) = \frac{1}{c}\text{MS}(T)$, $\widetilde{\text{MS}}\,[R\,(T)] = \frac{1}{c}\text{MS}[R\,(T)]$

   (f) $F = \frac{\widetilde{\text{MS}}(T)}{\widetilde{\text{MS}}[R(T)] + U}$ where $U = \frac{1}{c}\left\{\text{MS}\,(T * C) - \text{MS}\,[R * C\,(T)]\right\}$

   (g) $E\left\{\text{MS}\,(T * C)\right\} = r\sigma^2_{TC} + \sigma^2$, $E\left\{\text{MS}\,[R * C\,(T)]\right\} = \sigma^2 \implies E\,(U) = \frac{1}{c}\left(r\sigma^2_{TC}\right) = r\,(\text{Cov}_2 - \text{Cov}_3)$.

   (h) $F^*_{\text{OR}} = \frac{\widetilde{\text{MS}}(T)}{\widetilde{\text{MS}}[R(T)] + r(\text{Cov}_2 - \text{Cov}_3)}$

   (i) $F_{\text{OR}} = \frac{\widetilde{\text{MS}}(T)}{\widetilde{\text{MS}}[R(T)] + r\max\left(\widehat{\text{Cov}}_2 - \widehat{\text{Cov}}_3, 0\right)}$

   (j) Under H_0, $F_{\text{OR}} \dot{\sim} F_{t-1,\text{df}_2}$ where $\text{df}_2 = \frac{\left[\widetilde{\text{MS}}[R(T)] + r\max\left(\widehat{\text{Cov}}_2 - \widehat{\text{Cov}}_3, 0\right)\right]^2}{\left[\widetilde{\text{MS}}[R(T)]\right]^2 / [t(r-1)]}$

3. Derive confidence intervals

   (a) Mm-ANOVA model test accuracy performance parameters: $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_t)'$, with: $\theta_i = E\left(\tilde{Y}_{i\bullet}\right), i = 1, \ldots, t$

   (b) Corresponding conventional ANOVA parameters: $\theta_i = E\left(Y_{i\bullet\bullet}\right) = \mu + \tau_i$

   (c) Conventional ANOVA estimate: $\hat{\theta}_i = Y_{i\bullet\bullet}$

   CI for $\boldsymbol{l}'\,(\boldsymbol{\theta})$ with $\boldsymbol{l} = (l_1, \ldots, l_t)'$, $\sum_{i=1}^t l_i = 0$:

   (d) $\boldsymbol{l}'\left(\hat{\boldsymbol{\theta}}\right) = \sum_{i=1}^t l_i \hat{\boldsymbol{\theta}}_i = \sum_{i=1}^t l_i Y_{i\bullet\bullet} = \sum_{i=1}^t l_i \tau_i + \sum_{i=1}^t l_i \left[R_{(i)\bullet} + (\tau C)_{i\bullet} + (RC)_{(i)\bullet\bullet} + \varepsilon_{i\bullet\bullet}\right]$
   $\Rightarrow V = \sum_{i=1}^t l_i^2 \left[\frac{\sigma^2_{R(T)}}{r} + \frac{\sigma^2_{TC}}{c} + \frac{\sigma^2}{rc}\right] = \frac{1}{rc}\sum_{i=1}^t l_i^2 \left[c\sigma^2_{R(T)} + r\sigma^2_{TC} + \sigma^2\right]$

   (e) $V = \frac{1}{rc}\sum_{i=1}^t l_i^2 E\left[\text{MS}\,[R\,(T)] + \text{MS}\,(T * C) - \text{MS}\,[R * C\,(T)]\right]$

   (f) $V = \frac{1}{r}\sum_{i=1}^t l_i^2 E\left[\widetilde{\text{MS}}\,[R\,(T)] + U\right]$ where $U = \frac{1}{c}\left\{\text{MS}\,(T * C) - \text{MS}\,[R * C\,(T)]\right\}$

   (g) $E\,(U) = \frac{r\sigma^2_{TC}}{c} = r\,(\text{Cov}_2 - \text{Cov}_3) \Rightarrow V = \frac{1}{r}\sum_{i=1}^t l_i^2 \left\{E\left[\widetilde{\text{MS}}\,[R\,(T)]\right] + r\,(\text{Cov}_2 - \text{Cov}_3)\right\}$

(h) $\hat{V} = \frac{1}{r}\sum_{i=1}^{t} l_i^2 \left\{\widetilde{\text{MS}}\left[R\left(T\right)\right] + r\max\left(\widehat{\text{Cov}}_2 - \widehat{\text{Cov}}_3, 0\right)\right\}$

(i) $\text{df}_2 = \frac{\left[\widetilde{\text{MS}}[R(T)] + r\max\left(\widehat{\text{Cov}}_2 - \widehat{\text{Cov}}_3, 0\right)\right]^2}{\left[\widetilde{\text{MS}}[R(T)]\right]^2/[t(r-1)]}$ (same as df_2 in step 2j)

(j) $\hat{\theta}_i = \tilde{Y}_{i\bullet}$

(k) CI: $\sum_{i=1}^{t} l_i \tilde{Y}_{i\bullet} \pm t_{\alpha/2;\text{df}}\sqrt{\frac{1}{r}\sum_{i=1}^{t} l_i^2 \left\{\widetilde{\text{MS}}\left[R\left(T\right)\right] + r\max\left(\widehat{\text{Cov}}_2 - \widehat{\text{Cov}}_3, 0\right)\right\}}$

CI for θ_i

(d) $\hat{\theta}_i = Y_{i\bullet\bullet} = \mu + \tau_i + R_{(i)\bullet} + C_\bullet + (\tau C)_{i\bullet} + (RC)_{(i)\bullet\bullet} + \varepsilon_{i\bullet\bullet} \Rightarrow V = \frac{1}{rc}\left(c\sigma_{R(T)}^2 + r\sigma_C^2 + r\sigma_{TC}^2 + \sigma^2\right)$

(e) $V = \frac{1}{trc}E\left\{t\,\text{MS}\left[R\left(T\right)\right] + \text{MS}\left(C\right) + \left(t-1\right)\text{MS}\left(T*C\right) - t\,\text{MS}\left[R*C\left(T\right)\right]\right\}$

(f) $V = \frac{1}{tr}E\left\{t\widetilde{\text{MS}}\left[R\left(T\right)\right] + U\right\}$ where $U = \frac{1}{c}\left(\text{MS}\left(C\right) + \left(t-1\right)\text{MS}\left(T*C\right) - t\,\text{MS}\left[R*C\left(T\right)\right]\right)$

(g) $E\left(U\right) = \frac{1}{c}tr\left(\sigma_C^2 + \sigma_{TC}^2\right) = tr\,\text{Cov}_2 \Rightarrow V = \frac{1}{r}\left\{\widetilde{\text{MS}}\left[R\left(T\right)\right] + r\,\text{Cov}_2\right\}$

(h) $\hat{V} = \frac{1}{r}\left\{\widetilde{\text{MS}}\left[R\left(T\right)\right] + \max\left(r\widehat{\text{Cov}}_2, 0\right)\right\}$

(i) $\text{df}_2 = \frac{\left\{\widetilde{\text{MS}}[R(T)] + \max\left(r\widehat{\text{Cov}}_2, 0\right)\right\}^2}{\frac{\left\{\widetilde{\text{MS}}[R(T)]\right\}^2}{t(r-1)}}$

(j) $\hat{\theta}_i = \tilde{Y}_{i\bullet}$

(k) CI: $\tilde{Y}_{i\bullet} \pm t_{\alpha/2;\text{df}}\sqrt{\frac{1}{r}\left\{\widetilde{\text{MS}}\left[R\left(T\right)\right] + \max\left(r\widehat{\text{Cov}}_2, 0\right)\right\}}$

4. Derive the non-null distribution $F_{\text{df}_1,\text{df}_2;\lambda}$ of the step-2 F statistic

(a) Step 2d F numerator: $\text{MS}_{\text{num}} = \text{MS}\left(T\right), E\left[\text{MS}\left(T\right)\right] = \frac{rc}{(t-1)}\sum_{i=1}^{t}\tau_i^2 + c\sigma_{R(T)}^2 + r\sigma_{TC}^2 + \sigma^2$, $\text{df}\left(\text{MS}\left(T\right)\right) = t-1$, $E\left(Y_{ijk}\right) = \mu + \tau_i \Rightarrow \lambda = \frac{\text{df}(\text{MS}_{\text{num}})\,\text{MS}_{\text{num}}|_{\mathbf{Y}=E(\mathbf{Y})}}{E(\text{MS}_{\text{num}}\,|\text{H}_0)} = \frac{rc\sum_{i=1}^{t}\tau_i^2}{c\sigma_{R(T)}^2 + r\sigma_{TC}^2 + \sigma^2}$

(b) $r\sigma_{TC}^2 + \sigma^2 = c\left[\sigma_{\tilde{\varepsilon}}^2 + \left(r-1\right)\text{Cov}_2 - r\,\text{Cov}_3\right] \Rightarrow \lambda = \frac{r\sum_{i=1}^{t}\tau_i^2}{\sigma_{R(T)}^2 + \sigma_{\tilde{\varepsilon}}^2 + (r-1)\,\text{Cov}_2 - r\,\text{Cov}_3}$

(c) Step 2h F_{OR}^* denominator $= \widetilde{\text{MS}}\left[R\left(T\right)\right] + r\left(\widehat{\text{Cov}}_2 - \widehat{\text{Cov}}_3\right)$, $E\left\{\widetilde{\text{MS}}\left[R\left(T\right)\right]\right\} = \frac{1}{c}E\left\{\text{MS}\left[R\left(T\right)\right]\right\} = \frac{1}{c}\left(c\sigma_{R(T)}^2 + \sigma^2\right) = \sigma_{R(T)}^2 + \sigma_{\tilde{\varepsilon}}^2 - \text{Cov}_2 \Rightarrow \text{df}_2 = \frac{\left[\sigma_{R(T)}^2 + \sigma_{\tilde{\varepsilon}}^2 + (r-1)\,\text{Cov}_2 - r\,\text{Cov}_3\right]^2}{\frac{\left(\sigma_{R(T)}^2 + \sigma_{\tilde{\varepsilon}}^2 - \text{Cov}_2\right)^2}{t(r-1)}}$

(d) $F_{\text{OR}} \dot{\sim} F_{t-1,\text{df}_2;\lambda}$

Table S3. Mm-ANOVA approach for case-nested-within-test split-plot study design. Because step 1 shows that this model is the same as the factorial model (Table 6) except that Cov_1 and Cov_3 are constrained to zero, there is no need to continue with the other steps.

1. Derive the mm-ANOVA model
   (a) Conventional ANOVA model: $Y_{ijk} = \mu + \tau_i + R_j + C_{(i)k} + (\tau R)_{ij} + (RC)_{(i)jk} + \varepsilon_{ijk},\ i = 1, \ldots, t; j = 1, \ldots, r; k = 1, \ldots c,$ with variance components $\sigma^2_R, \sigma^2_{C(T)}, \sigma^2_{TR}, \sigma^2_{RC(T)}, \sigma^2_\varepsilon$ and constraint $\sum_{i=1}^t \tau_i = 0$. Define $\sigma^2 = \sigma^2_{RC(T)} + \sigma^2_\varepsilon$.
   (b) Mm-ANOVA model $(\tilde{Y}_{ij} = Y_{ij\bullet}) : \tilde{Y}_{ij} = \mu + \tau_i + R_j + (\tau R)_{ij} + \tilde{\varepsilon}_{ij}$ where $\tilde{\varepsilon}_{ij} = C_{(i)\bullet} + (RC)_{(i)j\bullet} + \varepsilon_{ij\bullet}$ and $\sum_{i=1}^t \tau_i = 0$
   (c) Mm-ANOVA covariances expressed in terms of conventional ANOVA variance components: $\sigma^2_{\tilde{\varepsilon}} = \frac{1}{c}\left(\sigma^2_{C(T)} + \sigma^2\right)$, $\text{Cov}_2 \equiv \text{cov}(\tilde{\varepsilon}_{ij}, \tilde{\varepsilon}_{ij'}) = \frac{1}{c}\sigma^2_{C(T)}$, $\text{Cov}_1 \equiv \text{cov}(\tilde{\varepsilon}_{ij}, \tilde{\varepsilon}_{i'j}) = 0$, $\text{Cov}_3 \equiv \text{cov}(\tilde{\varepsilon}_{ij}, \tilde{\varepsilon}_{i'j'}) = 0$, where $i \neq i', j \neq j'$
   (d) Covariance constraints: $\text{Cov}_2 \geq 0$, $\text{Cov}_1 = \text{Cov}_3 = 0$

Table S4 Mm-ANOVA approach for case-nested-within-reader split-plot design. Because step 1 shows that this model is the same as the factorial model (Table 6) except that Cov_2 and Cov_3 are constrained to zero, there is no need to continue with the other steps.

1. Derive the mm-ANOVA model

   (a) Conventional ANOVA model: $Y_{ijk} = \mu + \tau_i + R_j + C_{(j)k} + (\tau R)_{ij} + (\tau C)_{i(j)k} + \varepsilon_{ijk}$, $i = 1, \ldots, t; j = 1, \ldots, r; k = 1, \ldots c$, with variance components $\sigma^2_R, \sigma^2_{C(R)}, \sigma^2_{TR}, \sigma^2_{TC(R)}, \sigma^2_\varepsilon$ and constraint $\sum_{i=1}^t \tau_i = 0$. Define $\sigma^2 = \sigma^2_{TC(R)} + \sigma^2_\varepsilon$.

   (b) Mm-ANOVA model ($\tilde{Y}_{ij} = Y_{ij\bullet}$): $\tilde{Y}_{ij} = \mu + \tau_i + R_j + (\tau R)_{ij} + \tilde{\varepsilon}_{ij}$ where $\tilde{\varepsilon}_{ij} = C_{(j)\bullet} + (\tau C)_{i(j)\bullet} + \varepsilon_{ij\bullet}$ and $\sum_{i=1}^t \tau_i = 0$

   (c) Mm-ANOVA covariances expressed in terms of conventional ANOVA variance components: $\sigma^2_{\tilde{\varepsilon}} = \frac{1}{c}\left(\sigma^2_{C(R)} + \sigma^2\right)$, $\text{Cov}_1 \equiv \text{cov}(\tilde{\varepsilon}_{ij}, \tilde{\varepsilon}_{i'j}) = \frac{1}{c}\sigma^2_{C(R)}$, $\text{Cov}_2 \equiv \text{cov}(\tilde{\varepsilon}_{ij}, \tilde{\varepsilon}_{ij'}) = 0$, $\text{Cov}_3 \equiv \text{cov}(\tilde{\varepsilon}_{ij}, \tilde{\varepsilon}_{i'j'}) = 0$, where $i \neq i', j \neq j'$

   (d) Covariance constraints: $\text{Cov}_1 \geq 0$, $\text{Cov}_2 = \text{Cov}_3 = 0$

Table S5. Mm-ANOVA approach for "mixed" split-plot design (reader and case crossed and nested within group)

1. Derive the mm-ANOVA model

   (a) Conventional ANOVA model: $Y_{hijk} = \mu + \gamma_h + \tau_i + (\gamma\tau)_{hi} + R_{(h)j} + C_{(h)k} + (\tau R)_{(h)ij} + (\tau C)_{(h)ik} + (RC)_{(h)jk} + (\tau RC)_{(h)ijk} + \varepsilon_{hijk}, h = 1,\ldots,g; i = 1,\ldots,t; j = 1,\ldots,r; k = 1,\ldots c,$ with variance components $\sigma^2_{R(G)}, \sigma^2_{C(G)}, \sigma^2_{TR(G)}, \sigma^2_{TC(G)}, \sigma^2_{RC(G)}, \sigma^2_{TRC(G)},\ \sigma^2_\varepsilon$ and constraints $\sum_{i=1}^t \tau_i = \sum_{h=1}^g \gamma_h = \sum_{h=1}^g (\gamma\tau)_{hi} = \sum_{i=1}^t (\gamma\tau)_{hi} = 0$; γ_h denotes the effect of group level h and $R(G)$ is read "reader nested within group," etc. Define $\sigma^2 \equiv \sigma^2_{TRC(G)} + \sigma^2_\varepsilon$.
   (b) Mm-ANOVA model (note: $\tilde{Y}_{hij} = Y_{hij\bullet}$): $\tilde{Y}_{hij} = \mu + \gamma_h + \tau_i + (\gamma\tau)_{hi} + R_{(h)i} + (\tau R)_{(h)ij} + \tilde{\varepsilon}_{hij}$ where $\tilde{\varepsilon}_{hij} = C_{(h)\bullet} + (\tau C)_{(h)i\bullet} + (RC)_{(h)j\bullet} + (\tau RC)_{(h)ij\bullet} + \varepsilon_{hij\bullet}$ and $\sum_{i=1}^t \tau_i = \sum_{h=1}^g \gamma_h = \sum_{h=1}^g \sum_{i=1}^t (\gamma\tau)_{hi} = 0$
   (c) Mm-ANOVA error variance and covariances expressed in terms of conventional ANOVA variance-components: $\sigma^2_{\tilde{\varepsilon}} = \frac{1}{c}\left(\sigma^2_{C(G)} + \sigma^2_{TC(G)} + \sigma^2_{RC(G)} + \sigma^2\right)$, $\text{Cov}_1 \equiv \text{cov}(\tilde{\varepsilon}_{hij}, \tilde{\varepsilon}_{hi'j}) = \frac{1}{c}\left(\sigma^2_{C(G)} + \sigma^2_{RC(G)}\right)$, $\text{Cov}_2 \equiv \text{cov}(\tilde{\varepsilon}_{hij}, \tilde{\varepsilon}_{hij'}) = \frac{1}{c}\left(\sigma^2_{C(G)} + \sigma^2_{TC(G)}\right)$, $\text{Cov}_3 \equiv \text{cov}(\tilde{\varepsilon}_{hij}, \tilde{\varepsilon}_{hi'j'}) = \frac{1}{c}\sigma^2_{C(G)}$, where $i \neq i', j \neq j'$
   (d) Covariance constraints: $\text{Cov}_1 \geq \text{Cov}_3$, $\text{Cov}_2 \geq \text{Cov}_3$, $\text{Cov}_3 \geq 0$

2. Derive the mm-ANOVA test statistic and its null distribution

   (a) Mm-ANOVA hypothesis of equal test accuracies: $\text{H}_0 : \theta_1 = \cdots = \theta_t$ where $\theta_i = E\left(\tilde{Y}_{\bullet i \bullet}\right)$
   (b) Conventional ANOVA hypothesis: $\theta_i = E\left(Y_{\bullet i \bullet\bullet}\right) = \mu + \gamma_\bullet + \tau_i + (\gamma\tau)_{\bullet i} \implies \text{H}_0 : \tau_1 = \cdots = \tau_t = 0$
   (c) Conventional ANOVA expected mean squares

Mean square	Expected mean square
MS(G)	$\frac{trc}{(g-1)}\sum_{i=1}^t \gamma_h^2 + tc\sigma^2_{R(G)} + tr\sigma^2_{C(G)} + c\sigma^2_{TR(G)} + r\sigma^2_{TC(G)} + t\sigma^2_{RC(G)} + \sigma^2$
MS(T)	$\frac{grc}{(t-1)}\sum_{i=1}^t \tau_i^2 + c\sigma^2_{TR(G)} + r\sigma^2_{TC(G)} + \sigma^2$
MS$(G*T)$	$\frac{rc}{(g-1)(t-1)}\sum_{h=1}^g \sum_{i=1}^t (\gamma\tau)^2_{hi} + \sigma^2_{TR(G)} + \sigma^2_{TC(G)} + \sigma^2$
MS$[R(G)]$	$tc\sigma^2_{R(G)} + c\sigma^2_{TR(G)} + t\sigma^2_{RC(G)} + \sigma^2$
MS$[C(G)]$	$tr\sigma^2_{C(G)} + r\sigma^2_{TC(G)} + t\sigma^2_{RC(G)} + \sigma^2$
MS$[T*R(G)]$	$c\sigma^2_{TR(G)} + \sigma^2$
MS$[T*C(G)]$	$r\sigma^2_{TC(G)} + \sigma^2$
MS$[R*C(G)]$	$t\sigma^2_{RC(G)} + \sigma^2$
MS$[T*R*C(G)]$	$\sigma^2 \equiv \sigma^2_{\tau RC(G)} + \sigma^2_\varepsilon$

   (d) Conventional model test statistic: $F = \frac{\text{MS}(T)}{\text{MS}[T*R(G)] + \text{MS}[T*C(G)] - \text{MS}[T*R*C(G)]}$
   (e) $\widetilde{\text{MS}}(T) = \frac{1}{c}\text{MS}(T)$, $\widetilde{\text{MS}}(G) = \frac{1}{c}\text{MS}(G)$, $\widetilde{\text{MS}}(G*T) = \frac{1}{c}\text{MS}(G*T)$, $\widetilde{\text{MS}}[R(G)] = \frac{1}{c}\text{MS}[R(G)]$, $\widetilde{\text{MS}}[T*R(G)] = \frac{1}{c}\text{MS}[T*R(G)]$
   (f) $F = \frac{\widetilde{\text{MS}}(T)}{\widetilde{\text{MS}}[T*R(G)] + U}$ where $U = \frac{1}{c}\left\{\text{MS}[T*C(G)] - \text{MS}[T*R*C(G)]\right\}$
   (g) $E(U) = \frac{r}{c}\sigma^2_{TC(G)} = r(\text{Cov}_2 - \text{Cov}_3)$
   (h) $F^*_{\text{OR}} = \frac{\widetilde{\text{MS}}(T)}{\widetilde{\text{MS}}[T*R(G)] + r(\text{Cov}_2 - \text{Cov}_3)}$
   (i) $F_{\text{OR}} = \frac{\widetilde{\text{MS}}(T)}{\widetilde{\text{MS}}[T*R(G)] + \max\left[r\left(\widehat{\text{Cov}}_2 - \widehat{\text{Cov}}_3\right), 0\right]}$
   (j) Under H_0, $F \,\dot\sim\, F_{t-1,\text{df}_2}$ where $\text{df}_2 = \frac{\left\{\widetilde{\text{MS}}[T*R(G)] + \max\left[r\left(\widehat{\text{Cov}}_2 - \widehat{\text{Cov}}_3\right), 0\right]\right\}^2}{\frac{\left\{\widetilde{\text{MS}}[T*R(G)]\right\}^2}{g(t-1)(r-1)}}$

3. Derive confidence intervals

   (a) Mm-ANOVA model test accuracy parameters: $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_t)'$, with: $\theta_i = E\left(\tilde{Y}_{\bullet i \bullet}\right), i = 1, \ldots, t$
   (b) Corresponding conventional ANOVA parameters: $\theta_i = E\left(Y_{\bullet i \bullet\bullet}\right) = \mu + \tau_i$

(c) Conventional ANOVA estimate: $\hat{\theta}_i = Y_{\bullet i \bullet\bullet}$

CI for $\boldsymbol{l}'(\boldsymbol{\theta})$ with $\boldsymbol{l} = (l_1, \ldots, l_t)'$, $\sum_{i=1}^t l_i = 0$:

(d) $\boldsymbol{l}'\left(\hat{\boldsymbol{\theta}}\right) = \sum_{i=1}^t l_i \hat{\boldsymbol{\theta}}_i = \sum_{i=1}^t l_i Y_{\bullet i \bullet\bullet} = \sum_{i=1}^t l_i \tau_i + \sum_{i=1}^t l_i \left[(\tau R)_{(\bullet)i\bullet} + (\tau C)_{(\bullet)i\bullet} + (\tau RC)_{(\bullet)i\bullet\bullet} + \varepsilon_{\bullet i \bullet\bullet}\right]$
$\Rightarrow V = \sum_{i=1}^t l_i^2 \left[\frac{\sigma^2_{TR(G)}}{gr} + \frac{\sigma^2_{TC(G)}}{gc} + \frac{\sigma^2}{grc}\right] = \frac{1}{grc}\sum_{i=1}^t l_i^2 \left[c\sigma^2_{TR(G)} + r\sigma^2_{TC(G)} + \sigma^2\right]$

(e) $V = \frac{1}{grc}\sum_{i=1}^t l_i^2 E\left\{\text{MS}\left[T*R(G)\right] + \text{MS}\left[T*C(G)\right] - \text{MS}\left[T*R*C(G)\right]\right\}$

(f) $V = \frac{1}{gr}\sum_{i=1}^t l_i^2 E\left[\widetilde{\text{MS}}\left[T*R(G)\right] + U\right]$ where $U = \frac{1}{c}\left\{\text{MS}\left[T*C(G)\right] - \text{MS}\left[T*R*C(G)\right]\right\}$

(g) $E(U) = \frac{r\sigma^2_{TC(G)}}{c} = r\left(\text{Cov}_2 - \text{Cov}_3\right) \Rightarrow V = \frac{1}{gr}\sum_{i=1}^t l_i^2 \left\{E\left[\widetilde{\text{MS}}\left[T*R(G)\right] + r\left(\text{Cov}_2 - \text{Cov}_3\right)\right]\right\}$

(h) $\hat{V} = \frac{1}{gr}\sum_{i=1}^t l_i^2 \left\{\widetilde{\text{MS}}\left[T*R(G)\right] + \max\left[r\left(\widehat{\text{Cov}}_2 - \widehat{\text{Cov}}_3\right), 0\right]\right\}$

(i) $\text{df}_2 = \frac{\left\{\widetilde{\text{MS}}[T*R(G)] + \max\left[r\left(\widehat{\text{Cov}}_2 - \widehat{\text{Cov}}_3\right), 0\right]\right\}^2}{\left[\widetilde{\text{MS}}[T*R(G)]\right]^2 / [g(t-1)(r-1)]}$ (same as df_2 in step 2j)

(j) $\hat{\theta}_i = \tilde{Y}_{\bullet i \bullet}$

(k) CI: $\sum_{i=1}^t l_i \tilde{Y}_{\bullet i \bullet} \pm t_{\alpha/2;\text{df}_2}\sqrt{\frac{1}{gr}\sum_{i=1}^t l_i^2 \left\{\widetilde{\text{MS}}\left[T*R(G)\right] + \max\left[r\left(\widehat{\text{Cov}}_2 - \widehat{\text{Cov}}_3\right), 0\right]\right\}}$

CI for θ_i

(d) $\hat{\theta}_i = Y_{\bullet i \bullet\bullet} = \mu + \tau_i + R_{(\bullet)\bullet} + (\tau R)_{(\bullet)i\bullet} + C_{(\bullet)\bullet} + (\tau C)_{(\bullet)i\bullet} + (RC)_{(\bullet)\bullet\bullet} + (\tau RC)_{(\bullet)i\bullet\bullet} + \varepsilon_{\bullet i \bullet\bullet} \Rightarrow$
$V = \left(\frac{\sigma^2_{R(G)}}{gr} + \frac{\sigma^2_{TR(G)}}{gr} + \frac{\sigma^2_{C(G)}}{gc} + \frac{\sigma^2_{TC(G)}}{gc} + \frac{\sigma^2_{RC(G)}}{grc} + \frac{\sigma^2}{grc}\right)$
$= \frac{1}{grc}\left(c\sigma^2_{R(G)} + c\sigma^2_{TR(G)} + r\sigma^2_{C(G)} + r\sigma^2_{TC(G)} + \sigma^2_{RC(G)} + \sigma^2\right)$

(e) $V = \frac{1}{gtrc}E\left[\begin{array}{c}\text{MS}\left[R(G)\right] + (t-1)\,\text{MS}\left[T*R(G)\right] + \text{MS}\left[C(G)\right] - \text{MS}\left[R*C(G)\right] + \\ (t-1)\,\text{MS}\left[T*C(G)\right] - (t-1)\,\text{MS}\left[T*R*C(G)\right]\end{array}\right]$

(f) $V = \frac{1}{gtr}E\left[\widetilde{\text{MS}}\left[R(G)\right] + (t-1)\widetilde{\text{MS}}\left[T*R(G)\right] + U\right]$ where
$U = \frac{1}{c}\left\{\text{MS}\left[C(G)\right] - \text{MS}\left[R*C(G)\right] + (t-1)\,\text{MS}\left[T*C(G)\right] - (t-1)\,\text{MS}\left[T*R*C(G)\right]\right\}$

(g) $E(U) = \frac{tr}{c}\left(\sigma^2_{C(G)} + \sigma^2_{TC(G)}\right) = tr\,\text{Cov}_2 \Rightarrow V = \frac{1}{gtr}E\left\{\widetilde{\text{MS}}\left[R(G)\right] + (t-1)\widetilde{\text{MS}}\left[T*R(G)\right] + tr\,\text{Cov}_2\right\}$

(h) $\hat{V} = \frac{1}{gtr}\left[\widetilde{\text{MS}}\left[R(G)\right] + (t-1)\widetilde{\text{MS}}\left[T*R(G)\right] + tr\max\left(\widehat{\text{Cov}}_2, 0\right)\right]$

(i) $\text{df}_2 = \frac{\left\{\widetilde{\text{MS}}[R(G)] + (t-1)\widetilde{\text{MS}}[T*R(G)] + tr\max\left(\widehat{\text{Cov}}_2, 0\right)\right\}^2}{\left\{\widetilde{\text{MS}}[R(G)]\right\}^2/[g(r-1)] + \left\{(t-1)\widetilde{\text{MS}}[T*R(G)]\right\}^2/[g(t-1)(r-1)]}$

(j) $\hat{\theta}_i = \tilde{Y}_{\bullet i \bullet}$

(k) CI: $\tilde{Y}_{\bullet i \bullet} \pm t_{\alpha/2;\text{df}}\sqrt{\frac{1}{gtr}\left[\widetilde{\text{MS}}\left[R(G)\right] + (t-1)\widetilde{\text{MS}}\left[T*R(G)\right] + tr\max\left(\widehat{\text{Cov}}_2, 0\right)\right]}$

4. Derive the non-null distribution $F_{\text{df}_1,\text{df}_2;\lambda}$ of the step-2 F statistic

(a) Step 2d F numerator: $\text{MS}_{\text{num}} = \text{MS}(T)$, $E\left[\text{MS}(T)\right] = \frac{grc}{(t-1)}\sum_{i=1}^t \tau_i^2 + c\sigma^2_{TR(G)} + r\sigma^2_{TC(G)} + \sigma^2$, $\text{df}\left(\text{MS}(T)\right) = t-1$, $E\left(Y_{hijk}\right) = \mu + \gamma_h + \tau_i + (\gamma\tau)_{hi} \Rightarrow \lambda = \frac{\text{df(MS)}\,\text{MS}|_{\mathbf{Y}=E(\mathbf{Y})}}{E(\text{MS}\,|\text{H}_0)} = \frac{grc\sum_{i=1}^t \tau_i^2}{c\sigma^2_{TR(G)} + r\sigma^2_{TC(G)} + \sigma^2}$

(b) $r\sigma^2_{TC(G)} + \sigma^2 = c\left[\sigma^2_{\bar{\varepsilon}} - \text{Cov}_1 + (r-1)\left(\text{Cov}_2 - \text{Cov}_3\right)\right] \Rightarrow \lambda = \frac{gr\sum_{i=1}^t \tau_i^2}{\sigma^2_{TR(G)} + \sigma^2_{\bar{\varepsilon}} - \text{Cov}_1 + (r-1)(\text{Cov}_2 - \text{Cov}_3)}$

(c) Step 2h F^*_{OR} denominator $=$ $\widetilde{\text{MS}}\left[T*R(G)\right] + r\left(\text{Cov}_2 - \text{Cov}_3\right)$, $E\left(\widetilde{\text{MS}}\left[T*R(G)\right]\right) = \frac{1}{c}E\left(\text{MS}\left[T*R(G)\right]\right) = \frac{1}{c}\left(c\sigma^2_{TR(G)} + \sigma^2\right) = \left(\sigma^2_{TR(G)} + \sigma^2_{\bar{\varepsilon}} - \text{Cov}_1 - \text{Cov}_2 + \text{Cov}_3\right) \Rightarrow \text{df}_2 = \frac{\left[\sigma^2_{TR(G)} + \sigma^2_{\bar{\varepsilon}} - \text{Cov}_1 + (r-1)(\text{Cov}_2 - \text{Cov}_3)\right]^2}{\frac{\left[\sigma^2_{TR(G)} + \sigma^2_{\bar{\varepsilon}} - \text{Cov}_1 - \text{Cov}_2 + \text{Cov}_3\right]^2}{g(t-1)(r-1)}}$

(d) $F_{\text{OR}} \dot{\sim} F_{t-1,\text{df}_2;\lambda}$

Table S6. Mm-ANOVA approach for replicated test×reader×case factorial study design

1. Derive the mm-ANOVA model

   (a) Conventional ANOVA model: $Y_{ijkm} = \mu + \tau_i + R_j + C_k + (\tau R)_{ij} + (\tau C)_{ik} + (RC)_{jk} + (\tau RC)_{ijk} + \varepsilon_{ijkm},\ i = 1, \dots, t; j = 1, \dots, r; k = 1, \dots c, m = 1, \dots, n$ with variance components $\sigma^2_R,\ \sigma^2_C,\ \sigma^2_{TR},\ \sigma^2_{TC},\ \sigma^2_{RC},\ \sigma^2_{\tau RC}$, and σ^2_ε and constraint $\sum_{i=1}^t \tau_i = 0$.
   (b) Mm-ANOVA model (note: $\tilde{Y}_{ijm} = Y_{ij\bullet m}$):
   $\tilde{Y}_{ijm} = \mu + \tau_i + R_j + (\tau R)_{ij} + \tilde{\varepsilon}_{ijm}$ where $\tilde{\varepsilon}_{ijm} = C_\bullet + (\tau C)_{i\bullet} + (RC)_{j\bullet} + (\tau RC)_{ij\bullet} + \varepsilon_{ij\bullet m}$ and $\sum_{i=1}^t \tau_i = 0$
   (c) Mm-ANOVA error variance and covariances expressed in terms of conventional ANOVA variance-components: $\widetilde{\sigma}^2_\varepsilon = \frac{1}{c}\left(\sigma^2_C + \sigma^2_{TC} + \sigma^2_{RC} + \sigma^2_{TRC} + \sigma^2_\varepsilon\right)$, $\text{Cov}_1 \equiv \text{cov}(\tilde{\varepsilon}_{ijm}, \tilde{\varepsilon}_{i'jm'}) = \frac{1}{c}\left(\sigma^2_C + \sigma^2_{RC}\right)$, $\text{Cov}_2 \equiv \text{cov}\left(\tilde{\varepsilon}_{ijm}, \tilde{\varepsilon}'_{ij'm'}\right) = \frac{1}{c}\left(\sigma^2_C + \sigma^2_{TC}\right)$, $\text{Cov}_3 \equiv \text{cov}(\tilde{\varepsilon}_{ijm}, \tilde{\varepsilon}_{i'j'm'}) = \frac{1}{c}\sigma^2_C$, where $i \neq i', j \neq j', m = m'$ or $m \neq m'$; $\text{Cov}_0 \equiv \text{cov}\left(\tilde{\varepsilon}_{ijm}, \tilde{\varepsilon}_{ijm'}\right) = \frac{1}{c}\left(\sigma^2_C + \sigma^2_{TC} + \sigma^2_{RC} + \sigma^2_{TRC}\right), m \neq m'$
   (d) Covariance constraints: $\text{Cov}_0 \geq \text{Cov}_1 \geq \text{Cov}_3$; $\text{Cov}_0 \geq \text{Cov}_2 \geq \text{Cov}_3$; $\text{Cov}_3 \geq 0$

2. Derive the mm-ANOVA test statistic and its null distribution

   (a) Mm-ANOVA model hypothesis of equal test accuracies: $\text{H}_0 : \theta_1 = \cdots = \theta_t$ where $\theta_i = E\left(\tilde{Y}_{i\bullet\bullet}\right)$
   (b) Conventional ANOVA model hypothesis: $\theta_i = E\left(Y_{i\bullet\bullet\bullet}\right) = \mu + \tau_i \implies \text{H}_0 : \tau_1 = \cdots = \tau_t$
   (c) Conventional ANOVA expected mean squares

Mean square	Expected mean square
MS(T)	$\frac{nrc}{(t-1)}\sum_{i=1}^t \tau_i^2 + nc\sigma^2_{TR} + nr\sigma^2_{TC} + n\sigma^2_{TRC} + \sigma^2_\varepsilon$
MS(R)	$ntc\sigma^2_R + nc\sigma^2_{TR} + nt\sigma^2_{RC} + n\sigma^2_{TRC} + \sigma^2_\varepsilon$
MS(C)	$ntr\sigma^2_C + nr\sigma^2_{TC} + nt\sigma^2_{RC} + n\sigma^2_{TRC} + \sigma^2_\varepsilon$
MS($T*R$)	$nc\sigma^2_{TR} + n\sigma^2_{TRC} + \sigma^2_\varepsilon$
MS($T*C$)	$nr\sigma^2_{TC} + n\sigma^2_{TRC} + \sigma^2_\varepsilon$
MS($R*C$)	$nt\sigma^2_{RC} + n\sigma^2_{TRC} + \sigma^2_\varepsilon$
MS($T*R*C$)	$n\sigma^2_{TRC} + \sigma^2_\varepsilon$
MS(error)	σ^2_ε

   (d) Conventional ANOVA test statistic: $F = \frac{\text{MS}(T)}{\text{MS}(T*R)+\text{MS}(T*C)-\text{MS}(T*R*C)}$
   (e) $\widetilde{\text{MS}}\,(T) = \frac{1}{c}\text{MS}(T)$, $\widetilde{\text{MS}}\,(T*R) = \frac{1}{c}\text{MS}(T*R)$, $\widetilde{\text{MS}}\,(R) = \frac{1}{c}\text{MS}(R)$
   (f) $F = \frac{\widetilde{\text{MS}}(T)}{\widetilde{\text{MS}}(T*R)+U}$ where $U = \frac{1}{c}\left\{\text{MS}\,(T*C) - \text{MS}\,(T*R*C)\right\}$
   (g) $E\left\{\text{MS}\,(T*C)\right\} = nr\sigma^2_{TC} + n\sigma^2_{TRC} + \sigma^2_\varepsilon,\ E\left\{\text{MS}\,[T*R*C]\right\} = n\sigma^2_{TRC} + \sigma^2_\varepsilon \Longrightarrow E\,(U) = \frac{1}{c}\left(nr\sigma^2_{TC}\right) = nr\left(\text{Cov}_2 - \text{Cov}_3\right)$.
   (h) $F^*_{\text{OR}} = \frac{\widetilde{\text{MS}}(T)}{\widetilde{\text{MS}}(T*R)+nr(\text{Cov}_2-\text{Cov}_3)}$
   (i) $F_{\text{OR}} = \frac{\widetilde{\text{MS}}(T)}{\widetilde{\text{MS}}(T*R)+nr\max\left(\widehat{\text{Cov}}_2-\widehat{\text{Cov}}_3,0\right)}$
   (j) Under H_0, $F_{\text{OR}} \mathrel{\dot\sim} F_{t-1,\text{df}_2}$ where $\text{df}_2 = \frac{\left[\widetilde{\text{MS}}[T*R]+nr\max\left(\widehat{\text{Cov}}_2-\widehat{\text{Cov}}_3,0\right)\right]^2}{\left[\widetilde{\text{MS}}[T*R]\right]^2/[(t-1)(r-1)]}$

3. Derive confidence intervals

   (a) Mm-ANOVA test accuracy parameters: $\boldsymbol{\theta} = (\theta_1, \dots, \theta_t)'$, with $\theta_i = E\left(\tilde{Y}_{i\bullet\bullet}\right), i = 1, \dots, t$
   (b) Corresponding conventional ANOVA parameters: $\theta_i = E\left(Y_{i\bullet\bullet\bullet}\right) = \mu + \tau_i$
   (c) Conventional ANOVA estimate: $\hat{\theta}_i = Y_{i\bullet\bullet\bullet}$

   CI for $\boldsymbol{l}'\left(\boldsymbol{\theta}\right)$ with $\boldsymbol{l} = (l_1, \dots, l_t)'$, $\sum_{i=1}^t l_i = 0$:
   (d) $\boldsymbol{l}'\left(\hat{\boldsymbol{\theta}}\right) = \sum_{i=1}^t l_i\hat{\boldsymbol{\theta}}_i = \sum_{i=1}^t l_i Y_{i\bullet\bullet\bullet} = \sum_{i=1}^t l_i\tau_i + \sum_{i=1}^t l_i\left[(\tau R)_{i\bullet\bullet} + (\tau C)_{i\bullet\bullet} + (\tau RC)_{i\bullet\bullet} + \varepsilon_{i\bullet\bullet\bullet}\right]$
   $\Rightarrow V = \sum_{i=1}^t l_i^2\left[\frac{\sigma^2_{TR}}{r} + \frac{\sigma^2_{TC}}{c} + \frac{\sigma^2_{TRC}}{rc} + \frac{\sigma^2_\varepsilon}{nrc}\right] = \frac{1}{rc}\sum_{i=1}^t l_i^2\left[c\sigma^2_{TR} + r\sigma^2_{TC} + \sigma^2_{TRC} + \frac{\sigma^2_\varepsilon}{n}\right]$

(e) $V = \frac{1}{nrc}\left(\sum_{i=1}^{t} l_i^2\right) E\left[\text{MS}(T*R) + \text{MS}(T*C) - \text{MS}(T*R*C)\right]$

(f) $V = \frac{1}{nr}\left(\sum_{i=1}^{t} l_i^2\right) E\left[\widetilde{\text{MS}}(T*R) + U\right]$ where $U = \frac{1}{c}\left\{\text{MS}(T*C) - \text{MS}(T*R*C)\right\}$

(g) $E(U) = \frac{nr\sigma_{TC}^2}{c} = nr(\text{Cov}_2 - \text{Cov}_3) \Rightarrow V = \frac{1}{nr}\left(\sum_{i=1}^{t} l_i^2\right)\left\{E\left[\widetilde{\text{MS}}(T*R)\right] + nr(\text{Cov}_2 - \text{Cov}_3)\right\}$

(h) $\hat{V} = \frac{1}{nr}\left(\sum_{i=1}^{t} l_i^2\right)\left\{\widetilde{\text{MS}}(T*R) + \max\left[nr\left(\widehat{\text{Cov}}_2 - \widehat{\text{Cov}}_3\right), 0\right]\right\}$

(i) $\text{df}_2 = \frac{\left[\widetilde{\text{MS}}(T*R) + nr\max\left(\widehat{\text{Cov}}_2 - \widehat{\text{Cov}}_3, 0\right)\right]^2}{\left[\widetilde{\text{MS}}(T*R)\right]^2/[(t-1)(r-1)]}$ (same as df_2 in step 2j)

(j) $\hat{\theta}_i = \tilde{Y}_{i\bullet\bullet}$

(k) CI: $\sum_{i=1}^{t} l_i \tilde{Y}_{i\bullet\bullet} \pm t_{\alpha/2;\text{df}_2}\sqrt{\frac{1}{nr}\left(\sum_{i=1}^{t} l_i^2\right)\left\{\widetilde{\text{MS}}(T*R) + \max\left[nr\left(\widehat{\text{Cov}}_2 - \widehat{\text{Cov}}_3\right), 0\right]\right\}}$

CI for θ_i

(d) $\hat{\theta}_i = Y_{i\bullet\bullet\bullet} = \tau_i + R_\bullet + C_\bullet + (\tau R)_{i\bullet} + (\tau C)_{i\bullet} + (RC)_{\bullet\bullet} + (\tau RC)_{i\bullet\bullet} + \varepsilon_{i\bullet\bullet\bullet}$
$\Rightarrow V = \frac{\sigma_R^2}{r} + \frac{\sigma_C^2}{c} + \frac{\sigma_{TR}^2}{r} + \frac{\sigma_{TC}^2}{c} + \frac{\sigma_{RC}^2}{rc} + \frac{\sigma_{TRC}^2}{rc} + \frac{\sigma_\varepsilon^2}{nrc} = \frac{1}{rc}\left(c\sigma_R^2 + r\sigma_C^2 + c\sigma_{TR}^2 + r\sigma_{TC}^2 + \sigma_{RC}^2 + \sigma_{TRC}^2 + \frac{\sigma_\varepsilon^2}{n}\right)$

(e) $V = \frac{1}{ntrc}E[\text{MS}(R) + (t-1)\text{MS}(T*R) + \text{MS}(C) - \text{MS}(R*C)$
$+ (t-1)\text{MS}(T*C) - (t-1)\text{MS}(T*R*C)]$

(f) $V = \frac{1}{ntr}E\left[\widetilde{\text{MS}}(R) + (t-1)\widetilde{\text{MS}}(T*R) + U\right]$ where
$U = \frac{1}{c}\left\{\text{MS}(C) - \text{MS}(R*C) + (t-1)\text{MS}(T*C) - (t-1)\text{MS}(T*R*C)\right\}$

(g) $E(U) = \frac{ntr}{c}\left(\sigma_C^2 + \sigma_{\tau C}^2\right) = ntr\,\text{Cov}_2 \Rightarrow V = \frac{1}{ntr}\left\{E\left[\widetilde{\text{MS}}(R) + (t-1)\widetilde{\text{MS}}(T*R)\right] + ntr\,\text{Cov}_2\right\}$

(h) $\hat{V} = \frac{1}{ntr}\left[\widetilde{\text{MS}}(R) + (t-1)\widetilde{\text{MS}}(T*R) + ntr\max\left(\widehat{\text{Cov}}_2, 0\right)\right]$

(i) $\text{df}_2 = \frac{\left[\widetilde{\text{MS}}(R) + (t-1)\widetilde{\text{MS}}(T*R) + ntr\max\left(\widehat{\text{Cov}}_2, 0\right)\right]^2}{\left[\widetilde{\text{MS}}(R)\right]^2/(r-1) + \left[(t-1)\widetilde{\text{MS}}(T*R)\right]^2/[(t-1)(r-1)]}$

(j) $\hat{\theta}_i = \tilde{Y}_{i\bullet\bullet}$

(k) CI: $\tilde{Y}_{i\bullet\bullet} \pm t_{\alpha/2;\text{df}_2}\sqrt{\frac{1}{ntr}\left[\widetilde{\text{MS}}(R) + (t-1)\widetilde{\text{MS}}(T*R) + ntr\max\left(\widehat{\text{Cov}}_2, 0\right)\right]}$

4. Derive the non-null distribution $F_{\text{df}_1,\text{df}_2;\lambda}$ of the step-2 F statistic

(a) Step 2d F numerator: $\text{MS}_{\text{num}} = \text{MS}(T)$, $E[\text{MS}(T)] = \frac{nrc}{(t-1)}\sum_{i=1}^{t}\tau_i^2 + nc\sigma_{TR}^2 + nr\sigma_{TC}^2 + n\sigma_{TRC}^2 + \sigma_\varepsilon^2$,
$\text{df}[\text{MS}(T)] = t-1$, $E(Y_{ijkm}) = \mu + \tau_i \Rightarrow \lambda = \frac{\text{df}(\text{MS}_{\text{num}})\,\text{MS}_{\text{num}}|_{\mathbf{Y}=E(\mathbf{Y})}}{E(\text{MS}_{\text{num}}\,|\text{H}_0)} = \frac{nrc\sum_{i=1}^{t}\tau_i^2}{nc\sigma_{TR}^2 + nr\sigma_{TC}^2 + n\sigma_{TRC}^2 + \sigma_\varepsilon^2}$

(b) $nr\sigma_{TC}^2 + n\sigma_{TRC}^2 + \sigma_\varepsilon^2 = nc\left\{-\text{Cov}_1 + (r-1)(\text{Cov}_2 - \text{Cov}_3) + \sigma_{\tilde{\varepsilon}}^2/n + [(n-1)/n]\,\text{Cov}_0\right\} \Rightarrow \lambda =$
$\frac{r\sum_{i=1}^{t}\tau_i^2}{\sigma_{TR}^2 - \text{Cov}_1 + (r-1)(\text{Cov}_2 - \text{Cov}_3) + \sigma_{\tilde{\varepsilon}}^2/n + [(n-1)/n]\,\text{Cov}_0}$

(c) Step 2h F_{OR}^* denominator $= \widetilde{\text{MS}}(T*R) + nr(\text{Cov}_2 - \text{Cov}_3)$, $E\left(\widetilde{\text{MS}}(T*R)\right) = \frac{1}{c}E(\text{MS}(T*R)) =$
$\frac{1}{c}\left(nc\sigma_{TR}^2 + n\sigma_{TRC}^2 + \sigma_\varepsilon^2\right) = n\sigma_{TR}^2 + \sigma_{\tilde{\varepsilon}}^2 - n\text{Cov}_1 - n\text{Cov}_2 + n\text{Cov}_3 + (n-1)\,\text{Cov}_0 \qquad \Rightarrow \text{df}_2 =$
$\frac{\left[\sigma_{TR}^2 + \sigma_{\tilde{\varepsilon}}^2/n - \text{Cov}_1 + (r-1)(\text{Cov}_2 - \text{Cov}_3) + [(n-1)/n]\,\text{Cov}_0\right]^2}{\left[\sigma_{TR}^2 + \sigma_{\tilde{\varepsilon}}^2/n - \text{Cov}_1 - (\text{Cov}_2 - \text{Cov}_3) + [(n-1)/n]\,\text{Cov}_0\right]^2/[(t-1)(r-1)]}$

(d) $F_{\text{OR}} \mathrel{\dot\sim} F_{t-1,\text{df}_2;\lambda}$