Supplementary Material to

Title: Improved Statistical Significance of Clustering

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# A Proof of all Theorems

Before giving the proofs of our theorems, we provide the following two lemmas which will be used for our proofs of the theorems.

**Lemma 1.** *von Newmann's lower bound trace inequality: for d-dimensional positive semi-definite matrices A and B, denote $\lambda_j(A)$ $(j = 1, \cdots, d)$ as the jth eigenvalue of A, then $trace(AB) \geq \sum_{j=1}^{d} \lambda_j(A)\lambda_{d-j+1}(B)$.*

Proof. From von Newmann's trace inequality (Mirsky, 1975), we have

$$\begin{aligned}
\text{trace}(A[\lambda_1(B)I - B]) &\leq \sum_j \lambda_j(A)\lambda_j(\lambda_1(B)I - B) \\
&= \sum_j \lambda_j(A)[\lambda_1(B) - \lambda_{d-j+1}(B)] \\
&= \lambda_1(B)\sum_j \lambda_j(A) - \sum_j \lambda_j(A)\lambda_{d-j+1}(B),
\end{aligned}$$

which implies the lower bound

$$\text{trace}(AB) \geq \sum_j \lambda_j(A)\lambda_{d-j+1}(B)$$

since

$$\text{trace}(A(\lambda_1(B)I - B)) = \lambda_1(B)\text{trace}(A) - \text{trace}(AB).$$

This completes the proof of Lemma 1.

Define $\alpha = 1/\sigma_N^2$. Now consider the optimization problem

$$\min_{d_1,\cdots,d_d} \left[-\sum_{k=1}^{d} \ln(\alpha - d_k) + \sum_{k=1}^{d} \tilde{\lambda}_k(\alpha - d_k)\right] \tag{A.1}$$

$$\text{subject to } 0 \leq d_k \leq \alpha,\ \#\{k : d_k > 0\} \leq l,\ k = 1, \cdots, d. \tag{A.2}$$

Let $\hat{d}_1, \cdots, \hat{d}_d$ be the solution to (A.1) (A.2) and $\hat{D} = \text{diag}(\hat{d}_1, \cdots, \hat{d}_d)$. Then the following lemma establishes the relationship between $\hat{D}$ and the solution to (7) (8) (9).

**Lemma 2.** *The solution to (7) (8) (9) is* $\tilde{U}(\alpha I - \hat{D})\tilde{U}^T$.

Proof. Assume that the solution of (A.1) (A.2) leads to an objective value $o_1$, and the solution of (7) (8) (9) leads to an objective value $o_2$. From von Neumann's lower bound trace inequality, we have

$$\text{trace}(C\tilde{\Sigma}) \geq \sum_{k=1}^{d} \lambda_k(\tilde{\Sigma})\lambda_{d-k+1}(C) = \sum_{k=1}^{d} \tilde{\lambda}_k(\alpha - d_k)$$

for the second term of (7), since $\lambda_{d-k+1}(C) = \lambda_{d-k+1}(\alpha - W_0) = \alpha - d_k$. Consequently, we have $o_1 \leq o_2$. The proof is now completed because $\tilde{U}(\alpha I - \hat{D})\tilde{U}^T$ is in the feasible region of (8) and (9) and it can be verified that when plugging it into (7), the objective function reaches $o_1$, which must be a minimum of (7) since $o_1 \leq o_2$. This completes the proof of Lemma 2.

With the two lemmas in place, we are now ready to prove Theorem 1.

**Proof of Theorem 1**: First, note that the optimization problem in (A.1) and (A.2) can be reduced to $d$ subproblems indexed by $k$:

$$\min_{d_k} \left[ -\ln(\alpha - d_k) + \tilde{\lambda}_k(\alpha - d_k) \right] \quad \text{(A.3)}$$
$$\text{subject to } 0 \leq d_k \leq \alpha.$$

Note that when $\tilde{\lambda}_k \leq 1/\alpha = \sigma_N^2$, the objective function (A.3) is increasing in the feasible region. Therefore, we must have $\hat{d}_k = 0$. Otherwise, first order condition leads to $\hat{d}_k = \alpha - 1/\tilde{\lambda}_k$. To sum up, we have

$$\hat{d}_k = (\alpha - 1/\tilde{\lambda}_k)_+.$$

Note that the corresponding minimum

$$-\ln(\alpha - \hat{d}_k) + \tilde{\lambda}_k(\alpha - \hat{d}_k) = -\ln\left(\alpha \wedge \frac{1}{\tilde{\lambda}_k}\right) + (1 \wedge \alpha\tilde{\lambda}_k)$$

is an increasing function of $\tilde{\lambda}_k$, which suggests that the solution to (A.1) (A.2) is

$$\hat{d}_k = \begin{cases} \alpha - 1/\tilde{\lambda}_k & \text{if } k \leq l \text{ and } \tilde{\lambda}_k > \sigma_N^2 \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof of Theorem 1.

For Theorem 2, we consider an alternative optimization problem to (7), (8), and (11):

$$\min_{d_1,\cdots,d_d} \left[ -\textstyle\sum_{k=1}^{d} \ln(\alpha - d_k) + \sum_{k=1}^{d} \tilde{\lambda}_k(\alpha - d_k) \right] \quad \text{(A.4)}$$

$$\text{subject to } 0 \leq d_k \leq \alpha, \ \textstyle\sum d_k \leq M, \ k = 1, \cdots, d. \quad \text{(A.5)}$$

Let $\hat{d}_1, \cdots, \hat{d}_d$ be the solution to (A.4), (A.5), and $\hat{\Lambda} = \text{diag}(\hat{d}_1, \cdots, \hat{d}_d)$. Then we have the following lemma which can be proved in the same fashion as Lemma 2.

**Lemma 3.** *The solution to (7) (8) (11) is* $\tilde{U}(\alpha I - \hat{\Lambda})\tilde{U}^T$.

**Proof of Theorem 2:**

To solve (A.4) (A.5), we first write them in a Lagrange form

$$\min_{d_1,\cdots,d_k} \textstyle\sum_{k=1}^{d} [-\ln(\alpha - d_k) + \tilde{\lambda}_k(\alpha - d_k) + \tau d_k]$$
$$\text{subject to } 0 \leq d_k \leq \alpha,\ k = 1, \cdots, d.$$

which essentially can be solved as $d$ separate problems. When $\tilde{\lambda}_k \leq \tau + \sigma_N^2$,

$$-\ln(\alpha - d_k) + \tilde{\lambda}_k(\alpha - d_k) + \tau d_k$$

is an increasing function of $d_k$ in the feasible set $[0, \alpha]$. Therefore, $\hat{d}_k = 0$. Otherwise, the first order condition yields

$$\hat{d}_k = \alpha - \frac{1}{\tilde{\lambda}_k - \tau}.$$

In other words,

$$\hat{d}_k = \left(\alpha - \frac{1}{\tilde{\lambda}_k - \tau}\right)_+.$$

It is now clear that the tuning parameters $\tau$ and $M$ are related through

$$\sum_{k=1}^{d} \hat{d}_k = \left(\alpha - \frac{1}{\tilde{\lambda}_k - \tau}\right)_+ = M.$$

From Lemma 3, we have

$$\hat{\lambda}_k = \frac{1}{\alpha - \hat{d}_k} = (\tilde{\lambda}_k - \tau - \sigma_N^2)_+ + \sigma_N^2.$$

This completes the proof of Theorem 2.

**Proof of Theorem 3:**

Due to rotation invariance, it is enough to consider the situations with diagonal covariance matrix. Let's first calculate the total sum square TSS.

$$\begin{aligned}
\text{TSS} &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} ||\mathbf{x}||^2 \phi(\boldsymbol{x}) dx_1 \cdots dx_d \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{j=1}^{d} x_j^2 \left(\prod_{j=1}^{d} \varphi_{\lambda_j}(x_j)\right) dx_1 \cdots dx_d \\
&= \sum_{j=1}^{d} \int_{-\infty}^{\infty} x_j^2 \varphi_{\lambda_j}(x_j) dx_j = \sum_{j=1}^{d} \lambda_j,
\end{aligned}$$

where $\phi(\boldsymbol{x}) = \prod_{j=1}^{d} \varphi_{\lambda_j}(x_j) = \prod_{j=1}^{d} \frac{1}{\sqrt{2\pi\lambda_j}} e^{-x_j^2/2\lambda_j}$. Now assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$. Then $(1, 0, \cdots, 0)^T$ is the direction of the greatest variation. Thus it makes sense to define the 2-means separating plane as the plane which is through $\boldsymbol{\mu} = (0, \cdots, 0)^T$ and orthogonal to $(1, 0, \cdots, 0)^T$.

Let $\text{WSS}_1$ denote the sum square within Class 1 and $\text{WSS}_2$ the sum square within Class 2. First find center points: $\boldsymbol{\mu}_1 = (\mu_{11}, \mu_{12}, \cdots \mu_{1d})^T$. By symmetry, we have $\mu_{12} = \mu_{13} = \cdots = \mu_{1d} = 0$.

$$\mu_{11} = 2\int_0^\infty x_1 \varphi_{\lambda_1}(x_1) dx_1 = \sqrt{\frac{2\lambda_1}{\pi}}.$$

Here 2 is the normalization factor. So $\boldsymbol{\mu}_1 = \left(\sqrt{\frac{2\lambda_1}{\pi}}, 0, \cdots, 0\right)^T$. Similarly, $\boldsymbol{\mu}_2 = \left(-\sqrt{\frac{2\lambda_1}{\pi}}, 0, \cdots, 0\right)^T$. Next

$$\begin{aligned}
\text{WSS}_1 &= \int_0^\infty \cdots \int_{-\infty}^\infty ||\mathbf{x} - \boldsymbol{\mu}_1||^2 \phi(\mathbf{x}) dx_1 \cdots dx_d \\
&= \int_0^\infty \left(x_1 - \sqrt{\frac{2\lambda_1}{\pi}}\right)^2 \varphi_{\lambda_1}(x_1) dx_1 + \sum_{j=2}^{d} \int_0^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty x_j^2 \phi(\boldsymbol{x}) dx_1 \cdots dx_d \\
&= \left(\frac{\pi - 2}{2\pi}\right) \lambda_1 + \sum_{j=2}^{d} \frac{\lambda_j}{2}.
\end{aligned}$$

Similarly $\text{WWS}_2$=$\text{WWS}_1$. Thus

$$\text{TCI} = \frac{\text{WSS}_1 + \text{WSS}_2}{\text{TSS}} = 1 - \frac{2}{\pi} \frac{\lambda_1}{\sum_{j=1}^{d} \lambda_j}.$$

This completes of the proof of Theorem 3.

# B More Simulated Examples

We perform more simulations in this section. The simulation setting is the same as in Section 3.1 but with different dimensions. Tables A, B, and C summarize the results for $d = 5000$, $d = 300$, and $d = 200$ respectively. As dimension increases, there was less anti-conservatism for the hard method. The sample and the soft methods can control the type-I error in all situations.

# References

Mirsky, L. (1975). A trace inequality of John von Neumann. *Monatsh. Math. 79*, 303–306.

Table A: Summary table of empirical SigClust $p$-value distribution over 100 replications based on four methods under different settings in Simulation 3.1. The mean and the numbers of $p$-values which are less than 0.05 (denoted as N5) and 0.1 (denoted as N10) are reported (d=5000, n=100).

| | | True | | | Sample | | | Hard | | | Soft | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $v$ | $w$ | Mean | N5 | N10 | Mean | N5 | N10 | Mean | N5 | N10 | Mean | N5 | N10 |
| 1000 | 1 | 0.55 | 8 | 13 | 0.65 | 0 | 0 | 0.00 | 100 | 100 | 0.47 | 0 | 0 |
| 200 | 5 | 0.39 | 5 | 12 | 0.98 | 0 | 0 | 0.01 | 98 | 100 | 0.65 | 0 | 0 |
| 100 | 10 | 0.36 | 7 | 16 | 1.00 | 0 | 0 | 0.28 | 5 | 13 | 0.80 | 0 | 0 |
| 40 | 25 | 0.29 | 14 | 22 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 0.97 | 0 | 0 |
| 20 | 50 | 0.22 | 11 | 24 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |
| 10 | 100 | 0.20 | 19 | 31 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |
| 200 | 1 | 0.54 | 6 | 11 | 0.97 | 0 | 0 | 0.00 | 100 | 100 | 0.42 | 0 | 0 |
| 100 | 1 | 0.58 | 2 | 4 | 1.00 | 0 | 0 | 0.39 | 5 | 15 | 0.44 | 0 | 0 |
| 50 | 1 | 0.51 | 8 | 10 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 0.81 | 0 | 0 |
| 40 | 1 | 0.50 | 6 | 9 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 0.98 | 0 | 0 |
| 30 | 1 | 0.50 | 5 | 8 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |
| 20 | 1 | 0.48 | 4 | 6 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |
| 10 | 1 | 0.21 | 20 | 28 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |
| 50 | 10 | 0.35 | 11 | 20 | 1.00 | 0 | 0 | 0.97 | 0 | 0 | 0.84 | 0 | 0 |
| 40 | 10 | 0.38 | 5 | 15 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 0.92 | 0 | 0 |
| 30 | 10 | 0.26 | 23 | 36 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 0.98 | 0 | 0 |
| 20 | 10 | 0.36 | 7 | 13 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |
| 10 | 10 | 0.22 | 18 | 27 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |
| 50 | 5 | 0.41 | 5 | 17 | 1.00 | 0 | 0 | 0.98 | 0 | 0 | 0.78 | 0 | 0 |
| 40 | 5 | 0.41 | 6 | 13 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 0.91 | 0 | 0 |
| 30 | 5 | 0.35 | 10 | 16 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |
| 20 | 5 | 0.35 | 7 | 13 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |
| 10 | 5 | 0.24 | 13 | 27 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |
| 50 | 2 | 0.47 | 6 | 12 | 1.00 | 0 | 0 | 0.99 | 0 | 0 | 0.78 | 0 | 0 |
| 40 | 2 | 0.44 | 7 | 14 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 0.95 | 0 | 0 |
| 30 | 2 | 0.50 | 7 | 13 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |
| 20 | 2 | 0.43 | 4 | 9 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |
| 10 | 2 | 0.24 | 12 | 23 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |
| 5 | 1 | 0.14 | 26 | 50 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |
| 3 | 1 | 0.13 | 24 | 51 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |
| 1 | 1 | 0.14 | 24 | 47 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |

Table B: Summary table of empirical SigClust $p$-value distribution over 100 replications based on four methods under different settings in Simulation 3.1. The mean and the numbers of $p$-values which are less than 0.05 (denoted as N5) and 0.1 (denoted as N10) are reported (d=300, n=100).

| | | True | | | Sample | | | Hard | | | Soft | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $v$ | $w$ | Mean | N5 | N10 | Mean | N5 | N10 | Mean | N5 | N10 | Mean | N5 | N10 |
| 1000 | 1 | 0.46 | 6 | 18 | 0.48 | 4 | 7 | 0.04 | 77 | 88 | 0.45 | 4 | 9 |
| 200 | 5 | 0.37 | 9 | 14 | 0.75 | 0 | 0 | 0.30 | 1 | 7 | 0.69 | 0 | 0 |
| 100 | 10 | 0.35 | 7 | 12 | 0.90 | 0 | 0 | 0.52 | 0 | 0 | 0.84 | 0 | 0 |
| 40 | 25 | 0.26 | 7 | 24 | 0.99 | 0 | 0 | 0.84 | 0 | 0 | 0.96 | 0 | 0 |
| 20 | 50 | 0.26 | 14 | 26 | 1.00 | 0 | 0 | 0.97 | 0 | 0 | 0.99 | 0 | 0 |
| 10 | 100 | 0.22 | 18 | 34 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |
| 200 | 1 | 0.48 | 3 | 5 | 0.53 | 0 | 0 | 0.00 | 100 | 100 | 0.43 | 0 | 0 |
| 100 | 1 | 0.50 | 5 | 15 | 0.58 | 0 | 0 | 0.00 | 100 | 100 | 0.40 | 0 | 1 |
| 50 | 1 | 0.53 | 3 | 9 | 0.65 | 0 | 0 | 0.00 | 100 | 100 | 0.34 | 0 | 2 |
| 40 | 1 | 0.54 | 5 | 7 | 0.73 | 0 | 0 | 0.00 | 100 | 100 | 0.37 | 0 | 2 |
| 30 | 1 | 0.49 | 6 | 9 | 0.78 | 0 | 0 | 0.00 | 100 | 100 | 0.33 | 0 | 2 |
| 20 | 1 | 0.51 | 7 | 11 | 0.87 | 0 | 0 | 0.01 | 97 | 99 | 0.28 | 1 | 3 |
| 10 | 1 | 0.47 | 5 | 12 | 0.98 | 0 | 0 | 0.17 | 19 | 38 | 0.35 | 0 | 0 |
| 50 | 10 | 0.35 | 4 | 11 | 0.92 | 0 | 0 | 0.32 | 0 | 1 | 0.81 | 0 | 0 |
| 40 | 10 | 0.34 | 7 | 15 | 0.92 | 0 | 0 | 0.26 | 2 | 12 | 0.78 | 0 | 0 |
| 30 | 10 | 0.33 | 7 | 15 | 0.94 | 0 | 0 | 0.21 | 6 | 22 | 0.76 | 0 | 0 |
| 20 | 10 | 0.33 | 6 | 15 | 0.97 | 0 | 0 | 0.16 | 4 | 30 | 0.70 | 0 | 0 |
| 10 | 10 | 0.36 | 6 | 11 | 1.00 | 0 | 0 | 0.31 | 2 | 6 | 0.66 | 0 | 0 |
| 50 | 5 | 0.38 | 5 | 10 | 0.83 | 0 | 0 | 0.05 | 60 | 87 | 0.63 | 0 | 0 |
| 40 | 5 | 0.33 | 10 | 13 | 0.84 | 0 | 0 | 0.04 | 80 | 94 | 0.60 | 0 | 0 |
| 30 | 5 | 0.37 | 6 | 11 | 0.88 | 0 | 0 | 0.03 | 85 | 96 | 0.56 | 0 | 0 |
| 20 | 5 | 0.39 | 4 | 13 | 0.94 | 0 | 0 | 0.04 | 71 | 90 | 0.53 | 0 | 0 |
| 10 | 5 | 0.39 | 11 | 18 | 0.99 | 0 | 0 | 0.18 | 12 | 30 | 0.49 | 0 | 0 |
| 50 | 2 | 0.46 | 3 | 6 | 0.72 | 0 | 0 | 0.00 | 100 | 100 | 0.44 | 0 | 0 |
| 40 | 2 | 0.42 | 10 | 15 | 0.75 | 0 | 0 | 0.00 | 100 | 100 | 0.45 | 0 | 0 |
| 30 | 2 | 0.45 | 4 | 6 | 0.80 | 0 | 0 | 0.01 | 98 | 100 | 0.39 | 0 | 1 |
| 20 | 2 | 0.53 | 2 | 4 | 0.90 | 0 | 0 | 0.02 | 92 | 98 | 0.37 | 0 | 0 |
| 10 | 2 | 0.42 | 6 | 14 | 0.99 | 0 | 0 | 0.15 | 23 | 41 | 0.38 | 0 | 0 |
| 5 | 1 | 0.44 | 4 | 7 | 1.00 | 0 | 0 | 0.95 | 0 | 0 | 0.91 | 0 | 0 |
| 3 | 1 | 0.23 | 15 | 23 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |
| 1 | 1 | 0.17 | 23 | 38 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |

Table C: Summary table of empirical SigClust $p$-value distribution over 100 replications based on four methods under different settings in Simulation 3.1. The mean and the numbers of $p$-values which are less than 0.05 (denoted as N5) and 0.1 (denoted as N10) are reported (d=200, n=100).

| | | True | | | Sample | | | Hard | | | Soft | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $v$ | $w$ | Mean | N5 | N10 | Mean | N5 | N10 | Mean | N5 | N10 | Mean | N5 | N10 |
| 1000 | 1 | 0.49 | 5 | 10 | 0.50 | 4 | 10 | 0.14 | 44 | 62 | 0.48 | 6 | 11 |
| 200 | 5 | 0.37 | 3 | 14 | 0.76 | 0 | 0 | 0.47 | 0 | 0 | 0.71 | 0 | 0 |
| 100 | 10 | 0.32 | 10 | 18 | 0.87 | 0 | 0 | 0.65 | 0 | 0 | 0.83 | 0 | 0 |
| 40 | 25 | 0.28 | 9 | 20 | 0.98 | 0 | 0 | 0.90 | 0 | 0 | 0.96 | 0 | 0 |
| 20 | 50 | 0.27 | 14 | 25 | 1.00 | 0 | 0 | 0.98 | 0 | 0 | 0.99 | 0 | 0 |
| 10 | 100 | 0.20 | 21 | 34 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |
| 200 | 1 | 0.52 | 3 | 6 | 0.55 | 0 | 1 | 0.01 | 96 | 99 | 0.46 | 0 | 3 |
| 100 | 1 | 0.52 | 5 | 7 | 0.60 | 0 | 0 | 0.01 | 96 | 99 | 0.47 | 0 | 0 |
| 50 | 1 | 0.50 | 6 | 9 | 0.62 | 0 | 0 | 0.00 | 100 | 100 | 0.38 | 0 | 2 |
| 40 | 1 | 0.52 | 5 | 7 | 0.64 | 0 | 0 | 0.00 | 99 | 100 | 0.35 | 0 | 0 |
| 30 | 1 | 0.49 | 4 | 10 | 0.69 | 0 | 0 | 0.01 | 100 | 100 | 0.33 | 0 | 1 |
| 20 | 1 | 0.52 | 9 | 10 | 0.77 | 0 | 0 | 0.01 | 94 | 99 | 0.29 | 1 | 4 |
| 10 | 1 | 0.53 | 6 | 8 | 0.93 | 0 | 0 | 0.11 | 28 | 61 | 0.31 | 0 | 5 |
| 50 | 10 | 0.32 | 8 | 20 | 0.89 | 0 | 0 | 0.48 | 0 | 0 | 0.79 | 0 | 0 |
| 40 | 10 | 0.38 | 5 | 11 | 0.92 | 0 | 0 | 0.46 | 0 | 0 | 0.81 | 0 | 0 |
| 30 | 10 | 0.34 | 6 | 10 | 0.92 | 0 | 0 | 0.38 | 0 | 1 | 0.77 | 0 | 0 |
| 20 | 10 | 0.34 | 4 | 12 | 0.94 | 0 | 0 | 0.29 | 0 | 4 | 0.72 | 0 | 0 |
| 10 | 10 | 0.34 | 10 | 19 | 0.98 | 0 | 0 | 0.31 | 1 | 7 | 0.63 | 0 | 0 |
| 50 | 5 | 0.38 | 3 | 12 | 0.81 | 0 | 0 | 0.15 | 10 | 32 | 0.64 | 0 | 0 |
| 40 | 5 | 0.39 | 9 | 12 | 0.82 | 0 | 0 | 0.12 | 21 | 43 | 0.63 | 0 | 0 |
| 30 | 5 | 0.37 | 7 | 17 | 0.84 | 0 | 0 | 0.09 | 32 | 59 | 0.60 | 0 | 0 |
| 20 | 5 | 0.40 | 7 | 15 | 0.88 | 0 | 0 | 0.07 | 40 | 72 | 0.52 | 0 | 0 |
| 10 | 5 | 0.38 | 11 | 17 | 0.96 | 0 | 0 | 0.14 | 15 | 37 | 0.45 | 0 | 0 |
| 50 | 2 | 0.43 | 3 | 11 | 0.69 | 0 | 0 | 0.02 | 96 | 99 | 0.47 | 0 | 0 |
| 40 | 2 | 0.44 | 5 | 8 | 0.71 | 0 | 0 | 0.01 | 95 | 99 | 0.45 | 0 | 0 |
| 30 | 2 | 0.47 | 7 | 11 | 0.75 | 0 | 0 | 0.02 | 95 | 100 | 0.43 | 0 | 1 |
| 20 | 2 | 0.42 | 4 | 13 | 0.80 | 0 | 0 | 0.02 | 94 | 100 | 0.35 | 0 | 1 |
| 10 | 2 | 0.46 | 8 | 9 | 0.94 | 0 | 0 | 0.10 | 33 | 60 | 0.34 | 0 | 2 |
| 5 | 1 | 0.42 | 5 | 13 | 1.00 | 0 | 0 | 0.70 | 0 | 0 | 0.67 | 0 | 0 |
| 3 | 1 | 0.30 | 10 | 17 | 1.00 | 0 | 0 | 0.99 | 0 | 0 | 0.98 | 0 | 0 |
| 1 | 1 | 0.20 | 18 | 38 | 1.00 | 0 | 0 | 1.00 | 0 | 0 | 1.00 | 0 | 0 |