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SUPPLEMENT TO: A REFERENCE-INVARIANT HEALTH DISPARITY INDEX BASED ON RÉNYI DIVERGENCE—TECHNICAL APPENDIX

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Decomposability is the separation of the total or aggregate health disparity index (HDI) into between- and within-group components. As in the generalized entropy class of HDIs, decomposability allows for multiple predictors of individual-level disparities to be analyzed in succession. In this technical appendix, we examine the decomposition of the total Rényi (RI) and symmetrized Rényi (SRI) indices when groups are population-weighted—which is consistent with individuals being equally-weighted—as well as when groups are equally-weighted. In the latter case, only a weak decomposition of the aggregate RI and SRI holds. The derivation of the designed-based standard errors for the between-group RI and SRI is shown in Talih (2012a). This technical appendix contains the corresponding derivations for the total or aggregate RI and SRI as well as their within-group components.

1. Group-level vs. individual-level disparities. Consider individual-level disparities r_{ij} in the health outcome y_{ij}, given by $r_{ij} \propto y_{ij}$, $i = 1, 2, \cdots, n_j$. Typically, these are measured relative to the population average, $r_{ij} = y_{ij}/\bar{y}_{\cdot\cdot}$, although one might compare an individual's health outcome y_{ij} to another reference point. Following Talih (2012a), the generalized Rényi divergence between the p_{ij} and the $q_{ij} := p_{ij} r_{ij}$, $\alpha \neq 0, 1$, is given by

$$(1.1)\quad R_\alpha(p||q) = \frac{1}{\alpha(1-\alpha)} \ln\left\{ \frac{\left(\sum_{j=1}^m \sum_{i=1}^{n_j} p_{ij}\right)^\alpha \left(\sum_{j=1}^m \sum_{i=1}^{n_j} p_{ij} r_{ij}\right)^{1-\alpha}}{\sum_{j=1}^m \sum_{i=1}^{n_j} p_{ij} r_{ij}^{1-\alpha}} \right\}.$$

We refer to the resulting health disparity index (HDI) as the aggregate or total Rényi index (RI), because it measures disparities at the individual rather

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than the group level. Of course, the aggregate or total RI makes sense only when the individual's outcome y_{ij} is a positive scalar quantity,—e.g., total blood cholesterol level; see the supplemental case-study in Talih (2012b)—as opposed to a binary outcome variable—e.g., indicator of moderate or severe periodontitis; see the case-study in Talih (2012a). In this appendix, we use the notation $[\mathrm{RI}]_{\mathrm{B}}$ for the between-group RI and $[\mathrm{RI}]_{\mathrm{T}}$ for the aggregate or total RI.

Total and aggregate HDIs. When groups are population-weighted, $p_j = n_j/n$, the RI and symmetrized RI (SRI) are perfectly decomposable: as in the generalized entropy (GE) class, this allows for a 'top-down' separation of total inequality into successively nested between-group and within-group components, as seen in section 1.1, below. Thus, multiple predictors of individual-level inequality may be analyzed in succession, just as in multiple regression and analysis of variance. When groups are equally-weighted ($p_j = 1/m$), we see in section 1.2 that a weak form of decomposability also holds. However, only an aggregate 'bottom-up' HDI can be derived in that case. Decomposability and aggregation are important properties for summary measures of health inequality; see, for example, Asada (2010), Pradhan, Sahn and Younger (2003), Shorrocks (1984), and Bourguignon (1979).

1.1. *Population-weighted groups: $p_j = n_j/n$—equally-weighted individuals: $p_{ij} = 1/n$.* With $p_{ij} = 1/n$, $r_{ij} \propto y_{ij}$ in (1.1), and $[\mathrm{RI}_\alpha]_{\mathrm{T}}$ denoting the total HDI, we have:

$$[\mathrm{RI}_\alpha]_{\mathrm{T}} = \frac{1}{\alpha(1-\alpha)} \ln\left\{ \frac{n\bar{y}_{\cdot\cdot}^{1-\alpha}}{\sum_{j=1}^{m}\sum_{i=1}^{n_j} y_{ij}^{1-\alpha}} \right\} \tag{1.2}$$

It follows that

$$\begin{aligned}[\mathrm{RI}_\alpha]_{\mathrm{T}} &= \frac{1}{\alpha(1-\alpha)} \ln\left\{ \frac{n^\alpha\left(\sum_{j=1}^{m} n_j\bar{y}_{\cdot j}\right)^{1-\alpha}}{\sum_{j=1}^{m} n_j \bar{y}_{\cdot j}^{1-\alpha}} \right\} + \frac{1}{\alpha(1-\alpha)} \ln\left\{ \frac{\sum_{j=1}^{m} n_j \bar{y}_{\cdot j}^{1-\alpha}}{\sum_{j=1}^{m}\sum_{i=1}^{n_j} y_{ij}^{1-\alpha}} \right\} \\ &= \qquad [\mathrm{RI}_\alpha]_{\mathrm{B}} \qquad + \qquad [\mathrm{RI}_\alpha]_{\mathrm{W}}.\end{aligned} \tag{1.3}$$

The decomposition of the total SRI, $[\mathrm{SRI}_\alpha]_{\mathrm{T}}$, is derived in a similar manner. The between-group component $[\mathrm{RI}_\alpha]_{\mathrm{B}}$ of the total RI in the above decomposition is the population-weighted between-group RI of Talih (2012a). The within-group component $[\mathrm{RI}_\alpha]_{\mathrm{W}}$ compares, in the aggregate, the two measures of disparities in health outcome as assigned to the individual in a given group, namely $r_{ij} \propto y_{ij}$ vs. $r^*_{ij} \equiv r_j \propto \bar{y}_{\cdot j}$. For groups j where all the individual health outcomes y_{ij} are close together (and, therefore, close to the

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group average health outcome $\bar{y}_{\cdot j}$), the contributions to the sum in either the numerator or the denominator in the argument of the logarithm in $[\mathrm{RI}_\alpha]_{\mathrm{W}}$ will be comparable. Thus, positive values of the within-group index $[\mathrm{RI}_\alpha]_{\mathrm{W}}$ will capture large spreads in the health outcome y_{ij} within any given group j around the group mean $\bar{y}_{\cdot j}$.

Decomposition of the within-group component. Consider the group-specific indices

$$[\mathrm{RI}_\alpha]_{\mathrm{T},j} = \frac{1}{\alpha(1-\alpha)} \ln\left\{ \frac{n_j^\alpha \left(\sum_{i=1}^{n_j} y_{ij}\right)^{1-\alpha}}{\sum_{i=1}^{n_j} y_{ij}^{1-\alpha}} \right\}.$$

Note that the limiting cases of the group-specific indices are given by:

$$[\mathrm{RI}_1]_{\mathrm{T},j} = \lim_{\alpha\to 1} [\mathrm{RI}_\alpha]_{\mathrm{T},j} = -\frac{1}{n_j} \sum_{j=1}^{n_j} \ln y_{ij} + \ln \bar{y}_{\cdot j}$$

$$[\mathrm{RI}_0]_{\mathrm{T},j} = \lim_{\alpha\to 0} [\mathrm{RI}_\alpha]_{\mathrm{T},j} = \frac{1}{n_j \bar{y}_{\cdot j}} \sum_{j=1}^{n_j} y_{ij} \ln y_{ij} - \ln \bar{y}_{\cdot j}.$$

The within-group component $[\mathrm{RI}_\alpha]_{\mathrm{W}}$ can be obtained from the $[\mathrm{RI}_\alpha]_{\mathrm{T},j}$ as follows:

$$e^{\alpha(1-\alpha)[\mathrm{RI}_\alpha]_{\mathrm{W}}} = \sum_{j=1}^{m} \omega_j e^{\alpha(1-\alpha)[\mathrm{RI}_\alpha]_{\mathrm{T},j}},$$

with weights ω_j given by

$$\omega_j := \frac{\sum_{i=1}^{n_j} y_{ij}^{1-\alpha}}{\sum_{k=1}^{m} \sum_{i=1}^{n_k} y_{ik}^{1-\alpha}}.$$

This enables a recursive implementation of the RI when nested partitions are considered.

Limiting expressions for the within-group component. The limiting expressions for the within-group component $[\mathrm{RI}_\alpha]_{\mathrm{W}}$ of the population-weighted RI for $\alpha \to 1$ and $\alpha \to 0$ are:

$$[\mathrm{RI}_1]_{\mathrm{W}} = \frac{1}{n} \sum_{j=1}^{m} n_j \ln \bar{y}_{\cdot j} - \frac{1}{n} \sum_{j=1}^{m} \sum_{i=1}^{n_j} \ln y_{ij} = \frac{1}{n} \sum_{j=1}^{m} n_j [\mathrm{RI}_1]_{\mathrm{T},j}$$

$$[\mathrm{RI}_0]_{\mathrm{W}} = -\frac{1}{n\bar{y}_{\cdot\cdot}} \left\{ \sum_{j=1}^{m} n_j \bar{y}_{\cdot j} \ln \bar{y}_{\cdot j} - \sum_{j=1}^{m} \sum_{i=1}^{n_j} y_{ij} \ln y_{ij} \right\} = \frac{1}{n\bar{y}_{\cdot\cdot}} \sum_{j=1}^{m} n_j \bar{y}_{\cdot j} [\mathrm{RI}_0]_{\mathrm{T},j}.$$

Those are the expressions of the within-group components $[\mathrm{MLD}]_{\mathrm{W}}$ of the MLD and $[\mathrm{TI}]_{\mathrm{W}}$ of the TI, respectively, derived in Borrell and Talih (2011).

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The within-group component for the population-weighted SRI when $\alpha \to 1$ or 0, which is recognized as the within-group component $[\mathrm{STI}]_{\mathrm{W}}$ of the STI, easily follows; see Borrell and Talih (2011).

Limiting expressions for the total. The total population-weighted RI when $\alpha \to 1$ and $\alpha \to 0$ are obtained by taking limits in (1.2) above. Thus, $\alpha \to 1$ gives the total MLD,

$$[\mathrm{RI}_1]_{\mathrm{T}} := -\frac{1}{n}\sum_{j=1}^{m}\sum_{i=1}^{n_j} \ln y_{ij} + \ln \bar{y}_{\cdot\cdot},$$

whereas $\alpha \to 0$ gives the total TI,

$$[\mathrm{RI}_0]_{\mathrm{T}} := \frac{1}{n\bar{y}_{\cdot\cdot}}\sum_{j=1}^{m}\sum_{i=1}^{n_j} y_{ij} \ln y_{ij} - \ln \bar{y}_{\cdot\cdot}.$$

The total STI , which is the limit of the total SRI as $\alpha \to 1$ or 0, follows:

$$[\mathrm{SRI}_1]_{\mathrm{T}} := \frac{1}{2n\bar{y}_{\cdot\cdot}}\sum_{j=1}^{m}\sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{\cdot\cdot}) \ln y_{ij} =: [\mathrm{SRI}_0]_{\mathrm{T}}.$$

1.2. *Equally-weighted groups: $p_j = 1/m$—unequally-weighted individuals: $p_{ij} = 1/(mn_j)$.* The aggregate RI when groups are equally-weighted is given by

$$[\mathrm{RI}'_\alpha]_{\mathrm{T}} = \frac{1}{\alpha(1-\alpha)} \ln\left\{ \frac{m^\alpha \left(\sum_{j=1}^{m} \bar{y}_{\cdot j}\right)^{1-\alpha}}{\sum_{j=1}^{m} \frac{1}{n_j} \sum_{i=1}^{n_j} y_{ij}^{1-\alpha}} \right\}. \tag{1.4}$$

Unlike (1.2), where any partition of the population results in the same total population-weighted RI, here, the aggregate equally-weighted RI is partition-dependent. Nonetheless, a derivation similar to the above, using $p_{ij} = 1/(mn_j)$ and $r_{ij} \propto y_{ij}$ in (1.1), yields $[\mathrm{RI}'_\alpha]_{\mathrm{T}} = [\mathrm{RI}'_\alpha]_{\mathrm{B}} + [\mathrm{RI}'_\alpha]_{\mathrm{W}}$, where the between-group component $[\mathrm{RI}'_\alpha]_{\mathrm{B}}$ is the equally-weighted between-group RI of Talih (2012a) and the within-group component $[\mathrm{RI}'_\alpha]_{\mathrm{W}}$ compares the two individual-level measures of disparities $r_{ij} \propto y_{ij}$ vs. $r^*_{ij} \equiv r_j \propto \bar{y}_{\cdot j}$:

$$[\mathrm{RI}'_\alpha]_{\mathrm{W}} = \frac{1}{\alpha(1-\alpha)} \ln\left\{ \frac{\sum_{j=1}^{m} \bar{y}_{\cdot j}^{1-\alpha}}{\sum_{j=1}^{m} \frac{1}{n_j} \sum_{i=1}^{n_j} y_{ij}^{1-\alpha}} \right\}. \tag{1.5}$$

The decomposition of the aggregate equally-weighted SRI, $[\mathrm{SRI}'_\alpha]_{\mathrm{T}}$, is derived similarly.

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Weak decomposition of the within-group component. The within-group component $[\mathrm{RI}'_\alpha]_{\mathrm{W}}$ of the equally-weighted RI is obtained from the group-specific indices $[\mathrm{RI}_\alpha]_{\mathrm{T},j}$ (sic!) using

$$e^{\alpha(1-\alpha)[\mathrm{RI}'_\alpha]_{\mathrm{W}}} = \sum_{j=1}^{m} \varpi_j e^{\alpha(1-\alpha)[\mathrm{RI}_\alpha]_{\mathrm{T},j}},$$

with weights ϖ_j given by

$$\varpi_j := \frac{\frac{1}{n_j}\sum_{i=1}^{n_j} y_{ij}^{1-\alpha}}{\sum_{k=1}^{m}\frac{1}{n_k}\sum_{i=1}^{n_k} y_{ik}^{1-\alpha}}.$$

Just as with the population-weighted RI, this also enables a recursive implementation.

Limiting expressions for the within-group component. The expressions for the within-group component of the equally-weighted RI for $\alpha \to 1$ or 0 are obtained by taking limits in (1.5):

$$[\mathrm{RI}'_1]_{\mathrm{W}} = \frac{1}{m}\sum_{j=1}^{m} \ln \bar{y}_{\cdot j} - \frac{1}{m}\sum_{j=1}^{m}\tfrac{1}{n_j}\sum_{i=1}^{n_j}\ln y_{ij} = \frac{1}{m}\sum_{j=1}^{m}[\mathrm{RI}_1]_{\mathrm{T},j}$$

$$[\mathrm{RI}'_0]_{\mathrm{W}} = -\frac{1}{\sum_{k=1}^{m}\bar{y}_{\cdot k}}\left\{\sum_{j=1}^{m}\bar{y}_{\cdot j}\ln\bar{y}_{\cdot j} - \sum_{j=1}^{m}\tfrac{1}{n_j}\sum_{i=1}^{n_j} y_{ij}\ln y_{ij}\right\} = \frac{1}{\sum_{k=1}^{m}\bar{y}_{\cdot k}}\sum_{j=1}^{m}\bar{y}_{\cdot j}[\mathrm{RI}_0]_{\mathrm{T},j}.$$

Limiting expressions for the aggregate. The expressions for the aggregate equally-weighted RI when $\alpha \to 1$ and $\alpha \to 0$ are obtained by taking limits in (1.4). Thus, $\alpha \to 1$ yields,

$$[\mathrm{RI}'_1]_{\mathrm{T}} := -\frac{1}{m}\sum_{j=1}^{m}\tfrac{1}{n_j}\sum_{i=1}^{n_j}\ln y_{ij} + \ln\left(\sum_{j=1}^{m}\bar{y}_{\cdot j}\right) - \ln m,$$

whereas $\alpha \to 0$ yields,

$$[\mathrm{RI}'_0]_{\mathrm{T}} := \frac{1}{\sum_{k=1}^{m}\bar{y}_{\cdot k}}\sum_{j=1}^{m}\tfrac{1}{n_j}\sum_{i=1}^{n_j} y_{ij}\ln y_{ij} - \ln\left(\sum_{j=1}^{m}\bar{y}_{\cdot j}\right) + \ln m.$$

It follows that the aggregate equally-weighted SRI is given by:

$$[\mathrm{SRI}'_1]_{\mathrm{T}} := \frac{1}{2\sum_{k=1}^{m}\bar{y}_{\cdot k}}\sum_{j=1}^{m}\tfrac{1}{n_j}\sum_{i=1}^{n_j}\left(y_{ij} - \tfrac{1}{m}\sum_{k=1}^{m}\bar{y}_{\cdot k}\right)\ln y_{ij} =: [\mathrm{SRI}'_0]_{\mathrm{T}}.$$

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2. Variance calculations.

2.1. *Total RI and SRI and their within-group components—population-weighted groups.*

Total. Consider the total population-weighted RI, which is given in Talih (2012a) by

$$[\mathrm{RI}_\alpha]_\mathrm{T} = \frac{1}{\alpha(1-\alpha)} \ln\left\{ \frac{n\bar{y}_{\cdot\cdot}^{1-\alpha}}{\sum_{j=1}^{m}\sum_{i=1}^{n_j} y_{ij}^{1-\alpha}} \right\}.$$

For any real number a, Talih (2012a) defines:

$$U_{a,j} = \sum_{s=1}^{S}\sum_{c=1}^{C_s}\sum_{i=1}^{l_{cs}} \delta_{icsj} w_{ics} y_{ics}^{a} \tag{2.1}$$

$$U_{a,\cdot} = \sum_{j=1}^{m} U_{a,j}. \tag{2.2}$$

Thus, when $\alpha \neq 0, 1$, the partial derivatives with respect to $U_{0,k}$, $U_{1,k}$, $U_{\alpha,k}$, and $U_{1-\alpha,k}$ are:

$$\begin{aligned}
\frac{\partial}{\partial U_{0,k}}[\mathrm{RI}_\alpha]_\mathrm{T} &= \frac{1}{(1-\alpha)\,n} \\
\frac{\partial}{\partial U_{1,k}}[\mathrm{RI}_\alpha]_\mathrm{T} &= \frac{1}{\alpha\, n\bar{y}_{\cdot\cdot}} \\
\frac{\partial}{\partial U_{1-\alpha,k}}[\mathrm{RI}_\alpha]_\mathrm{T} &= -\frac{1}{\alpha(1-\alpha)\,U_{1-\alpha,\cdot}} \\
\frac{\partial}{\partial U_{\alpha,k}}[\mathrm{RI}_\alpha]_\mathrm{T} &= 0.
\end{aligned}$$

Note that, for $[\mathrm{RI}_{1-\alpha}]_\mathrm{T}$, the roles of α and $1-\alpha$ above are interchanged: the partial derivative with respect to $U_{1-\alpha,k}$ is zero, whereas that with respect to $U_{\alpha,k}$ is not. The σ_{icsk}, representing the variance contribution from each sample observation, are obtained from the dot product of the above vector of partial derivatives with the vector of summands in (2.1):

$$\sigma_{icsk} = \delta_{icsk} w_{ics} \left\{ \frac{\partial[\mathrm{RI}_\alpha]_\mathrm{T}}{\partial U_{0,k}} + y_{ics}\frac{\partial[\mathrm{RI}_\alpha]_\mathrm{T}}{\partial U_{1,k}} + y_{ics}^{1-\alpha}\frac{\partial[\mathrm{RI}_\alpha]_\mathrm{T}}{\partial U_{1-\alpha,k}} + y_{ics}^{\alpha}\frac{\partial[\mathrm{RI}_\alpha]_\mathrm{T}}{\partial U_{\alpha,k}} \right\}. \tag{2.3}$$

Thus, the sampling variance of $\sum_{k=1}^{m}\sum_{i=1}^{l_{cs}} \sigma_{icsk}$ provides an estimate of the sample variance of $[\mathrm{RI}_\alpha]_\mathrm{T}$. That for the SRI, with $[\mathrm{SRI}_\alpha]_\mathrm{T} = \big([\mathrm{RI}_\alpha]_\mathrm{T} + [\mathrm{RI}_{1-\alpha}]_\mathrm{T}\big)/2$, follows readily.

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Within-group component. The within-group component $[\mathrm{RI}_\alpha]_\mathrm{W}$ is given in Talih (2012a):

$$[\mathrm{RI}_\alpha]_\mathrm{W} = \frac{1}{\alpha(1-\alpha)} \ln\left\{ \frac{\sum_{j=1}^m n_j \bar{y}_{\cdot j}^{1-\alpha}}{\sum_{j=1}^m \sum_{i=1}^{n_j} y_{ij}^{1-\alpha}} \right\}.$$

When $\alpha \neq 0, 1$, the partial derivatives are:

$$\begin{aligned}
\frac{\partial}{\partial U_{0,k}}[\mathrm{RI}_\alpha]_\mathrm{W} &= \frac{\bar{y}_{\cdot k}^{1-\alpha}}{(1-\alpha)\sum_{j=1}^m n_j \bar{y}_{\cdot j}^{1-\alpha}} \\
\frac{\partial}{\partial U_{1,k}}[\mathrm{RI}_\alpha]_\mathrm{W} &= \frac{\bar{y}_{\cdot k}^{-\alpha}}{\alpha \sum_{j=1}^m n_j \bar{y}_{\cdot j}^{1-\alpha}} \\
\frac{\partial}{\partial U_{1-\alpha,k}}[\mathrm{RI}_\alpha]_\mathrm{W} &= -\frac{U_{1-\alpha,\cdot}}{\alpha(1-\alpha)} \\
\frac{\partial}{\partial U_{\alpha,k}}[\mathrm{RI}_\alpha]_\mathrm{W} &= 0.
\end{aligned}$$

The variance elements σ_{icsk} for the within-group component of the population-weighted RI are as in (2.3), only where $[\mathrm{RI}_\alpha]_\mathrm{T}$ is replaced with $[\mathrm{RI}_\alpha]_\mathrm{W}$. Those for $[\mathrm{SRI}_\alpha]_\mathrm{W}$ readily follow.

Limiting cases. The partial derivatives when $\alpha \to 1$ (MLD) and $\alpha \to 0$ (TI) were computed in Borrell and Talih (2011) and implemented in the companion R code to that article. We group them here for completeness. The variance calculations for the MLD and the TI require the introduction of the additional sufficient statistics $T_{a,j}$, given by

$$T_{a,j} = \sum_{s=1}^{S} \sum_{c=1}^{C_s} \sum_{i=1}^{l_{cs}} \delta_{icsj} w_{ics} y_{ics}^{a} \ln y_{ics}, \tag{2.4}$$

$$\text{and} \quad T_{a,\cdot} = \sum_{j=1}^{m} T_{a,j}. \tag{2.5}$$

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Thus, for $[\mathrm{RI}_1]_\mathrm{T}$, i.e., the total MLD,

$$\begin{aligned}
\frac{\partial}{\partial U_{0,k}}[\mathrm{RI}_1]_\mathrm{T} &= \frac{1}{n}\left\{\frac{T_{0,\cdot}}{n}-1\right\}\\
\frac{\partial}{\partial U_{1,k}}[\mathrm{RI}_1]_\mathrm{T} &= \frac{1}{n\bar{y}_{\cdot\cdot}}\\
\frac{\partial}{\partial T_{0,k}}[\mathrm{RI}_1]_\mathrm{T} &= -\frac{1}{n}\\
\frac{\partial}{\partial T_{1,k}}[\mathrm{RI}_1]_\mathrm{T} &= 0.
\end{aligned}$$

The σ_{icsk}, representing the variance contribution from each sample observation, are obtained by taking the dot product of the above vector of partial derivatives with the vector of summands in the sufficient statistics in (2.1) and (2.4), for $a = 0, 1$. Thus, for $[\mathrm{RI}_1]_\mathrm{T}$,

(2.6)

$$\sigma_{icsk} = \delta_{icsk} w_{ics} \times \left\{\frac{\partial[\mathrm{RI}_1]_\mathrm{T}}{\partial U_{0,k}} + y_{ics}\frac{\partial[\mathrm{RI}_1]_\mathrm{T}}{\partial U_{1,k}} + \ln y_{ics}\frac{\partial[\mathrm{RI}_1]_\mathrm{T}}{\partial T_{0,k}} + y_{ics}\ln y_{ics}\frac{\partial[\mathrm{RI}_1]_\mathrm{T}}{\partial T_{1,k}}\right\}.$$

Similarly, for $[\mathrm{RI}_0]_\mathrm{T}$, i.e., the total TI, using the following partial derivatives in (2.6) instead:

$$\begin{aligned}
\frac{\partial}{\partial U_{0,k}}[\mathrm{RI}_0]_\mathrm{T} &= \frac{1}{n}\\
\frac{\partial}{\partial U_{1,k}}[\mathrm{RI}_0]_\mathrm{T} &= -\frac{1}{n\bar{y}_{\cdot\cdot}}\left\{\frac{T_{1,\cdot}}{n\bar{y}_{\cdot\cdot}}+1\right\}\\
\frac{\partial}{\partial T_{0,k}}[\mathrm{RI}_0]_\mathrm{T} &= 0\\
\frac{\partial}{\partial T_{1,k}}[\mathrm{RI}_0]_\mathrm{T} &= \frac{1}{n\bar{y}_{\cdot\cdot}}.
\end{aligned}$$

On the other hand, for the within-group component $[\mathrm{RI}_1]_\mathrm{W}$ of the MLD:

$$\begin{aligned}
\frac{\partial}{\partial U_{0,k}}[\mathrm{RI}_1]_\mathrm{W} &= \frac{1}{n^2}\left\{T_{0,\cdot}-\sum_{j=1}^{m} n_j[1+\ln(\bar{y}_{\cdot j}/\bar{y}_{\cdot k})]\right\}\\
\frac{\partial}{\partial U_{1,k}}[\mathrm{RI}_1]_\mathrm{W} &= \frac{1}{n\bar{y}_{\cdot k}}\\
\frac{\partial}{\partial T_{0,k}}[\mathrm{RI}_1]_\mathrm{W} &= -\frac{1}{n}\\
\frac{\partial}{\partial T_{1,k}}[\mathrm{RI}_1]_\mathrm{W} &= 0.
\end{aligned}$$

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Again, the variance elements σ_{icsk} are as in (2.6), only where $[\mathrm{RI}_1]_\mathrm{T}$ is replaced with $[\mathrm{RI}_1]_\mathrm{W}$. Likewise for within-group component $[\mathrm{RI}_0]_\mathrm{W}$ of the TI, where the partial derivatives are:

$$\begin{aligned}
\frac{\partial}{\partial U_{0,k}}[\mathrm{RI}_0]_\mathrm{W} &= \frac{\bar{y}_{\cdot k}}{n\bar{y}_{\cdot\cdot}} \\
\frac{\partial}{\partial U_{1,k}}[\mathrm{RI}_0]_\mathrm{W} &= -\frac{1}{(n\bar{y}_{\cdot\cdot})^2}\left\{T_{1,\cdot} + \sum_{j=1}^{m} n_j\bar{y}_{\cdot j}[1-\ln(\bar{y}_{\cdot j}/\bar{y}_{\cdot k})]\right\} \\
\frac{\partial}{\partial T_{0,k}}[\mathrm{RI}_0]_\mathrm{W} &= 0 \\
\frac{\partial}{\partial T_{1,k}}[\mathrm{RI}_0]_\mathrm{W} &= \frac{1}{n\bar{y}_{\cdot\cdot}}.
\end{aligned}$$

2.2. *Aggregate RI and SRI and their within-group components—equally-weighted groups.*

Aggregate. The aggregate equally-weighted RI, $[\mathrm{RI}'_\alpha]_\mathrm{T}$, was given in Talih (2012a) by

$$[\mathrm{RI}'_\alpha]_\mathrm{T} = \frac{1}{\alpha(1-\alpha)}\ln\left\{\frac{m^\alpha\left(\sum_{j=1}^m \bar{y}_{\cdot j}\right)^{1-\alpha}}{\sum_{j=1}^m \frac{1}{n_j}\sum_{i=1}^{n_j} y_{ij}^{1-\alpha}}\right\}.$$

When $\alpha \neq 0, 1$, the partial derivatives with respect to $U_{0,k}$, $U_{1,k}$, $U_{\alpha,k}$, and $U_{1-\alpha,k}$ are:

$$\begin{aligned}
\frac{\partial}{\partial U_{0,k}}[\mathrm{RI}'_\alpha]_\mathrm{T} &= -\frac{\bar{y}_{\cdot k}}{\alpha\, n_k\sum_{j=1}^m \bar{y}_{\cdot j}} + \frac{\frac{1}{n_k}U_{1-\alpha,k}}{\alpha(1-\alpha)\, n_k\sum_{j=1}^m \frac{1}{n_j}U_{1-\alpha,j}} \\
\frac{\partial}{\partial U_{1,k}}[\mathrm{RI}'_\alpha]_\mathrm{T} &= \frac{1}{\alpha\, n_k\sum_{j=1}^m \bar{y}_{\cdot j}} \\
\frac{\partial}{\partial U_{1-\alpha,k}}[\mathrm{RI}'_\alpha]_\mathrm{T} &= -\frac{1}{\alpha(1-\alpha)\, n_k\sum_{j=1}^m \frac{1}{n_j}U_{1-\alpha,j}} \\
\frac{\partial}{\partial U_{\alpha,k}}[\mathrm{RI}'_\alpha]_\mathrm{T} &= 0.
\end{aligned}$$

An expression identical to (2.3) is derived, only where RI is replaced with RI'.

Within-group component. The within-group component $[\mathrm{RI}'_\alpha]_\mathrm{W}$ is given in Talih (2012a):

$$[\mathrm{RI}'_\alpha]_\mathrm{W} = \frac{1}{\alpha(1-\alpha)}\ln\left\{\frac{\sum_{j=1}^m \bar{y}_{\cdot j}^{1-\alpha}}{\sum_{j=1}^m \frac{1}{n_j}\sum_{i=1}^{n_j} y_{ij}^{1-\alpha}}\right\}.$$

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Thus, the partial derivatives when $\alpha \neq 0, 1$, are:

$$
\begin{aligned}
\frac{\partial}{\partial U_{0,k}}[\mathrm{RI}'_\alpha]_\mathrm{W} &= -\frac{\bar{y}_{\cdot k}^{1-\alpha}}{\alpha\, n_k \sum_{j=1}^m \bar{y}_{\cdot j}^{1-\alpha}} + \frac{\frac{1}{n_k} U_{1-\alpha,k}}{\alpha(1-\alpha)\, n_k \sum_{j=1}^m \frac{1}{n_j} U_{1-\alpha,j}}\\
\frac{\partial}{\partial U_{1,k}}[\mathrm{RI}'_\alpha]_\mathrm{W} &= \frac{\bar{y}_{\cdot k}^{-\alpha}}{\alpha\, n_k \sum_{j=1}^m \bar{y}_{\cdot j}^{1-\alpha}}\\
\frac{\partial}{\partial U_{1-\alpha,k}}[\mathrm{RI}'_\alpha]_\mathrm{W} &= -\frac{1}{\alpha(1-\alpha)\, n_k \sum_{j=1}^m \frac{1}{n_j} U_{1-\alpha,j}}\\
\frac{\partial}{\partial U_{\alpha,k}}[\mathrm{RI}'_\alpha]_\mathrm{W} &= 0.
\end{aligned}
$$

Again, for $[\mathrm{RI}'_{1-\alpha}]_\mathrm{W}$, the roles of α and $1-\alpha$ above are interchanged, whence the partial derivative with respect to $U_{1-\alpha,k}$ is zero whereas that with respect to $U_{\alpha,k}$ is not. The variance elements σ_{icsk} for the within-group component of the equally-weighted RI remain as in (2.3), only where $[\mathrm{RI}_\alpha]_\mathrm{T}$ is replaced with $[\mathrm{RI}'_\alpha]_\mathrm{W}$.

Limiting cases. The partial derivatives of $[\mathrm{RI}'_1]_\mathrm{T}$ are given by:

$$
\begin{aligned}
\frac{\partial}{\partial U_{0,k}}[\mathrm{RI}'_1]_\mathrm{T} &= -\frac{\bar{y}_{\cdot k}}{n_k \sum_{j=1}^m \bar{y}_{\cdot j}} + \frac{T_{0,k}}{m\, n_k^2}\\
\frac{\partial}{\partial U_{1,k}}[\mathrm{RI}'_1]_\mathrm{T} &= \frac{1}{n_k \sum_{j=1}^m \bar{y}_{\cdot j}}\\
\frac{\partial}{\partial T_{0,k}}[\mathrm{RI}'_1]_\mathrm{T} &= -\frac{1}{m\, n_k}\\
\frac{\partial}{\partial T_{1,k}}[\mathrm{RI}'_1]_\mathrm{T} &= 0.
\end{aligned}
$$

As before, the variance elements σ_{icsk} follow from (2.6), only where $[\mathrm{RI}_1]_\mathrm{T}$ is replaced with $[\mathrm{RI}'_1]_\mathrm{T}$. Similarly for $[\mathrm{RI}'_0]_\mathrm{T}$:

$$
\begin{aligned}
\frac{\partial}{\partial U_{0,k}}[\mathrm{RI}'_0]_\mathrm{T} &= \frac{\bar{y}_{\cdot k} - \frac{1}{n_k} T_{1,k}}{n_k \sum_{j=1}^m \bar{y}_{\cdot j}} + \frac{\bar{y}_{\cdot k} \sum_{j=1}^m \frac{1}{n_j} T_{1,j}}{n_k \left(\sum_{l=1}^m \bar{y}_{\cdot l}\right)^2}\\
\frac{\partial}{\partial U_{1,k}}[\mathrm{RI}'_0]_\mathrm{T} &= -\frac{1}{n_k \sum_{j=1}^m \bar{y}_{\cdot j}} - \frac{\sum_{j=1}^m \frac{1}{n_j} T_{1,j}}{n_k \left(\sum_{l=1}^m \bar{y}_{\cdot l}\right)^2}\\
\frac{\partial}{\partial T_{0,k}}[\mathrm{RI}'_0]_\mathrm{T} &= 0\\
\frac{\partial}{\partial T_{1,k}}[\mathrm{RI}'_0]_\mathrm{T} &= \frac{1}{n_k \sum_{j=1}^m \bar{y}_{\cdot j}}.
\end{aligned}
$$

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On the other hand, for the within-group component $[\mathrm{RI}'_1]_{\mathrm{W}}$:

$$\begin{aligned}
\frac{\partial}{\partial U_{0,k}}[\mathrm{RI}'_1]_{\mathrm{W}} &= \frac{T_{0,k}-n_k}{mn_k^2} \\
\frac{\partial}{\partial U_{1,k}}[\mathrm{RI}'_1]_{\mathrm{W}} &= \frac{1}{mn_k\bar{y}_{\cdot k}} \\
\frac{\partial}{\partial T_{0,k}}[\mathrm{RI}'_1]_{\mathrm{W}} &= -\frac{1}{mn_k} \\
\frac{\partial}{\partial T_{1,k}}[\mathrm{RI}'_1]_{\mathrm{W}} &= 0.
\end{aligned}$$

The variance elements σ_{icsk} follow from (2.6), only where $[\mathrm{RI}_1]_{\mathrm{T}}$ is replaced with $[\mathrm{RI}'_1]_{\mathrm{W}}$. Similarly, the partials derivatives for $[\mathrm{RI}'_0]_{\mathrm{W}}$ are:

$$\begin{aligned}
\frac{\partial}{\partial U_{0,k}}[\mathrm{RI}'_0]_{\mathrm{W}} &= -\frac{\frac{1}{n_k}T_{1,k}-\bar{y}_{\cdot k}(1+\ln\bar{y}_{\cdot k})}{n_k\sum_{l=1}^m\bar{y}_{\cdot l}} + \frac{\bar{y}_{\cdot k}}{n_k\left(\sum_{l=1}^m\bar{y}_{\cdot l}\right)^2}\left\{\sum_{j=1}^m\frac{1}{n_j}T_{1,j}-\sum_{j=1}^m\bar{y}_{\cdot j}\ln\bar{y}_{\cdot j}\right\} \\
\frac{\partial}{\partial U_{1,k}}[\mathrm{RI}'_0]_{\mathrm{W}} &= -\frac{1+\ln\bar{y}_{\cdot k}}{n_k\sum_{l=1}^m\bar{y}_{\cdot l}} - \frac{1}{n_k\left(\sum_{l=1}^m\bar{y}_{\cdot l}\right)^2}\left\{\sum_{j=1}^m\frac{1}{n_j}T_{1,j}-\sum_{j=1}^m\bar{y}_{\cdot j}\ln\bar{y}_{\cdot j}\right\} \\
\frac{\partial}{\partial T_{0,k}}[\mathrm{RI}'_0]_{\mathrm{W}} &= 0 \\
\frac{\partial}{\partial T_{1,k}}[\mathrm{RI}'_0]_{\mathrm{W}} &= \frac{1}{n_k\sum_{l=1}^m\bar{y}_{l\cdot}}.
\end{aligned}$$

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