Supporting Information and Appendices

Appendix A: Sensitivity of the stochastic growth rate

The stochastic growth rate of the perturbed population, $\bar{a}(s)$, can be calculated from the products of the perturbed matrices X_j, where at time t: $X_j(t) = X^*(t) + sH_j(t)$. Given M independent sample paths of T time steps each, for any sample path j:

$$\Lambda_j(T) = V_{j,T}^T X_j(T) X_j(T-1) \dots X_j(1) U_j(0)$$
$$a_j = \lim_{T\to\infty} \frac{\log \Lambda_j(T)}{T}$$

Now if we expand in the above product in orders of s, $O(s)$,we have:

$$\Lambda_j(s) = \Lambda_j(0) + sV_j^T(T)\left(\sum_{i=1} X_j(T)\dots H(i)X_j(i-1)\dots X_j(1)\right)U_j(0) + O(s^2)$$

Therefore,

$$\bar{a}(s) = \frac{1}{M}\sum_{j=1}^{M} \lim_{T\to\infty} \frac{1}{T} \log\left(\Lambda_j(0) + sV_j^T(T)\left(\sum_{i=1}^{T} X_j(T)\dots H_j(i)X_j(i-1)\dots X_j(1)\right)U(0) + O(s^2)\right)$$

Retaining only terms $O(s)$:

$$\bar{a}(s) = \log(\Lambda_o + \delta\Lambda) \backsimeq \log\Lambda_o + \frac{\delta\Lambda}{\Lambda_o}$$

This leads to, for any sample path:

$$\begin{aligned}
\log\Lambda(s) &= \log\Lambda_o + s\lim_{T\to\infty}\frac{1}{T}\left(\frac{(V^T(T)\sum_{i=1}^{T} X(T)X(T-1)\dots H(i)X(i-1)\dots X(1)U(0)}{V^T(T)X(T)X(T-1)\dots X(i)X(i-1)\dots X(1)U(0)}\right)\\
&= \log\Lambda_o + \lim_{T\to\infty}\frac{1}{T}\sum_{i=1}^{T}\left(\frac{(V(i)^T H(i)U_{i-1}}{V(i)^T X(i)U_{i-1}}\right)\\
&= \log\Lambda_o + \lim_{T\to\infty}\frac{1}{T}\sum_{i=1}^{T}\left(\frac{(V(i)^T H(i)U_{i-1}}{\lambda(i)V(i)^T U(i)}\right)\\
&= \log\Lambda_o + E\left[\frac{V(t)^T H(t)U(t-1)}{\lambda(t)V(t)^T U(t)}\right]
\end{aligned}$$

For sample path j, at time t, define:

$$\xi_{j,t} = \frac{V_j(t)^T H_j(t) U_j(t-1)}{\lambda_j(t) V_j(t)^T U_j(t)}$$

The mean of these for that sample path is:

$$\bar{\xi}_j = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \frac{V_j(t)^T H_j(t) U_j(t-1)}{\lambda_j(t) V_j(t)^T U_j(t)}$$

and the mean of these quantities across all runs is:

$$S_a = \bar{\xi} = \frac{1}{M} \sum_{i=j}^{M} \bar{\xi}_j$$

Appendix B: Sensitivity of the variance in long run population growth

We take the variance of the stochastic growth of our reference population to be:

$$v \backsimeq \frac{1}{MT} \sum_{j=1}^{M} (\log \Lambda_j(T) - T\bar{a})^2 \tag{12}$$

After a small perturbation, s, the value of $v(s)$ will be changed by some small value δv where $v(s) = v(0) + s\delta v + O(s^2)$. We wish to estimate the change δv. For simplicity, hereafter take $\log \Lambda_j$ to indicate $\log \Lambda_j(T)$, and retain only terms of $O(s)$.

After perturbation, the new values of $\log \Lambda_j$ will be:

$$\begin{aligned} \log \Lambda_j(s) =& \log \Lambda_j(0) + sT\bar{\xi}_j \\ \bar{a}(s) =& \bar{a}(0) + s\bar{\xi} \\ =& \bar{a}(0) + \frac{s}{M} \sum_{j=1}^{M} \bar{\xi}_j \end{aligned}$$

and

$$
\begin{aligned}
\Lambda_j &= \sum_{t=1}^{T} \lambda_j(t) \\
\bar{\xi}_j &= \frac{1}{T} \sum_{t=1}^{T} \xi_j(t)
\end{aligned}
$$

Then we can approximate the variance of the perturbed population as:

$$
\begin{aligned}
v(s) =& \frac{1}{MT} \sum_j (\log \Lambda_j(T)(s) - T\bar{a}(s))^2 \\
=& \frac{1}{MT} \sum_{j=1}^{M} \left(\left(\sum_{t=1}^{T} (\log \lambda_j(t) + s \sum_{i=t}^{T} \xi_j(t)) \right) - T \left(\bar{a}(0) + s\bar{\xi} \right) \right)^2 \\
=& \frac{1}{MT} \sum_{j=1}^{M} \left(\log \Lambda_j(0) + Ts\bar{\xi}_j) - T(\bar{a}(0) + Ts\bar{\xi}) \right)^2 \\
=& \frac{T}{MT} \sum_{j=1}^{M} \left((a_j(0) - \bar{a}(0)) + s(\bar{\xi}_j - \bar{\xi}) \right)^2 \\
=& \frac{1}{M} \sum_{j=1}^{M} (a_j(0) - a(0))^2 + 2(a_j(0) - \bar{a}(0) s(\bar{\xi}_j - \bar{\xi}) + O(s^2) \\
=& v(0) + \frac{2s}{M} \sum_{j=1}^{M} (a_j(0) - \bar{a}(0)(\bar{\xi}_j - \bar{\xi}) + O(s^2)
\end{aligned}
$$

and thus, the rate of change in the variance due to the perturbation is:

$$
S_v = \frac{2s}{M} \sum_{j=1}^{M} (a_j - \bar{a})(\bar{\xi}_j - \bar{\xi}) \tag{13}
$$

Appendix C. Probability of quasiextinction and its sensitivity

We define a population to be quasi-extinct if it falls to 1 percent of its current size. Call this quasi-extinction threshold θ. Then the probability of quasi-extinction will be (after

Caswell 2001):

$$P_q = \begin{cases} 1 & \text{if } a < 0 \\ e^{(\frac{2alog\theta}{v})} & \text{if } a > 0 \end{cases}$$

By taking the log and applying the chain rule to the above, we get the sensitivity of the log extinction probability when $a > 0$:

$$\begin{aligned} \log P_q =& 2\log\theta + \frac{a}{v} \\ S_{\log P_q} =& \frac{S_a}{v} - \frac{a}{v^2} S_v \end{aligned}$$

Since we are interested in S_{P_q} we write:

$$\frac{S_{P_q}}{P_q} = S_{\log P_q} = \frac{S_a}{v} - \frac{a}{v^2} S_v$$

and rearrange to get:

$$S_{P_q} = P_q(\frac{S_a}{v} - \frac{a}{v^2} S_v)$$

When dealing with elasticities of a and v instead of sensitivities, (recalling that $S_a = aE_a$), this becomes:

$$\begin{aligned} S_{P_q} &= P_q(\frac{aE_a}{v} - \frac{aE_v}{v}) \\ &= \frac{a}{v} P_q(E_a - E_v) \end{aligned}$$

Appendix D. Cumulative Extinction Risk

The probablity that a population will ever reach a given extinction threshold, (say, $\theta = N_e/N_o$) is $P_q = e^{(\frac{2a\log\theta}{v})}$ when $a > 0$. In practice, when a is often less than 0 and extinction is certain, it is more useful to know the probability that a population will reach the threshold before some time horizon, t. If we condition on the quasiextinction threshold eventually being reached, time to extinction (T_q) is a positive real-valued random variable with a continuous probability distribution that can be written in terms of a standard normal cdf (Lande and Orzack 1988, Dennis et al 1991):

$$P(T_q \leq t) = G(t;\theta,a,v) = \Phi\Big(\frac{\log\theta - at}{\sqrt{vt}}\Big) + e^{(\frac{2\log\theta a}{v})}\Phi\Big(\frac{\log\theta + at}{\sqrt{vt}}\Big) \tag{14}$$

where Φ is the standard normal probability integral:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2}dz$$

Now we define $P_q(t)$ to be the probability of quasiextinction, P_q, before some time horizon, t. For any given t, this probability of quasiextinction, $P_q(t)$, is the cumulative probability defined above as $P(T_q \leq t)$ (Lande and Orzack 1988, Dennis et al 1991, Morris and Doak 2002).

The sensitivities of this time-horizon-specific P_q are its derivatives with respect to some perturbation, call it α.

For now, let's also define $x = \frac{\log\theta - at}{\sqrt{vt}}$, $y = \frac{2\log\theta a}{v}$ and $z = \frac{\log\theta + at}{\sqrt{vt}}$ such that $P(T_q \leq t) = \Phi(x) + e^y\Phi(z)$

Now we take the derivative to find that:

$$\frac{dP_q(t)}{d\alpha} = \frac{d\Phi(x)}{dx}\frac{dx}{d\alpha} + \frac{dy}{d\alpha}e^y\Phi(z) + e^y\frac{d\Phi(z)}{dz}\frac{dz}{d\alpha} \tag{15}$$

Since the normal pdf is the derivative of the cdf, we can simplify:

$$\frac{dP_q(t)}{d\alpha} = \phi(x)\frac{dx}{d\alpha} + \frac{dy}{d\alpha}e^y\Phi(z) + \phi(z)\frac{dz}{d\alpha} \tag{16}$$

Now we find expressions for $dx/d\alpha$, $dy/d\alpha$ and $dz/d\alpha$:

$$\begin{aligned} x =& \frac{\log\theta - at}{\sqrt{vt}} \\ \frac{dx}{d\alpha} =& \frac{-t\frac{da}{d\alpha}(\sqrt{vt}) - \sqrt{t}\frac{d\sqrt{v}}{d\alpha}(\log\theta - at)}{vt} \\ =& -\frac{1}{\sqrt{vt}}\Big(t\frac{da}{d\alpha} + \frac{(\log\theta - at)}{2v}\frac{dv}{d\alpha}\Big) \end{aligned}$$

$$\begin{aligned} y =& \frac{2\log\theta a}{v} \\ \frac{dy}{d\alpha} =& \big((2\log\theta\frac{da}{d\alpha})v - (2\log\theta a)(\frac{dv}{d\alpha})\big)(v^{-2}) \\ =& \frac{2\log\theta}{v^2}(v\frac{da}{d\alpha} - a\frac{dv}{d\alpha}) \end{aligned}$$

$$\begin{aligned} z =& \frac{\log\theta + at}{\sqrt{vt}} \\ \frac{dz}{d\alpha} =& \frac{t\frac{da}{d\alpha}(\sqrt{vt}) - \sqrt{t}\frac{d\sqrt{v}}{d\alpha}(\log\theta + at)}{vt} \\ =& \frac{t\sqrt{vt}\frac{da}{d\alpha} - \sqrt{t}(\log\theta + at)\frac{1}{2\sqrt{v}}\frac{dv}{d\alpha}}{vt} \\ =& \sqrt{\frac{t}{v}}\frac{da}{d\alpha} - (\frac{\log\theta + at}{2v\sqrt{vt}})\frac{dv}{d\alpha} \\ =& \frac{1}{\sqrt{vt}}(t\frac{da}{d\alpha} - \frac{(\log\theta + at)}{2v}\frac{dv}{d\alpha}) \end{aligned}$$

Subbing back in our expressions for x, y, z and their derivatives, we get a general expression for the sensitivity of $P_q(t)$ to a perturbation α:

$$\begin{aligned} \frac{dP_q(t)}{d\alpha} &= \phi(\frac{\log\theta - at}{\sqrt{vt}})\frac{-1}{\sqrt{vt}}(t\frac{da}{d\alpha} + \frac{(\log\theta - at)}{2v}\frac{dv}{d\alpha}) \\ &+ e^{\frac{2\log\theta a}{v}}\Big(\frac{2\log\theta}{v^2}(v\frac{da}{d\alpha} - a\frac{dv}{d\alpha})\Phi(\frac{\log\theta + at}{\sqrt{vt}})\Big) + \frac{1}{\sqrt{vt}}(t\frac{da}{d\alpha} - \frac{(\log\theta + at)}{2v}\frac{dv}{d\alpha})\phi(\frac{\log\theta + at}{\sqrt{vt}}) \end{aligned}$$

Note that terms $\frac{da}{d\alpha}$ and $\frac{dv}{d\alpha}$ are the sensitivities of a and v (S_a and S_v) to the same perturbation. A change of notation clarifies our final expression for the sensitivity of cumulative extinction probability:

$$\begin{aligned} S_{P_q}(t) &= \frac{-1}{\sqrt{vt}}\phi(\frac{\log\theta - at}{\sqrt{vt}})\big(tS_a + \frac{(\log\theta - at)}{2v}S_v\big) \\ &+ e^{\frac{2\log\theta a}{v}}\Big(\frac{2\log\theta}{v^2}(vS_a - aS_v)\Phi\big(\frac{\log\theta + at}{\sqrt{vt}}\big)\Big) + \frac{1}{\sqrt{vt}}\big(tS_a - \frac{(\log\theta + at)}{2v}S_v\big)\phi\big(\frac{\log\theta + at}{\sqrt{vt}}\big) \end{aligned} \quad (17)$$

Appendix E. Supplementary Figures.

Figure E: Upper quantile of stationary stage distributions

Figure F: Probability of and expected time to quasiextinction as a function of q

Figure G: Stochastic growth rate and its variance as a function of q