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Supplementary material for 'On random-effects meta-analysis'

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EXISTENCE OF THE MAXIMUM LIKELIHOOD ESTIMATORS

We will show that with probability tending to 1, there exists a local maximizer of $\sum_{k=1}^{K} l_k(\beta, \sigma^2, \eta_k)$ in the neighbourhood $\mathcal{N}$. The proof consists of four main steps.

Step 1. We use a Laplace approximation for the integral in $l_k(\beta, \sigma^2, \eta_k)$. For each choice of $(\beta, \sigma^2, \eta_k)$ in $\mathcal{N}$, the Taylor series expansion at $\xi = 0$ yields

$$\begin{aligned} &\mathcal{P}_{nk}\{-\log f_k(\mathcal{O}; \beta + \xi_k, \eta_k)\} + \frac{n\xi_k^2}{2n_k\sigma^2} \\ &\quad = \mathcal{P}_{nk}\{-\log f_k(\mathcal{O}; \beta, \eta_k)\} + \mathcal{P}_{nk}\{-\partial \log f_k(\mathcal{O}; \beta, \eta_k)/\partial\beta\}\xi_k \\ &\qquad + \frac{1}{2}\left[\mathcal{P}_{nk}\{-\partial^2 \log f_k(\mathcal{O}; \beta + \tilde{\xi}_k, \eta_k)/\partial\beta^2\} + \frac{n}{2n_k\sigma^2}\right]\xi_k^2 \end{aligned}$$

for some $\tilde{\xi}_k$ between 0 and ξ_k. By Conditions 2 and 4,

$$\begin{aligned} &\mathcal{P}_{nk}\{-\log f_k(\mathcal{O}; \beta + \xi_k, \eta_k)\} + \frac{n\xi_k^2}{2n_k\sigma^2} \\ &\quad > \mathcal{P}_{nk}\{-\log f_k(\mathcal{O}; \beta, \eta_k)\} + \mathcal{P}_{nk}\{-\partial \log f_k(\mathcal{O}; \beta, \eta_k)/\partial\beta\}\xi_k + \frac{q_0}{8\sigma_M^2}\xi_k^2, \end{aligned}$$

where q_0 is some positive constant. If $|\xi_k| = (8\sigma_M^2/q_0)|\mathcal{P}_{nk}\{-\partial \log f_k(\mathcal{O}; \beta, \eta_k)/\partial\beta\}|$, then

$$\mathcal{P}_{nk}\{-\log f_k(\mathcal{O}; \beta + \xi_k, \eta_k)\} + \frac{n\xi_k^2}{2n_k\sigma^2} > \mathcal{P}_{nk}\{-\log f_k(\mathcal{O}; \beta, \eta_k)\}.$$

This implies that there exists a local minimizer, $\hat{\xi}_k(\beta, \eta_k, \sigma^2)$, which minimizes

$$\mathcal{P}_{nk}\{-\log f_k(\mathcal{O}; \beta, \eta_k)\} + \mathcal{P}_{nk}\{-\partial \log f_k(\mathcal{O}; \beta, \eta_k)/\partial\beta\}\xi_k$$

and satisfies

$$\left|\hat{\xi}_k(\beta, \eta_k, \sigma^2)\right| \leq \frac{4\sigma_M^2}{q_0}\left|\mathcal{P}_{nk}\{-\partial \log f_k(\mathcal{O}; \beta, \eta_k)/\partial\beta\}\right|.$$

Since $\log f_k(\mathcal{O}; \beta, \eta_k)$ and its derivative are uniformly Donsker,

$$\sup_{\beta, \eta_1, \ldots, \eta_K}\left|(\mathcal{P}_{nk} - \mathcal{P}_k)\{-\partial \log f_k(\mathcal{O}; \beta, \eta_k)/\partial\beta\}\right| = O_{\mathrm{p}}(1)n^{-1/2},$$

where $O_{\mathrm{p}}(1)$ denotes some random variable that is bounded uniformly over $k = 1, \ldots, K$. In addition, $|\beta_{k0} - \beta_0| = O_{\mathrm{p}}(n^{-1/2})$ and $\mathcal{P}_k\{-\partial \log f_k(\mathcal{O}; \beta_{k0}, \eta_{k0})/\partial\beta\} = 0$. Thus,

$$\begin{aligned}
&\left|\mathcal{P}_k\{-\partial \log f_k(\mathcal{O}; \beta, \eta_k)/\partial\beta\}\right| \\
&\quad \le \left|\mathcal{P}_k\{-\partial \log f_k(\mathcal{O}; \beta_0, \eta_{k0})/\partial\beta\}\right| + O_{\mathrm{p}}(1)\{|\beta - \beta_0| + \|\eta_k - \eta_{k0}\|\} \\
&\quad \le O_{\mathrm{p}}(1)\big\{n^{-1/2} + M(Kn)^{-1/2} + \|\eta_k - \eta_{k0}\|\big\}.
\end{aligned}$$

As a result,

$$\left|\hat{\xi}_k(\beta, \eta_k, \sigma^2)\right| \le O_{\mathrm{p}}(1)\big\{n^{-1/2} + M(Kn)^{-1/2}\big\}. \tag{S1}$$

With the above bound for $\hat{\xi}_k(\beta, \eta_k, \sigma^2)$, we can apply the Laplace approximation to $l_k(\beta, \sigma^2, \eta_k)$. By Corollary 4.8 of Evans & Swartz (2000),

$$\begin{aligned}
\sup_{\substack{|\beta-\beta_0| \le M(Kn)^{-1/2}, \\ \|\eta_k - \eta_{k0}\| \le Mn^{-1/2}, \sigma^2 \in \mathcal{C}_3}} &\Bigg| l_k(\beta, \sigma^2, \eta_k) + \frac{1}{2}\log\sigma^2 \\
&- \Bigg(-\frac{1}{2}\log\left[\frac{n_k}{n}\frac{\partial^2}{\partial\beta^2}\mathcal{P}_{nk}\big\{-\log f_k(\mathcal{O}; \beta + \hat{\xi}_k(\beta, \eta_k, \sigma^2), \eta_k)\big\} + \frac{1}{\sigma^2}\right] \\
&+ n_k\left[\mathcal{P}_{nk}\big\{\log f_k(\mathcal{O}; \beta + \hat{\xi}_k(\beta, \eta_k, \sigma^2), \eta_k)\big\} - \frac{n\hat{\xi}_k^2(\beta, \eta_k, \sigma^2)}{2n_k\sigma^2}\right]\Bigg)\Bigg| \\
&\le O_{\mathrm{p}}(1)\, n^{-1/2}.
\end{aligned}$$

It follows from (S1) and the Glivenko–Cantelli theorem that

$$\begin{aligned}
&\Bigg|\log\left[\frac{n_k}{n}\frac{\partial^2}{\partial\beta^2}\mathcal{P}_{nk}\big\{-\log f_k(\mathcal{O}; \beta + \hat{\xi}_k(\beta, \eta_k, \sigma^2), \eta_k)\big\} + \frac{1}{\sigma^2}\right] \\
&\quad - \log\left[p_k\frac{\partial^2}{\partial\beta^2}E_{0k}\big\{-\log f_k(\mathcal{O}; \beta_0, \eta_{k0})\big\} + \frac{1}{\sigma^2}\right]\Bigg| \le O_{\mathrm{p}}(1)Mn^{-1/2},
\end{aligned}$$

where E_{0k} denotes the expectation under the density $f_k(\cdot\,; \beta_0, \eta_{k0})$. Thus,

$$\begin{aligned}
\sup_{\substack{|\beta-\beta_0| \le M(Kn)^{-1/2}, \\ \sigma^2, \eta_1, \ldots, \eta_K}} &\Bigg| l_k(\beta, \sigma^2, \eta_k) - B_k(\sigma^2) \\
&+ n_k\left[\mathcal{P}_{nk}\big\{-\log f_k(\mathcal{O}; \beta + \hat{\xi}_k(\beta, \eta_k, \sigma^2), \eta_k)\big\} + \frac{n\hat{\xi}_k^2(\beta, \eta_k, \sigma^2)}{2n_k\sigma^2}\right]\Bigg| \\
&\le O_{\mathrm{p}}(1)Mn^{-1/2},
\end{aligned}$$

where

$$B_k(\sigma^2) = -\frac{1}{2}\log\sigma^2 + \frac{1}{2}\log\left[p_k\frac{\partial^2}{\partial\beta^2}E_{0k}\{-\log f_k(\mathcal{O}; \beta_0, \eta_{k0})\} + \frac{1}{\sigma^2}\right].$$

Equivalently,

$$\sup_{\substack{|\beta-\beta_0|\le M(Kn)^{-1/2},\\ \sigma^2,\eta_1,\ldots,\eta_K}} \Bigg| l_k(\beta,\sigma^2,\eta_k) - B_k(\sigma^2) - \max_{\xi_k}\left(n_k\left[\mathcal{P}_{nk}\big\{\log f_k(\mathcal{O};\beta+\xi_k,\eta_k)\big\} - \frac{n\xi_k^2}{2n_k\sigma^2}\right]\right)\Bigg| \le O_{\rm p}(1)\big(Mn^{-1/2} + \|\hat{\eta}_k - \eta_{k0}\|\big). \tag{S2}$$

Step 2. We show the existence of a local estimator for η_k for each (β,σ^2) in the neighbourhood $\mathcal{N}$. To this end, we examine the function $\mathcal{P}_{nk}\{\log f_k(\mathcal{O};\beta+\xi_k,\eta_k)\} - n\xi_k^2/(2n_k\sigma^2)$. It follows from Conditions 2 and 4 that this function has a negative-definite Hessian matrix at $\xi_k = 0$ and $\eta_k = \eta_{k0}$. In addition, the Hessian matrix is continuous in a neighbourhood of β_0, and its eigenvalues are bounded away from zero uniformly for $k = 1,\ldots,K$. Thus, it follows from Taylor series expansion that when $|\xi_k| + \|\eta_k - \eta_{k0}\| < \epsilon_0$ for a small ϵ_0,

$$\begin{aligned}
&\left[\mathcal{P}_{nk}\big\{\log f_k(\mathcal{O};\beta+\xi_k,\eta_k)\big\} - \frac{n\xi_k^2}{2n_k\sigma^2}\right] - \mathcal{P}_{nk}\big\{\log f_k(\mathcal{O};\beta,\eta_{k0})\big\}\\
&\quad\le O_{\rm p}(1)\big\{n^{-1/2} + M(Kn)^{-1/2}\big\}\left(|\xi_k| + \|\eta_k-\eta_{k0}\|\right) - c_0\left(\xi_k^2 + \|\eta_k-\eta_{k0}\|^2\right)\\
&\quad\le O_{\rm p}(1)\big\{n^{-1/2} + M(Kn)^{-1/2}\big\}\left(|\xi_k| + \|\eta_k-\eta_{k0}\|\right) - c_0\|\eta_k-\eta_{k0}\|^2
\end{aligned}$$

for some positive constant c_0 independent of k. It then follows from (S1) that

$$\begin{aligned}
&\max_{\xi_k}\left[\mathcal{P}_{nk}\big\{\log f_k(\mathcal{O};\beta+\xi_k,\eta_k)\big\} - \frac{n\xi_k^2}{2n_k\sigma^2}\right] - \mathcal{P}_{nk}\big\{\log f_k(\mathcal{O};\beta,\eta_{k0})\big\}\\
&= \left(\mathcal{P}_{nk}\Big[\log f_k\big\{\mathcal{O};\beta+\hat{\xi}_k(\beta,\sigma^2,\eta_k),\eta_k\big\}\Big] - \frac{n\hat{\xi}_k^2(\beta,\sigma^2,\eta_k)}{2n_k\sigma^2}\right) - \mathcal{P}_{nk}\big\{\log f_k(\mathcal{O};\beta,\eta_{k0})\big\}\\
&\le O_{\rm p}(1)\big\{n^{-1/2} + M(Kn)^{-1/2}\big\}\big\{n^{-1/2} + M(nK)^{-1/2} + \|\eta_k-\eta_{k0}\|\big\} - c_0\|\eta_k-\eta_{k0}\|^2.
\end{aligned}$$

In addition,

$$\begin{aligned}
&\max_{\xi_k}\left[\mathcal{P}_{nk}\big\{\log f_k(\mathcal{O};\beta+\xi_k,\eta_{k0})\big\} - \frac{n\xi_k^2}{2n_k\sigma^2}\right] - \mathcal{P}_{nk}\big\{\log f_k(\mathcal{O};\beta,\eta_{k0})\big\}\\
&= \left(\mathcal{P}_{nk}\Big[\log f_k\big\{\mathcal{O};\beta+\hat{\xi}_k(\beta,\sigma^2,\eta_{k0}),\eta_{k0}\big\}\Big] - \frac{n\hat{\xi}_k^2(\beta,\sigma^2,\eta_{k0})}{2n_k\sigma^2}\right) - \mathcal{P}_{nk}\big\{\log f_k(\mathcal{O};\beta,\eta_{k0})\big\}\\
&\ge -O_{\rm p}(1)\big\{n^{-1/2} + M(Kn)^{-1/2}\big\}\big\{n^{-1/2} + M(nK)^{-1/2}\big\}.
\end{aligned}$$

Thus, it follows from (S2) that

$$\begin{aligned}
&n_k^{-1}l_k(\beta,\sigma^2,\eta_k) - n_k^{-1}l_k(\beta,\sigma^2,\eta_{k0})\\
&\quad\le O_{\rm p}(1)\big(Mn^{-3/2} + n^{-1}\|\eta_k-\eta_{k0}\|\big)\\
&\qquad + O_{\rm p}(1)\big\{n^{-1/2} + M(Kn)^{-1/2}\big\}\big\{n^{-1/2} + M(nK)^{-1/2} + \|\eta_k-\eta_{k0}\|\big\}\\
&\qquad - c_0\|\eta_k-\eta_{k0}\|^2.
\end{aligned}$$

If $\|\eta_k - \eta_{k0}\| = c_1 n^{-1/2}$ for some large constant c_1, then $l_k(\beta,\sigma^2,\eta_k) < l_k(\beta,\sigma^2,\eta_{k0})$; that is, there exists a local maximizer, denoted by $\hat{\eta}_k(\beta,\sigma^2)$, which maximizes $l_k(\beta,\sigma^2,\eta_k)$, such that

$$\sup_{k=1,\ldots,K}\left\|\hat{\eta}_k(\beta,\sigma^2)-\eta_{k0}\right\| \le O_{\mathrm{p}}(1)n^{-1/2}.$$

Thus, we define a local profile loglikelihood function as $\mathrm{pl}_k(\beta,\sigma^2)=l_k\{\beta,\sigma^2,\hat{\eta}_k(\beta,\sigma^2)\}$.

Step 3. We show that for each fixed σ^2 there exists some local maximizer for the profile loglikelihood function for (β,σ^2). In light of (S1) and (S2),

$$\begin{aligned}\sup_{|\beta-\beta_0|\le M(Kn)^{-1/2},\sigma^2}\Bigg|&\mathrm{pl}_k(\beta,\sigma^2)-B_k(\sigma^2)\\ &-\max_{\substack{\|\eta_k-\eta_{k0}\|\le O_{\mathrm{p}}(1)n^{-1/2},\\ |\xi_k|\le O_{\mathrm{p}}(1)Mn^{-1/2}}} n_k\left[\mathcal{P}_{nk}\big\{\log f_k(\mathcal{O};\beta+\xi_k,\eta_k)\big\}-\frac{n\xi_k^2}{2n_k\sigma^2}\right]\Bigg|\\ &\le O_{\mathrm{p}}(1)\frac{M}{n^{1/2}}.\end{aligned}$$

We obtain a quadratic expansion for the third term on the left-hand side. Specifically, define

$$S_{nk1}=\partial\mathcal{P}_{nk}\{\log f_k(\mathcal{O};\beta_{k0},\eta_{k0})\}/\partial\beta,\quad S_{nk2}=\partial\mathcal{P}_{nk}\{\log f_k(\mathcal{O};\beta_{k0},\eta_{k0})\}/\partial\eta_k,$$

where $\beta_{k0}=\beta_0+\xi_{k0}$ with ξ_{k0} being the true value of ξ_k, and let $\mathcal{I}_k$ denote the information matrix of $f_k(\mathcal{O}_{k1};\beta_{k0},\eta_{k0})$. By Taylor series expansion,

$$\begin{aligned}&\mathcal{P}_{nk}\big\{\log f_k(\mathcal{O};\beta+\xi_k,\eta_k)\big\}-\frac{n\xi_k^2}{2n_k\sigma^2}\\ &\quad=\mathcal{P}_{nk}\big\{\log f_k(\mathcal{O};\beta_{k0},\eta_{k0})\big\}+S_{nk1}(\beta+\xi_k-\beta_{k0})+S_{nk2}(\eta_k-\eta_{k0})\\ &\qquad+\frac{1}{2}(\beta+\xi_k-\beta_{k0},\eta_k-\eta_{k0})\big\{-\mathcal{I}_k+O_{\mathrm{p}}(1)Mn^{-1/2}\big\}\begin{pmatrix}\beta+\xi_k-\beta_{k0}\\ \eta_k-\eta_{k0}\end{pmatrix}-\frac{n\xi_k^2}{2n_k\sigma^2}\end{aligned}$$

uniformly over $k=1,\ldots,K$. Thus,

$$\begin{aligned}\sup_{|\beta-\beta_0|\le M(Kn)^{-1/2},\sigma^2}\Bigg|&\mathrm{pl}_k(\beta,\sigma^2)-B_k(\sigma^2)\\ &-\max_{\substack{\|\eta_k-\eta_{k0}\|\le O_{\mathrm{p}}(1)n^{-1/2},\\ |\xi_k|\le O_{\mathrm{p}}(1)Mn^{-1/2}}}\Bigg[n_k\mathcal{P}_{nk}\big\{\log f_k(\mathcal{O};\beta_0,\eta_{k0})\big\}\\ &\qquad+n_k\Bigg\{S_{nk1}(\beta+\xi_k-\beta_{k0})+S_{nk2}(\eta_k-\eta_{k0})\\ &\qquad-\frac{1}{2}(\beta+\xi_k-\beta_{k0},\eta_k-\eta_{k0})\mathcal{I}_k\begin{pmatrix}\beta+\xi_k-\beta_{k0}\\ \eta_k-\eta_{k0}\end{pmatrix}-\frac{\xi_k^2}{2p_k\sigma^2}\Bigg\}\Bigg]\Bigg|\\ &\le O_{\mathrm{p}}(1)M^3n^{-1/2}.\end{aligned}$$

When n is large enough, the third term on the left-hand side is the global maximum

$$C_{nk}-\frac{n_k}{2}\frac{\{\beta-\beta_{k0}-v_k(\mathcal{P}_{nk}-\mathcal{P}_k)(S_k^*)\}^2}{v_k+p_k\sigma^2},$$

where C_{nk} is independent of β and σ^2, and S_k^* is the efficient score function for β_{k0} based on $\mathcal{O}_{k1}$. Thus,

$$\sup_{|\beta-\beta_0|\le M(Kn)^{-1/2},\sigma^2}\left|\sum_{k=1}^K \mathrm{pl}_k(\beta,\sigma^2)-\sum_{k=1}^K C_{nk}-\sum_{k=1}^K B_k(\sigma^2)\right.$$
$$\left.+\sum_{k=1}^K \frac{n_k}{2}\frac{\{\beta-\beta_{k0}-v_k(\mathcal{P}_{nk}-\mathcal{P}_k)(S_k^*)\}^2}{v_k+p_k\sigma^2}\right|$$
$$\le O_{\mathrm{p}}(1)M^3Kn^{-1/2}. \tag{S3}$$

For each σ^2, the value of β that maximizes

$$g_n(\beta,\sigma^2)=-\sum_{k=1}^K \frac{n_k}{2}\frac{\{\beta-\beta_{k0}-v_k(\mathcal{P}_{nk}-\mathcal{P}_k)(S_k^*)\}^2}{v_k+p_k\sigma^2}$$

is equal to

$$\check\beta=\beta_0+\left(\sum_{k=1}^K\frac{n_k}{v_k+p_k\sigma^2}\right)^{-1}\left[\sum_{k=1}^K\frac{n_k}{v_k+p_k\sigma^2}\{\xi_{k0}+v_k(\mathcal{P}_{nk}-\mathcal{P}_k)(S_k^*)\}\right].$$

Its variance is of the order of $(Kn)^{-1}$, such that $|\check\beta-\beta_0|\le M(Kn)^{-1/2}/2$ for large M and n. We consider $\beta=\check\beta\pm\epsilon(Kn)^{-1/2}$ for any small ϵ. By (S3) and Condition 5,

$$\begin{aligned}\sum_{k=1}^K \mathrm{pl}_k(\check\beta,\sigma^2)-\sum_{k=1}^K \mathrm{pl}_k(\beta,\sigma^2)&\ge -O_{\mathrm{p}}(1)M^3Kn^{-1/2}+g_n(\check\beta,\sigma^2)-g_n(\beta,\sigma)\\&\ge -O_{\mathrm{p}}(1)M^3Kn^{-1/2}+c_2nK(\beta-\check\beta)^2\\&\ge -O_{\mathrm{p}}(1)M^3Kn^{-1/2}+c_2\epsilon,\end{aligned}$$

where c_2 is a positive constant. Thus, $\sum_{k=1}^K \mathrm{pl}_k(\check\beta,\sigma^2)-\sum_{k=1}^K \mathrm{pl}_k(\beta,\sigma^2)>0$ when n is large. This implies that there exists a local maximizer $\hat\beta_{\mathrm{ML}}(\sigma^2)$ for $\sum_{k=1}^K \mathrm{pl}_k(\beta,\sigma^2)$, and

$$\hat\beta_{\mathrm{ML}}(\sigma^2)=\beta_0+\left(\sum_{k=1}^K\frac{n_k}{v_k+p_k\sigma^2}\right)^{-1}\left[\sum_{k=1}^K\frac{n_k}{v_k+p_k\sigma^2}\{\xi_{k0}+v_k(\mathcal{P}_{nk}-\mathcal{P}_k)(S_k^*)\}\right]+\frac{o_{\mathrm{p}}(1)}{(Kn)^{1/2}}. \tag{S4}$$

Step 4. We define our estimator for σ^2, denoted by $\hat\sigma^2_{\mathrm{ML}}$, as the argument maximizing $\sum_{k=1}^K \mathrm{pl}_k\{\hat\beta_{\mathrm{ML}}(\sigma^2),\sigma^2\}$. Then the estimator for β is given by $\hat\beta_{\mathrm{ML}}(\hat\sigma^2_{\mathrm{ML}})$, and the estimator for η_k is $\hat\eta_k(\hat\beta_{\mathrm{ML}},\hat\sigma^2_{\mathrm{ML}})$. From Step 3, it is clear that

$$\sup_{\sigma^2}\left|\sum_{k=1}^K \mathrm{pl}_k\{\hat\beta_{\mathrm{ML}}(\sigma^2),\sigma^2\}-\sum_{k=1}^K C_{nk}-Q_n(\sigma^2)\right|\le O_{\mathrm{p}}(1)Kn^{-1/2},$$

where

$$Q_n(\sigma^2) = -\frac{1}{2}\sum_{k=1}^{K}\log(p_k\sigma^2\mathcal{I}_{\beta_k}+1) - \frac{1}{2}\sum_{k=1}^{K}\frac{n_k\{\xi_{k0}+v_k(\mathcal{P}_{nk}-\mathcal{P}_k)(S_k^*)\}^2}{v_k+p_k\sigma^2}$$
$$+\frac{1}{2}\left(\sum_{k=1}^{K}\frac{n_k}{v_k+p_k\sigma^2}\right)^{-1}\left[\sum_{k=1}^{K}\frac{n_k\{\xi_{k0}+v_k(\mathcal{P}_{nk}-\mathcal{P}_k)(S_k^*)\}}{v_k+p_k\sigma^2}\right]^2. \qquad \text{(S5)}$$

REFERENCES

EVANS, M. & SWARTZ, T. (2000). *Approximating Integrals via Monte Carlo and Deterministic Methods*. Oxford: Oxford University Press.

VAN DER VAART, A. W. & WELLNER, J. A. (1996). *Weak Convergence and Empirical Processes*. New York: Springer.

ADDITIONAL SIMULATION RESULTS

Additional simulation results not shown in the main article are reported in Tables S1–S3.

Table S1. *Values of I^2 (%) in the simulation studies*

	Number of studies, K									
τ^2	5	10	15	20	25	30	35	40	45	50
0·00	0	0	0	0	0	0	0	0	0	0
0·01	12	13	14	14	14	14	15	15	15	15
0·02	22	23	24	25	25	25	25	26	26	26
0·03	29	31	33	33	33	34	34	34	34	34
0·04	35	38	39	40	40	40	41	41	41	41
0·05	40	43	45	45	46	46	46	46	46	46
0·06	45	48	49	50	50	50	51	51	51	51
0·07	49	52	53	54	54	54	55	55	55	55
0·08	52	55	56	57	57	58	58	58	58	58
0·09	55	58	59	60	60	60	61	61	61	61
0·10	57	61	62	62	63	63	63	63	63	63

Table S2. *Coverage probabilities* $(\%)$ *of* 95% *confidence intervals based on the new resampling method, compared with those of three other methods*

		Number of studies, K									
τ^2		5	10	15	20	25	30	35	40	45	50
0·00	New	98·9	98·5	98·2	98·1	97·8	97·7	97·5	97·5	97·6	97·2
	DL	96·4	96·0	96·2	96·1	96·2	96·1	95·8	96·2	96·3	96·0
	JB	99·4	99·0	98·8	98·6	98·5	98·3	98·2	98·0	98·3	97·7
	HT	97·4	96·8	96·6	96·3	96·3	96·3	96·1	96·2	96·3	96·0
0·01	New	97·9	96·8	96·5	96·2	95·9	95·8	95·6	95·4	95·6	95·1
	DL	94·2	93·6	94·0	94·0	94·1	94·2	93·9	93·9	94·4	93·9
	JB	99·1	98·2	97·9	97·7	97·6	97·4	97	96·9	97·3	96·8
	HT	96·1	95·0	95·0	95·0	94·7	94·7	94·7	94·7	95·0	94·6
0·02	New	96·9	95·5	95·1	95·0	94·9	94·5	94·4	94·3	94·7	94·3
	DL	92·4	92·4	92·5	92·8	93·0	93·3	93·0	93·2	93·8	93·4
	JB	98·8	97·6	97·4	97·3	97·3	97·1	96·7	96·7	97·1	96·8
	HT	95·1	94·0	94·0	94·2	94·3	94·3	94·0	94·3	94·5	94·4
0·03	New	96·0	94·6	94·3	94·2	94·1	93·9	93·8	93·9	94·3	94·0
	DL	90·9	91·4	91·7	92·3	92·6	93·0	92·6	93·0	93·6	93·5
	JB	98·4	97·3	97·1	96·9	97·3	97·0	96·8	96·9	97·3	96·9
	HT	94·2	93·3	93·5	93·7	94·0	94·0	93·8	94·0	94·4	94·3
0·04	New	95·3	93·9	93·7	93·7	93·7	93·7	93·7	93·7	94·2	94·1
	DL	89·8	90·7	91·4	92·0	92·7	92·8	92·7	93·3	93·6	93·7
	JB	98·1	97·0	96·8	97·1	97·4	97·2	97·1	97·1	97·6	97·3
	HT	93·5	92·7	93·3	93·5	93·8	93·8	93·7	94·0	94·2	94·3
0·05	New	94·6	93·4	93·1	93·3	93·5	93·6	93·6	93·8	94·1	94·1
	DL	89·2	90·3	91·4	92·1	92·7	92·9	93·0	93·3	93·7	93·8
	JB	97·8	96·9	96·9	97·1	97·5	97·3	97·3	97·3	97·8	97·5
	HT	93·1	92·5	93·0	93·5	93·9	93·9	93·7	94·0	94·3	94·4
0·06	New	94·0	93·0	92·8	93·2	93·5	93·5	93·6	93·8	94·2	94·1
	DL	88·7	90·1	91·2	92·0	92·8	93·1	93·1	93·4	93·8	93·9
	JB	97·7	96·8	97	97·2	97·6	97·5	97·6	97·6	98·1	97·8
	HT	92·6	92·2	92·9	93·3	93·8	93·9	93·9	94·0	94·3	94·4
0·07	New	93·4	92·6	92·6	93·1	93·5	93·6	93·6	93·8	94·2	94·2
	DL	88·4	89·9	91·1	92·1	92·8	93·2	93·1	93·5	93·9	93·9
	JB	97·4	96·6	97·0	97·3	97·8	97·6	97·8	97·8	98·3	97·9
	HT	92·2	92·1	92·9	93·4	93·8	93·9	94·0	94·0	94·4	94·4
0·08	New	92·8	92·3	92·5	93·0	93·6	93·7	93·6	93·9	94·2	94·2
	DL	88·0	89·7	91·1	92·2	92·9	93·3	93·2	93·6	94·0	94·0
	JB	97·3	96·6	97·0	97·4	97·9	97·9	97·9	97·9	98·3	98·1
	HT	91·8	92·0	92·9	93·4	93·8	94·0	93·9	94·1	94·4	94·4
0·09	New	92·4	92·0	92·5	93·0	93·6	93·7	93·6	93·8	94·3	94·1
	DL	87·9	89·8	91·1	92·2	93·0	93·3	93·3	93·6	94·1	94·0
	JB	97·1	96·6	97·1	97·6	98·0	98·0	98·1	98·0	98·4	98·2
	HT	91·6	92·0	93·0	93·4	93·7	94·0	94·0	94·1	94·5	94·5
0·10	New	92·0	91·7	92·5	93·0	93·5	93·7	93·6	93·8	94·3	94·1
	DL	87·6	89·8	91·2	92·3	93·0	93·4	93·3	93·6	94·1	94·0
	JB	96·9	96·6	97·2	97·7	98·0	98·1	98·2	98·1	98·6	98·2
	HT	91·4	91·9	93·0	93·5	93·7	94·0	94·0	94·1	94·5	94·5

New, the new resampling method proposed in this paper; DL, the DerSimonian–Laird method; JB, the Jackson–Bowden method; HT, the Hardy–Thompson method.

Table S3. *Mean widths of* 95% *confidence intervals based on the new resampling method, compared with those of three other methods*

τ^2		Number of studies, K									
		5	10	15	20	25	30	35	40	45	50
0·00	New	0·666	0·406	0·312	0·26	0·227	0·204	0·186	0·172	0·161	0·151
	DL	0·553	0·354	0·278	0·235	0·207	0·188	0·172	0·16	0·15	0·142
	JB	0·883	0·493	0·367	0·3	0·258	0·229	0·206	0·189	0·175	0·164
	HT	0·618	0·375	0·288	0·241	0·211	0·19	0·173	0·161	0·15	0·142
0·01	New	0·684	0·421	0·326	0·273	0·239	0·215	0·197	0·182	0·171	0·161
	DL	0·575	0·374	0·296	0·252	0·223	0·203	0·187	0·174	0·163	0·154
	JB	0·927	0·528	0·398	0·328	0·284	0·253	0·229	0·211	0·196	0·184
	HT	0·646	0·401	0·312	0·263	0·231	0·209	0·192	0·178	0·167	0·158
0·02	New	0·703	0·437	0·341	0·287	0·253	0·229	0·21	0·195	0·183	0·173
	DL	0·598	0·394	0·315	0·27	0·24	0·219	0·202	0·189	0·178	0·168
	JB	0·97	0·563	0·429	0·357	0·31	0·278	0·253	0·234	0·218	0·205
	HT	0·672	0·425	0·334	0·283	0·25	0·227	0·209	0·194	0·183	0·173
0·03	New	0·722	0·455	0·357	0·302	0·267	0·242	0·223	0·208	0·195	0·185
	DL	0·62	0·415	0·334	0·288	0·257	0·235	0·217	0·203	0·191	0·182
	JB	1·012	0·597	0·459	0·384	0·336	0·302	0·276	0·255	0·239	0·225
	HT	0·697	0·447	0·354	0·301	0·267	0·243	0·224	0·209	0·196	0·186
0·04	New	0·74	0·472	0·372	0·317	0·281	0·255	0·235	0·22	0·207	0·196
	DL	0·641	0·435	0·352	0·304	0·272	0·249	0·231	0·216	0·204	0·193
	JB	1·052	0·629	0·487	0·41	0·36	0·324	0·297	0·275	0·257	0·243
	HT	0·722	0·468	0·373	0·318	0·283	0·257	0·238	0·222	0·209	0·198
0·05	New	0·759	0·488	0·388	0·331	0·294	0·268	0·247	0·231	0·218	0·206
	DL	0·662	0·454	0·369	0·32	0·287	0·262	0·243	0·228	0·215	0·204
	JB	1·091	0·66	0·515	0·435	0·382	0·345	0·316	0·294	0·275	0·259
	HT	0·745	0·488	0·391	0·334	0·297	0·271	0·25	0·233	0·22	0·208
0·06	New	0·777	0·505	0·402	0·345	0·307	0·28	0·259	0·242	0·228	0·216
	DL	0·683	0·472	0·385	0·335	0·3	0·275	0·255	0·239	0·225	0·214
	JB	1·128	0·69	0·541	0·458	0·404	0·365	0·335	0·311	0·291	0·275
	HT	0·767	0·507	0·407	0·349	0·311	0·283	0·262	0·244	0·23	0·218
0·07	New	0·795	0·521	0·417	0·358	0·319	0·291	0·269	0·252	0·237	0·225
	DL	0·703	0·489	0·401	0·349	0·313	0·287	0·266	0·249	0·235	0·223
	JB	1·164	0·718	0·565	0·48	0·424	0·383	0·352	0·327	0·307	0·289
	MLE	0·619	0·454	0·381	0·335	0·303	0·28	0·261	0·245	0·231	0·22
	HT	0·789	0·525	0·423	0·363	0·323	0·295	0·273	0·254	0·24	0·227
0·08	New	0·813	0·536	0·43	0·371	0·33	0·302	0·279	0·261	0·246	0·233
	DL	0·722	0·506	0·415	0·362	0·325	0·297	0·276	0·258	0·244	0·231
	JB	1·199	0·746	0·589	0·502	0·443	0·401	0·368	0·343	0·321	0·303
	HT	0·81	0·542	0·438	0·376	0·335	0·306	0·283	0·264	0·248	0·235
0·09	New	0·83	0·551	0·444	0·383	0·341	0·312	0·289	0·27	0·254	0·241
	DL	0·741	0·522	0·429	0·374	0·336	0·308	0·286	0·267	0·252	0·239
	JB	1·232	0·772	0·612	0·522	0·461	0·418	0·384	0·357	0·335	0·316
	HT	0·831	0·559	0·452	0·388	0·346	0·316	0·292	0·273	0·257	0·243
0·10	New	0·847	0·566	0·457	0·394	0·352	0·321	0·298	0·278	0·262	0·249
	DL	0·759	0·538	0·443	0·386	0·346	0·318	0·295	0·276	0·26	0·247
	JB	1·265	0·797	0·633	0·541	0·478	0·434	0·399	0·371	0·348	0·329
	HT	0·85	0·575	0·465	0·4	0·357	0·326	0·301	0·281	0·265	0·251

New, the new resampling method proposed in this paper; DL, the DerSimonian–Laird method; JB, the Jackson–Bowden method; HT, the Hardy–Thompson method.