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Web-based Supplementary Materials for "Quantile Regression Analysis of Censored Longitudinal Data with Irregular Outcome-Dependent Follow-Up"

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Web Appendix A: Notation, Regularity Conditions, and Prelim

Define $\zeta_i^{\tau}(\boldsymbol{\beta};\boldsymbol{\alpha}) = \int_0^{\infty} \rho_{\tau}\left[Y_i(t) - \max\left\{c, \mathbf{X}_i(t)^{\top}\boldsymbol{\beta}\right\}\right]\left[dN_i^L(t) + \exp\{-\mathbf{h}_i(t)^{\top}\boldsymbol{\alpha}\}dN_i(t)\right]$, and

$$\boldsymbol{\ell}_i^{\tau}(\boldsymbol{\beta};\boldsymbol{\alpha}) = \int_0^{\infty} \mathbf{X}_i(t) I\left\{\mathbf{X}_i(t)^{\top}\boldsymbol{\beta} > c\right\}\left[I\left\{Y_i(t) < \mathbf{X}_i(t)^{\top}\boldsymbol{\beta}\right\} - \tau\right]\left[dN_i^L(t) + \exp\{-\mathbf{h}_i(t)^{\top}\boldsymbol{\alpha}\}dN_i(t)\right].$$

Then $\Psi_{\tau}(\boldsymbol{\beta};\boldsymbol{\alpha}) = n^{-1/2}\sum_{i=1}^{n}\zeta_i^{\tau}(\boldsymbol{\beta};\boldsymbol{\alpha})$ and $\mathbf{U}_{\tau}(\boldsymbol{\beta};\boldsymbol{\alpha}) = n^{-1/2}\sum_{i=1}^{n}\boldsymbol{\ell}_i^{\tau}(\boldsymbol{\beta};\boldsymbol{\alpha})$. Let $\psi_{\tau}(\boldsymbol{\beta};\boldsymbol{\alpha}) = E\{n^{-1/2}\Psi_{\tau}(\boldsymbol{\beta};\boldsymbol{\alpha})\}$, and $\boldsymbol{\mu}_{\tau}(\boldsymbol{\beta};\boldsymbol{\alpha}) = E\left\{n^{-1/2}\mathbf{U}_{\tau}(\boldsymbol{\beta};\boldsymbol{\alpha})\right\}$. Let $f_{Y(t)}\{y|\mathbf{Z}(t)\}$ denote the conditional density function of $Y(t)$ given $\mathbf{Z}(t)$.

We assume the following regularity conditions:

C1. (a) There exists $\gamma \in (0,1)$ such that $E[\int_0^{\infty} I\{\mathbf{X}(t)^{\top}\boldsymbol{\beta}_0(\gamma) > c\}\mathbf{X}(t)^{\otimes 2}\{dN^L(t) + I(L < t \leq R)\lambda_0(t)dt\}]$ is positive definite;

(b) The conditional density function $f_{Y(t)}\{y|\mathbf{Z}(t)\}$ is continuous and positive at $y = \mathbf{X}(t)^{\top}\boldsymbol{\beta}_0(\tau)$ for any $\tau \in [\gamma,\gamma']$, where $0 < \gamma < \gamma' < 1$.

C2. $\boldsymbol{\beta}_0(\tau)$ is continuously differentiable in τ and lies in the interior of a compact parameter space $\mathcal{B}$ for all $\tau \in [\gamma,\gamma']$.

C3. There exists a neighborhood of $\boldsymbol{\alpha}_0$, denoted by $\mathcal{A}$, such that

$$\frac{\partial\psi_{\tau}(\boldsymbol{\beta};\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}} = E\Big(\int_0^{\infty}\rho_{\tau}\left[Y(t) - \max\left\{c, \mathbf{X}(t)^{\top}\boldsymbol{\beta}\right\}\right]\{-\mathbf{h}(t)\}\exp\{-\mathbf{h}(t)^{\top}\boldsymbol{\alpha}\}dN(t)\Big)$$

is bounded uniformly in $\boldsymbol{\beta} \in \mathcal{B}$, $\boldsymbol{\alpha} \in \mathcal{A}$, and $\tau \in [\gamma,\gamma']$.

C4. $\zeta^{\tau}(\boldsymbol{\beta};\boldsymbol{\alpha})$ has finite first and second moments for any $\boldsymbol{\beta} \in \mathcal{B}$, $\boldsymbol{\alpha} \in \mathcal{A}$, and $\tau \in [\gamma,\gamma']$, where $\zeta^{\tau}(\boldsymbol{\beta};\boldsymbol{\alpha}) = \int_0^{\infty}\rho_{\tau}\left[Y(t) - \max\left\{c, \mathbf{X}(t)^{\top}\boldsymbol{\beta}\right\}\right]\left[dN^L(t) + \exp\{-\mathbf{h}(t)^{\top}\boldsymbol{\alpha}\}dN(t)\right]$.

C5. (a) The covariate space $\mathcal{Z}$ is compact, that is, $\sup_t \|\mathbf{Z}(t)\| < \infty$, where $\|\cdot\|$ stands for Euclidean norm;

(b) $\sup_{\boldsymbol{\alpha}\in\mathcal{A}}\int_0^{\infty}\exp\left\{-\mathbf{h}(t)^{\top}\boldsymbol{\alpha}\right\}dN(t)$ is bounded;

(c) $f_{Y(t)}\left\{\mathbf{X}(t)^{\top}\boldsymbol{\beta}_0(\tau)|\mathbf{Z}(t)\right\}$ is uniformly bounded for any $\mathbf{Z}(t) \in \mathcal{Z}$ and $\tau \in [\gamma,\gamma']$;

(d) For any $d \geq 0$, there exists a positive constant M^+ such that

$$\sup_{\tau\in[\gamma,\gamma']} E\left|\int_0^{\infty} I\left\{|\mathbf{X}(t)^{\top}\boldsymbol{\beta}_0(\tau) - c| \leq \|\mathbf{X}(t)\|d\right\}\left\{dN^L(t) + I(L < t \leq R)\lambda_0(t)dt\right\}\right| \leq M^+ \cdot d;$$

(e) $E\left[\int_0^\infty \mathbf{h}(t)\exp\left\{-\mathbf{h}(t)^\top\boldsymbol{\alpha}\right\}dN(t)\right]$ is uniformly bounded for $\boldsymbol{\alpha}\in\mathcal{A}$.

C6. $\inf_{\tau\in[\gamma,\gamma']} eigmin \mathbf{B}_\tau(\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0)>0$, where

$$\begin{aligned}\mathbf{B}_\tau(\boldsymbol{\beta};\boldsymbol{\alpha}_0) &= \frac{\partial\boldsymbol{\mu}_\tau(\boldsymbol{\beta};\boldsymbol{\alpha}_0)}{\partial\boldsymbol{\beta}}\\ &= E\left[\int_0^\infty \mathbf{X}(t)^{\otimes 2} I\left\{\mathbf{X}(t)^\top\boldsymbol{\beta}>c\right\} f_{Y(t)}\left\{\mathbf{X}(t)^\top\boldsymbol{\beta}|\mathbf{X}(t)\right\}\left\{dN^L(t)+I(L<t\le R)\lambda_0(t)dt\right\}\right].\end{aligned}$$

and $eigmin(\cdot)$ denotes the minimum eigenvalue of a matrix.

The assumed regularity conditions are reasonable in real settings. Condition C1 is critical to ensure that $\boldsymbol{\beta}=\boldsymbol{\beta}_0(\tau)$ is identifiable from the data and is a unique minimizer of $\Psi_\tau(\boldsymbol{\beta};\boldsymbol{\alpha}_0)$. Condition C2 assumes the smoothness of $\boldsymbol{\beta}_0(\tau)$. By Condition C3, the variability associated with $\hat{\boldsymbol{\alpha}}$ has only tractable impact on the estimation of $\boldsymbol{\beta}_0(\tau)$. Condition C4 is a trivial condition to attain the convergence of the proposed objective function $\Psi_\tau(\boldsymbol{\beta},\boldsymbol{\alpha})$ to some limit pointwisely in τ. Condition C5 mainly requires the compactness of covariate space and some density functions. Such requirements are commonly seen in censored quantile regression literature. By Condition C6, matrix $\mathbf{B}_\tau(\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0)$ is invertible for all $\tau\in[\gamma,1)$, and moreover its inverse matrix is uniformly bounded. This helps justify the tightness of the limit process of $\sqrt{n}(\hat{\boldsymbol{\beta}}(\tau)-\boldsymbol{\beta}_0(\tau))$. With Conditions C5 and C6, the arguments for the asymptotic distribution of the proposed estimator are much simplified.

Given that the intensity ratio weights in the proposed estimating equation involve $\hat{\boldsymbol{\alpha}}$, the large sample studies of $\hat{\boldsymbol{\beta}}(\tau)$ need to be concerned with the asymptotic properties of $\hat{\boldsymbol{\alpha}}$. By following the arguments of Andersen and Gill (1982) with slightly stronger conditions imposed, we can show that $\hat{\boldsymbol{\alpha}}$ converges to $\boldsymbol{\alpha}_0$ almost surely and

$$\sqrt{n}(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_0)+n^{-1/2}\mathbf{J}(\boldsymbol{\alpha}_0)^{-1}\sum_{i=1}^n \boldsymbol{\iota}_i(\boldsymbol{\alpha}_0)\xrightarrow{d}0,$$

where

$$\mathbf{J}(\boldsymbol{\alpha}) = -E\Bigg[\frac{1}{n}\sum_{i=1}^{n}\int_0^\infty\Bigg\{\frac{\sum_{j=1}^n I(L_j < t \leq R_j)\mathbf{h}_j(t)^{\otimes 2}e^{\mathbf{h}_j(t)^\top\boldsymbol{\alpha}}}{\sum_{j=1}^n I(L_j < t \leq R_j)e^{\mathbf{h}_j(t)^\top\boldsymbol{\alpha}}} - \left(\frac{\sum_{j=1}^n I(L_j < t \leq R_j)\mathbf{h}_j(t)e^{\mathbf{h}_j(t)^\top\boldsymbol{\alpha}}}{\sum_{j=1}^n I(L_j < t \leq R_j)e^{\mathbf{h}_j(t)^\top\boldsymbol{\alpha}}}\right)^{\otimes 2}\Bigg\}dN_i(t)\Bigg] \tag{A.1}$$

and

$$\boldsymbol{\iota}_i(\boldsymbol{\alpha}) = \int_0^\infty\left\{\mathbf{h}_i(t) - \frac{\sum_{j=1}^n I(L_j < t \leq R_j)\mathbf{h}_j(t)e^{\mathbf{h}_j(t)^\top\boldsymbol{\alpha}}}{\sum_{j=1}^n I(L_j < t \leq R_j)e^{\mathbf{h}_j(t)^\top\boldsymbol{\alpha}}}\right\}\left(dN_i(t) - I(L_i < t \leq R_i)\lambda_0(t)e^{\mathbf{h}_i(t)^\top\boldsymbol{\alpha}}dt\right). \tag{A.2}$$

These results on $\hat{\boldsymbol{\alpha}}$ will be used in the proofs of both Theorems 1 and 2.

Web Appendix B: Proof of Theorem 1

Proof of Theorem 1: Our first step is to prove that $\psi_\tau(\boldsymbol{\beta};\boldsymbol{\alpha}_0)$ has a unique minimizer at $\boldsymbol{\beta} = \boldsymbol{\beta}_0(\tau)$. Define $\nu_\tau\{\boldsymbol{\beta};\mathbf{Z}(t)\} = E\left(\rho_\tau\left[Y(t) - \max\left\{c, \mathbf{X}(t)^\top\boldsymbol{\beta}\right\}\right]\big|\mathbf{Z}(t)\right)$. We will show that $\nu_\tau\{\boldsymbol{\beta};\mathbf{Z}(t)\} \geq \nu_\tau\{\boldsymbol{\beta}_0(\tau);\mathbf{Z}(t)\}$ for any given $\boldsymbol{\beta} \neq \boldsymbol{\beta}_0(\tau)$ by examining all possible situations listed below.

(A) When $\mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau) \leq c$ and $\mathbf{X}(t)^\top\boldsymbol{\beta} \leq c$, $\nu_\tau\{\boldsymbol{\beta}_0(\tau);\mathbf{Z}(t)\} = \nu_\tau\{\boldsymbol{\beta};\mathbf{Z}(t)\}$.

(B) When $\mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau) \leq c$ and $\mathbf{X}(t)^\top\boldsymbol{\beta} > c$,

$$\begin{aligned}
&\nu_\tau\{\boldsymbol{\beta}_0(\tau);\mathbf{Z}(t)\} - \nu_\tau\{\boldsymbol{\beta};\mathbf{Z}(t)\} \\
&= E\left[I\{Y(t) = c\}(\tau - 1)\left\{\mathbf{X}(t)^\top\boldsymbol{\beta} - c\right\}\big|\mathbf{Z}(t)\right] \\
&\quad + E\left(I\left\{c < Y(t) \leq \mathbf{X}(t)^\top\boldsymbol{\beta}\right\}\left[\tau\left\{\mathbf{X}(t)^\top\boldsymbol{\beta} - c\right\} + Y(t) - \mathbf{X}(t)^\top\boldsymbol{\beta}\right]\big|\mathbf{Z}(t)\right) \\
&\quad + E\left[I\left\{Y(t) > \mathbf{X}(t)^\top\boldsymbol{\beta}\right\}\tau\left\{\mathbf{X}(t)^\top\boldsymbol{\beta} - \tau\right\}\big|\mathbf{Z}(t)\right] \\
&\leq E\left([I\{Y(t) = c\}(\tau - 1) + \tau I\{Y(t) > c\}]\big|\mathbf{Z}(t)\right)\left\{\mathbf{X}(t)^\top\boldsymbol{\beta} - c\right\}.
\end{aligned}$$

Since $\mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau) \leq c$, we have that $E[I\{Y(t) = c\}|\mathbf{Z}(t)] \geq \tau$ and $E[I\{Y(t) > c\}|\mathbf{Z}(t)] \leq 1 - \tau$. Therefore, $\nu_\tau\{\boldsymbol{\beta}_0(\tau);\mathbf{Z}(t)\} - \nu_\tau\{\boldsymbol{\beta};\mathbf{Z}(t)\} \leq 0$.

(C) When $\mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau) > c$,

$$
\begin{aligned}
&\nu_\tau\{\boldsymbol{\beta}_0(\tau);\mathbf{Z}(t)\} - \nu_\tau\{\boldsymbol{\beta};\mathbf{Z}(t)\} \\
&= (1-\tau)P\left\{Y(t) \le \mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau)\right\}\left[\mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau) - \max\left\{c, \mathbf{X}(t)^\top\boldsymbol{\beta}\right\}\right] \\
&\quad - \tau P\left\{Y(t) > \mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau)\right\}\left[\mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau) - \max\left\{c, \mathbf{X}(t)^\top\boldsymbol{\beta}\right\}\right] \\
&\quad + E\left(\int_{\max\{c,\mathbf{X}(t)^\top\boldsymbol{\beta}\}}^{\mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau)} \left[y - \max\left\{c, \mathbf{X}(t)^\top\boldsymbol{\beta}\right\}\right] f_{Y(t)}\{y|\mathbf{Z}(t)\}dy\middle|\mathbf{Z}(t)\right) \\
&= E\left(\int_{\max\{c,\mathbf{X}(t)^\top\boldsymbol{\beta}\}}^{\mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau)} \left[\max\left\{c, \mathbf{X}(t)^\top\boldsymbol{\beta}\right\} - y\right] f_{Y(t)}\{y|\mathbf{Z}(t)\}dy\middle|\mathbf{Z}(t)\right) \\
&\le 0 \qquad \text{(B.1)}
\end{aligned}
$$

When $\mathbf{X}(t)^\top\boldsymbol{\beta} \neq \mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau)$, we would have $\max\{c, \mathbf{X}(t)^\top\boldsymbol{\beta}\} \neq \mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau)$. Under condition C1(b), there must exist an interval between $\max\{c, \mathbf{X}(t)^\top\boldsymbol{\beta}\}$ and $\mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau)$ such that for any y in this interval, $f_{Y(t)}\{y|\mathbf{Z}(t)\} > 0$ and $\left[\max\left\{c, \mathbf{X}(t)^\top\boldsymbol{\beta}\right\} - y\right] f_{Y(t)}\{y|\mathbf{Z}(t)\} < 0$, which would imply a strict inequality in (B.1). Hence, the equality in (B.1) holds if and only if $\mathbf{X}(t)^\top\boldsymbol{\beta} = \mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau)$.

Note that condition C1(a) implies that $E[\int_0^\infty I\{\mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau) > c\}\mathbf{X}(t)^{\otimes 2}\{dN^L(t) + I(L < t \le R)\lambda_0(t)dt\}]$ is positive definite for any $\tau \in [\gamma, \gamma']$. Hence, when $\boldsymbol{\beta} \neq \boldsymbol{\beta}_0(\tau)$,

$$
E\left[\int_0^\infty I\left\{\mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau) > c\right\}\left\{\mathbf{X}(t)^\top\boldsymbol{\beta} - \mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau)\right\}^2\left\{dN^L(t) + I(L < t \le R)\lambda_0(t)dt\right\}\right] > 0 \qquad \text{(B.2)}
$$

for $\tau \in [\gamma, \gamma']$. Because $\nu_\tau\{\boldsymbol{\beta}_0(\tau);\mathbf{Z}(t)\} < \nu_\tau\{\boldsymbol{\beta};\mathbf{Z}(t)\}$ when $\mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau) > c$ and $\mathbf{X}(t)^\top\boldsymbol{\beta} \neq \mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau)$, (B.2) implies

$$
E\left[\int_0^\infty I\left\{\mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau) > c\right\}\left[\nu_\tau\{\boldsymbol{\beta};\mathbf{Z}(t)\} - \nu_\tau\{\boldsymbol{\beta}_0(\tau);\mathbf{Z}(t)\}\right]\left\{dN^L(t) + I(L < t \le R)\lambda_0(t)dt\right\}\right]
$$

is also greater than 0.

Given the result that $\nu_\tau\{\boldsymbol{\beta};\mathbf{Z}(t)\} \geq \nu_\tau\{\boldsymbol{\beta}_0(\tau);\mathbf{Z}(t)\}$ for any given $\boldsymbol{\beta} \neq \boldsymbol{\beta}_0(\tau)$, we then have

$$
\begin{aligned}
&\psi_\tau(\boldsymbol{\beta};\boldsymbol{\alpha}_0) - \psi_\tau\left\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\right\} \\
&= E\Bigg(\int_0^\infty I\left\{\mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau) \leq c\right\} I\left\{\mathbf{X}(t)^\top\boldsymbol{\beta} \leq c\right\}\left[\nu_\tau\{\boldsymbol{\beta};\mathbf{Z}(t)\} - \nu_\tau\left\{\boldsymbol{\beta}_0(\tau);\mathbf{Z}(t)\right\}\right] \\
&\qquad\qquad \times \left\{dN^L(t) + I(L < t \leq R)\lambda_0(t)dt\right\}\Bigg) \\
&\quad + E\Bigg(\int_0^\infty I\left\{\mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau) \leq c\right\} I\left\{\mathbf{X}(t)^\top\boldsymbol{\beta} > c\right\}\left[\nu_\tau\{\boldsymbol{\beta};\mathbf{Z}(t)\} - \nu_\tau\left\{\boldsymbol{\beta}_0(\tau);\mathbf{Z}(t)\right\}\right] \\
&\qquad\qquad \times \left\{dN^L(t) + I(L < t \leq R)\lambda_0(t)dt\right\}\Bigg) \\
&\quad + E\Bigg(\int_0^\infty I\left\{\mathbf{X}(t)^\top\boldsymbol{\beta}_0(\tau) > c\right\}\left[\nu_\tau\{\boldsymbol{\beta};\mathbf{Z}(t)\} - \nu_\tau\left\{\boldsymbol{\beta}_0(\tau);\mathbf{Z}(t)\right\}\right] \\
&\qquad\qquad \times \left\{dN^L(t) + I(L < t \leq R)\lambda_0(t)dt\right\}\Bigg) \\
&> 0
\end{aligned}
$$

for any $\boldsymbol{\beta} \neq \boldsymbol{\beta}_0(\tau)$. Therefore, under condition C1, we prove that $\boldsymbol{\beta}_0(\tau)$ is a unique minimizer of $\psi_\tau(\boldsymbol{\beta};\boldsymbol{\alpha}_0)$.

Given $\hat{\boldsymbol{\alpha}} \xrightarrow{a.s.} \boldsymbol{\alpha}_0$, under condition C3, we have

$$
\sup_{\tau\in[\gamma,\gamma'],\ \boldsymbol{\beta}\in\mathcal{B}} \left|\psi_\tau(\boldsymbol{\beta};\hat{\boldsymbol{\alpha}}) - \psi_\tau(\boldsymbol{\beta};\boldsymbol{\alpha}_0)\right| \xrightarrow{a.s.} 0. \tag{B.3}
$$

Note that

$$
\begin{aligned}
\zeta_i^\tau(\boldsymbol{\beta};\boldsymbol{\alpha}) = \int_0^\infty \Big(&\tau\left[Y_i(t) - \max\left\{c, \mathbf{X}_i(t)^\top\boldsymbol{\beta}\right\}\right] \\
&- \tau I\left[Y_i(t) \leq \max\left\{c, \mathbf{X}_i(t)^\top\boldsymbol{\beta}\right\}\right]\left[Y_i(t) - \max\left\{c, \mathbf{X}_i(t)^\top\boldsymbol{\beta}\right\}\right] \\
&+ (1-\tau) I\left[Y_i(t) \leq \max\left\{c, \mathbf{X}_i(t)^\top\boldsymbol{\beta}\right\}\right]\left[\max\left\{c, \mathbf{X}_i(t)^\top\boldsymbol{\beta}\right\} - Y_i(t)\right]\Big) \\
&\times \left[dN_i^L(t) + \exp\left\{-\mathbf{h}_i(t)^\top\boldsymbol{\alpha}\right\} dN_i(t)\right]
\end{aligned}
$$

These three terms in the parenthesis $\Big(\cdot\Big)$ are either concave or convex functions of $\boldsymbol{\beta}$ and linear in τ, and $\exp\left\{-\mathbf{h}_i(t)^\top\boldsymbol{\alpha}\right\}$ is an either concave or convex function of $\boldsymbol{\alpha}$. This fact coupled with pointwise convergence by the strong law of large numbers given condition C4, implies the uniform convergence of $n^{-1/2}\Psi_\tau(\boldsymbol{\beta};\boldsymbol{\alpha})$ (Rockafellar, 1970 (Theorem 10.8)), i.e.

$$\sup_{\tau\in[\gamma,\gamma'],\ \boldsymbol{\beta}\in\mathcal{B},\ \boldsymbol{\alpha}\in\mathcal{A}} |n^{-1/2}\Psi_\tau(\boldsymbol{\beta};\boldsymbol{\alpha}) - \psi_\tau(\boldsymbol{\beta};\boldsymbol{\alpha})| \xrightarrow{a.s.} 0.$$

This, coupled with (B.3), gives

$$\sup_{\boldsymbol{\beta}\in\mathcal{B},\ \tau\in[\gamma,\gamma']} |n^{-1/2}\Psi_\tau(\boldsymbol{\beta};\hat{\boldsymbol{\alpha}}) - \psi_\tau(\boldsymbol{\beta};\boldsymbol{\alpha}_0)| \xrightarrow{a.s.} 0. \tag{B.4}$$

With $\psi_\tau\left\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\right\} = 0$ and $\Psi_\tau\left\{\hat{\boldsymbol{\beta}}(\tau);\hat{\boldsymbol{\alpha}}\right\} = 0$, some simple algebraic manipulation shows that

$$\sup_{\tau\in[\gamma,\gamma']}\left|\psi_\tau\left\{\hat{\boldsymbol{\beta}}(\tau);\boldsymbol{\alpha}_0\right\} - \psi_\tau\left\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\right\}\right| \le \sup_{\tau\in[\gamma,\gamma']}\left|\psi\left\{\hat{\boldsymbol{\beta}}(\tau);\boldsymbol{\alpha}_0\right\} - n^{-1/2}\Psi_\tau\left\{\hat{\boldsymbol{\beta}}(\tau);\hat{\boldsymbol{\alpha}}\right\}\right|.$$

By (B.4), we then have

$$\sup_{\tau\in[\gamma,\gamma']}\left|\psi_\tau\left\{\hat{\boldsymbol{\beta}}(\tau);\boldsymbol{\alpha}_0\right\} - \psi_\tau\left\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\right\}\right| \xrightarrow{a.s.} 0. \tag{B.5}$$

Based on (B.5), we can prove uniform strong convergency of $\hat{\boldsymbol{\beta}}(\tau)$ by following similar arguments in the proof of theorem 3 in Huang and Peng (2009). Specifically, we need to prove that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\sup_{\tau\in[\gamma,\gamma']}|\psi_\tau\left\{\boldsymbol{\beta}(\tau);\boldsymbol{\alpha}_0\right\} - \psi_\tau\left\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\right\}| < \delta$, then $\sup_{\tau\in[\gamma,\gamma']}\|\boldsymbol{\beta}(\tau)-\boldsymbol{\beta}_0(\tau)\| < \epsilon$. Suppose that this is not true. Then, there must exist a constant $\epsilon^* > 0$. For any $\left\{\frac{1}{k}: k = 1,2,\ldots\right\}$, there exists $(\boldsymbol{\beta}_k,\tau_k)$ such that $|\psi_{\tau_k}\left\{\boldsymbol{\beta}_k;\boldsymbol{\alpha}_0\right\} - \psi_{\tau_k}\left\{\boldsymbol{\beta}_0(\tau_k);\boldsymbol{\alpha}_0\right\}| < \frac{1}{k}$ but $\|\boldsymbol{\beta}_k - \boldsymbol{\beta}_0(\tau_k)\| > \epsilon^*$. Since $\boldsymbol{\mathcal{B}}$ is a compact space, there exists a subsequence of $(\boldsymbol{\beta}_k,\tau_k)$ that converges to, say, $(\boldsymbol{\beta}^*,\tau^*)$. Then, we have that $\psi_{\tau^*}(\boldsymbol{\beta}^*;\boldsymbol{\alpha}_0) = \psi_{\tau^*}\{\boldsymbol{\beta}_0(\tau^*);\boldsymbol{\alpha}_0\}$ but $\|\boldsymbol{\beta}^* - \boldsymbol{\beta}_0(\tau^*)\| \ge \epsilon^*$. This contradicts that $\boldsymbol{\beta}_0(\tau^*)$ is a unique minimizer of $\psi_{\tau^*}(\boldsymbol{\beta};\boldsymbol{\alpha}_0)$. Therefore, it is proved that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\sup_{\tau\in[\gamma,\gamma']}|\psi_\tau\left\{\boldsymbol{\beta}(\tau);\boldsymbol{\alpha}_0\right\} - \psi_\tau\left\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\right\}| < \delta$, then $\sup_{\tau\in[\gamma,\gamma']}\|\boldsymbol{\beta}(\tau)-\boldsymbol{\beta}_0(\tau)\| < \epsilon$. Consequently, given $\sup_{\tau\in[\gamma,\gamma']}\left|\psi_\tau\left\{\hat{\boldsymbol{\beta}}(\tau);\boldsymbol{\alpha}_0\right\} - \psi_\tau\left\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\right\}\right| \xrightarrow{a.s.} 0$, it follows that $\sup_{\tau\in[\gamma,\gamma']}\|\hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau)\| \xrightarrow{a.s.} 0$. The proof of Theorem 1 is completed.

Web Appendix C: Proof of Theorem 2

Lemma 1.

$$\sup_{\tau\in[\gamma,\gamma']}\left\|\mathbf{U}_\tau\left\{\hat{\boldsymbol{\beta}}(\tau);\hat{\boldsymbol{\alpha}}\right\}-\mathbf{U}_\tau\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\}-n^{1/2}\left[\boldsymbol{\mu}_\tau\left\{\hat{\boldsymbol{\beta}}(\tau);\hat{\boldsymbol{\alpha}}\right\}-\boldsymbol{\mu}_\tau\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\}\right]\right\|\xrightarrow{p}0.$$

Proof of Lemma 1:

This lemma can be proved by using the results in Alexander (1984) and the arguments for theorem 1 of Lai and Ying (1988). We only need to show that

$$\sup_{\tau\in[\gamma,\gamma']}Var\left[\boldsymbol{\ell}_i^\tau\left\{\hat{\boldsymbol{\beta}}(\tau);\hat{\boldsymbol{\alpha}}\right\}-\boldsymbol{\ell}_i^\tau\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\}\right]\xrightarrow{p}0. \tag{C.1}$$

Under condition C5(a) and (b), there exists a finite number M_1 such that when $\hat{\boldsymbol{\alpha}}\in\mathcal{A}$,

$$\begin{aligned}
&\sup_{\tau\in[\gamma,\gamma']}Var\left[\boldsymbol{\ell}_i^\tau\left\{\hat{\boldsymbol{\beta}}(\tau);\hat{\boldsymbol{\alpha}}\right\}-\boldsymbol{\ell}_i^\tau\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\}\right]\\
&\le E\left[\boldsymbol{\ell}_i^\tau\left\{\hat{\boldsymbol{\beta}}(\tau);\hat{\boldsymbol{\alpha}}\right\}-\boldsymbol{\ell}_i^\tau\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\}\right]^2\\
&\le M_1\cdot\sup_{\tau\in[\gamma,\gamma']}E\|\boldsymbol{\ell}_i^\tau\left\{\hat{\boldsymbol{\beta}}(\tau);\hat{\boldsymbol{\alpha}}\right\}-\boldsymbol{\ell}_i^\tau\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\}\|\\
&\le M_1\cdot\sup_{\tau\in[\gamma,\gamma']}E\bigg\|\int_0^\infty\mathbf{X}_i(t)I\left\{\mathbf{X}_i(t)^\top\hat{\boldsymbol{\beta}}(\tau)>c\right\}I\left\{\mathbf{X}_i(t)^\top\boldsymbol{\beta}_0(\tau)>c\right\}\\
&\quad\times\left[I\left\{Y_i(t)\le\mathbf{X}_i(t)^\top\hat{\boldsymbol{\beta}}(\tau)\right\}-I\left\{Y_i(t)\le\mathbf{X}_i(t)^\top\boldsymbol{\beta}_0(\tau)\right\}\right]\left\{dN_i^L(t)+I(L_i<t\le R_i)\lambda_0(t)dt\right\}\bigg\|\\
&+M_1\cdot\sup_{\tau\in[\gamma,\gamma']}E\bigg\|\int_0^\infty\mathbf{X}_i(t)I\left\{\mathbf{X}_i(t)^\top\hat{\boldsymbol{\beta}}(\tau)>c\right\}I\left\{\mathbf{X}_i(t)^\top\boldsymbol{\beta}_0(\tau)\le c\right\}\\
&\quad\times\left[I\left\{Y_i(t)\le\mathbf{X}_i(t)^\top\hat{\boldsymbol{\beta}}(\tau)\right\}-\tau\right]\left\{dN_i^L(t)+I(L_i<t\le R_i)\lambda_0(t)dt\right\}\bigg\|\\
&+M_1\cdot\sup_{\tau\in[\gamma,\gamma']}E\bigg\|\int_0^\infty\mathbf{X}_i(t)\left\{\mathbf{X}_i(t)^\top\hat{\boldsymbol{\beta}}(\tau)\le c\right\}I\left\{\mathbf{X}_i(t)^\top\boldsymbol{\beta}_0(\tau)>c\right\}\\
&\quad\times\left[I\left\{Y_i(t)\le\mathbf{X}_i(t)^\top\boldsymbol{\beta}_0(\tau)\right\}-\tau\right]\bigg]\left\{dN_i^L(t)+I(L_i<t\le R_i)\lambda_0(t)dt\right\}\bigg\|\\
&+M_1\cdot\sup_{\tau\in[\gamma,\gamma']}E\bigg\|\int_0^\infty\mathbf{X}_i(t)I\left\{\mathbf{X}_i(t)^\top\hat{\boldsymbol{\beta}}(\tau)>c\right\}\left[I\left\{Y_i(t)\le\mathbf{X}_i(t)^\top\hat{\boldsymbol{\beta}}(\tau)\right\}-\tau\right]\\
&\qquad\qquad\times\left[\exp\left\{-\mathbf{h}_i(t)^\top\boldsymbol{\alpha}_0\right\}-\exp\left\{-\mathbf{h}_i(t)^\top\hat{\boldsymbol{\alpha}}\right\}\right]dN_i(t)\bigg\|\\
&=(I)+(II)+(III)+(IV)
\end{aligned}$$

Under condition C5(a) - (c) and Theorem 1,

$$(I) \leq M_1 \cdot \sup_{\tau \in [\gamma, \gamma']} E \left\| \left[\int_0^\infty \mathbf{X}_i(t)^{\otimes 2} I \left\{ \mathbf{X}_i(t)^\top \boldsymbol{\beta} > c \right\} I \left\{ \mathbf{X}_i(t)^\top \boldsymbol{\beta}_0(\tau) > c \right\} f_{Y_i(t)}\{\mathbf{X}_i(t)^\top \boldsymbol{\beta}_0(\tau) | \mathbf{Z}_i(t)\} \right.\right.$$
$$\left.\left. \times \left\{ dN_i^L(t) + I(L_i < t \leq R_i)\lambda_0(t)dt \right\} + o_p(1) \right] \left\{ \hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau) \right\} \right\|$$
$$\xrightarrow{p} 0.$$

When $\left\{\mathbf{X}_i(t)^\top \hat{\boldsymbol{\beta}}(\tau) - c\right\}\left\{\mathbf{X}_i(t)^\top \boldsymbol{\beta}_0(\tau) - c\right\} \leq 0$, it is easy to see that $|\mathbf{X}_i(t)^\top \boldsymbol{\beta}_0(\tau) - c| \leq |\mathbf{X}_i(t)^\top \boldsymbol{\beta}_0(\tau) - \mathbf{X}_i(t)^\top \hat{\boldsymbol{\beta}}(\tau)| \leq \|\mathbf{X}_i(t)\| \|\boldsymbol{\beta}_0(\tau) - \hat{\boldsymbol{\beta}}(\tau)\|$. Under condition C5(a), (b), and (d) and Theorem 1,

$$(II) \leq \sup_{\tau \in [\gamma, \gamma']} E \left\| \int_0^\infty \mathbf{X}_i(t) I \left\{ |\mathbf{X}_i(t)^\top \boldsymbol{\beta}_0(\tau) - c| \leq \|\mathbf{X}_i(t)\| \|\boldsymbol{\beta}_0(\tau) - \hat{\boldsymbol{\beta}}(\tau)\| \right\} \right.$$
$$\left. \times \left[I \left\{ Y_i(t) \leq \mathbf{X}_i(t)^\top \hat{\boldsymbol{\beta}}(\tau) \right\} - \tau \right] \left\{ dN_i^L(t) + I(L_i < t \leq R_i)\lambda_0(t)dN_i(t) \right\} \right\|$$
$$\xrightarrow{p} 0.$$

Similarly, it can be shown that $(III) \xrightarrow{p} 0$.

Under condition C5(a) and (e) and the consistency of $\hat{\boldsymbol{\alpha}}$,

$$(IV) \leq M_1 \cdot \sup_{\tau \in [\gamma, \gamma']} \left\| \int_0^\infty \mathbf{X}_i(t) I \left\{ \mathbf{X}_i(t)^\top \hat{\boldsymbol{\beta}}(\tau) > c \right\} \left[I \left\{ Y_i(t) \leq \mathbf{X}_i(t)^\top \hat{\boldsymbol{\beta}}(\tau) \right\} - \tau \right] \right.$$
$$\left. \times \mathbf{h}_i(t) \exp(-\mathbf{h}_i(t)^\top \boldsymbol{\alpha}_0) dN_i(t) \right\| \|\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0\|$$
$$\xrightarrow{p} 0.$$

Therefore, we prove (C.1) and hence complete the proof of Lemma 1.

Proof of Theorem 2: According to Lemma 1 and $\mathbf{U}_\tau\left\{\hat{\boldsymbol{\beta}}(\tau);\hat{\boldsymbol{\alpha}}\right\}=\mathbf{0}$, we have

$$
\begin{aligned}
&-\mathbf{U}_\tau\left\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\right\}\\
&=n^{1/2}\left\{\boldsymbol{\mu}_\tau\left(\hat{\boldsymbol{\beta}};\hat{\boldsymbol{\alpha}}\right)-\boldsymbol{\mu}_\tau\left(\boldsymbol{\beta}_0;\boldsymbol{\alpha}_0\right)\right\}+o_{p:\tau\in[\gamma,\gamma']}(1)\\
&=\left[\mathbf{B}_\tau\left\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\right\}+o_p(1)\right]\cdot n^{1/2}\left\{\hat{\boldsymbol{\beta}}(\tau)-\boldsymbol{\beta}_0(\tau)\right\}+\mathbf{A}_\tau\left\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\right\}\cdot n^{1/2}\left\{\hat{\boldsymbol{\alpha}}(\tau)-\boldsymbol{\alpha}_0(\tau)\right\}\\
&\quad+o_{p:\tau\in[\gamma,\gamma']}(1)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{A}_\tau(\boldsymbol{\beta};\boldsymbol{\alpha})=\frac{\partial\boldsymbol{\mu}_\tau(\boldsymbol{\beta};\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}}=-\int_0^\infty \mathbf{X}_i(t)I\left\{\mathbf{X}_i(t)^\top\boldsymbol{\beta}>c\right\}\left[I\left\{Y_i(t)\le\mathbf{X}_i(t)^\top\boldsymbol{\beta}\right\}-\tau\right]&\\
\mathbf{h}_i(t)^\top\exp\left\{-\mathbf{h}_i(t)^\top\boldsymbol{\alpha}\right\}dN_i(t),&\qquad\text{(C.2)}
\end{aligned}
$$

and $o_{p:\tau\in[\gamma,\gamma']}(1)$ means uniform convergence in probability to zero over $\tau\in[\gamma,\gamma']$.

Under condition C6,

$$
\begin{aligned}
&n^{1/2}\left\{\hat{\boldsymbol{\beta}}(\tau)-\boldsymbol{\beta}_0(\tau)\right\}\\
&=-\mathbf{B}_\tau\left\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\right\}^{-1}\left[\mathbf{U}_\tau\left\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\right\}+\mathbf{A}_\tau\left\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\right\}\cdot n^{1/2}\left\{\hat{\boldsymbol{\alpha}}(\tau)-\boldsymbol{\alpha}_0(\tau)\right\}\right]+o_{p:\tau\in[\gamma,\gamma']}(1)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
n^{1/2}\{\hat{\boldsymbol{\beta}}(\tau)-\boldsymbol{\beta}_0(\tau)\}=n^{-1/2}\sum_{i=1}^n\Bigg[&-\mathbf{B}_\tau\left\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\right\}^{-1}\boldsymbol{\ell}_i^\tau\left\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\right\}\\
&+\mathbf{B}_\tau\left\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\right\}^{-1}\mathbf{A}_\tau\left\{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\right\}\mathbf{J}(\boldsymbol{\alpha}_0)^{-1}\boldsymbol{\iota}_i(\boldsymbol{\alpha}_0)\Bigg]+o_{p:\tau\in[\gamma,\gamma']}(1).
\end{aligned}
$$

According to the definition of quantile and the quantile regression model assumption, $\mathbf{X}_i(t)^\top\boldsymbol{\beta}_0(\tau)$ increases in τ. Since $\int_0^\infty \tau\mathbf{X}_i(t)I\{\mathbf{X}_i(t)^\top\boldsymbol{\beta}_0(\tau)>c\}\left[dN_i^L(t)+\exp\{-\mathbf{h}_i(t)^\top\boldsymbol{\alpha}_0\}dN_i(t)\right]$ and $\int_0^\infty \mathbf{X}_i(t)I\{\mathbf{X}_i(t)^\top\boldsymbol{\beta}_0(\tau)>c\}I\{Y_i(t)\le\mathbf{X}_i(t)^\top\boldsymbol{\beta}_0(\tau)\}\left[dN_i^L(t)+\exp\{-\mathbf{h}_i(t)^\top\boldsymbol{\alpha}_0\}dN_i(t)\right]$ are bounded and monotone functions on $\tau\in[\gamma,\gamma']$, $\{\boldsymbol{\ell}_i^\tau(\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0):\tau\in[\gamma,\gamma']\}$ is a Donsker class. By Donsker theorem and pointwise central limit theory, $n^{1/2}\left\{\hat{\boldsymbol{\beta}}(\tau)-\boldsymbol{\beta}_0(\tau)\right\}$ converges weakly to a Gaussian process with covariance matrix $\boldsymbol{\Sigma}(\tau_1,\tau_2)$ for $\tau\in[\gamma,\gamma']$, where

$$
\boldsymbol{\Sigma}(\tau_1,\tau_2)=E\left\{\boldsymbol{\xi}_i(\tau_1)\boldsymbol{\xi}_i(\tau_2)^\top\right\}\qquad\text{(C.3)}
$$

with

$$\boldsymbol{\xi}_i(\tau) = -\mathbf{B}_\tau \{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\}^{-1} \boldsymbol{\ell}_i^\tau \{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\} + \mathbf{B}_\tau \{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\}^{-1} \mathbf{A}_\tau \{\boldsymbol{\beta}_0(\tau);\boldsymbol{\alpha}_0\} \mathbf{J}(\boldsymbol{\alpha}_0)^{-1} \boldsymbol{\iota}_i(\boldsymbol{\alpha}_0).$$

Web Appendix D: Additional Simulation Results

Simulation results for larger τ's

Table F.1 present simulation results for Case 1 and Case 2 with $\tau = 0.85, 0.90, 0.95$ when $n = 200$.

[Table 1 about here.]

Simulation results with $n = 400$

Simulation results with $n = 400$ are presented in Table F.2.

[Table 2 about here.]

Robustness studies

We also investigated the robustness of the proposed estimation of model (1) to the potential mis-specification of the model for the follow-up time process. We consider three different scenarios of model mis-specification:

S1. The true follow-up intensity model is

$$P\{dN_i(t) = 1|\mathcal{H}_i(t)\} = I(L_i < t \leq R_i) v_i 0.2t \exp\left\{a_0 Y_i(t^-)\right\} dt,$$

which involves a subject-specific frailty v_i that follows $Gamma(2, 0.5)$ distribution but is not considered in the assumed model (7).

S2. The true follow-up intensity model is

$$P\{dN_i(t) = 1|\mathcal{H}_i(t)\} = I(L_i \leq t \leq R_i) 0.2t \exp\left\{a_0 Y_i(t^-) + a_1 Z_{i3}\right\} dt,$$

which contains a covariate Z_{i3} not included in the assumed model (7). We set a_1 as 0.5 or 1. Here $Z_{i3} \sim Bernoulli(0.5)$.

S3. The follow-up process does not follow a proportional intensity but a log-linear gap time model, by which, the gap times are generated by $Gamma(2, 0.5) \times \exp\{-0.2Y_i(t^-)\}$.

We performed the proposed estimation of model (1) assuming model (2) is the true model for the follow-up time process. In Tables F.3–F.5, we present the empirical bias and standard deviations of the proposed estimator along with those of the naive estimator which ignores outcome-dependent follow-up. The proposed estimator always has much smaller bias compared to that of the naive estimator. In Scenario S1, the magnitude of the empirical bias is mostly less than 10% of that of the true coefficient. In Scenario S2, when $a_1 = 0.5$ and $\tau = 0.25$ or 0.5, the bias of the proposed estimator is only slightly larger than the empirical bias observed in the case with correctly specified follow-up model. As expected, as a_1 is increased to postulate a larger departure from the assumed model, the bias of the proposed estimator becomes larger. When the type of the follow-up model is mis-specified, as in Scenario S3, the proposed estimator presents bias consistently across small or large τ's. The magnitudes of bias are not striking though, even smaller than those observed in Scenario S1.

[Table 3 about here.]

[Table 4 about here.]

[Table 5 about here.]

Web Appendix E: Additional Data Analyses for PBB Study

In Tables F.6 and F.7, we present the analysis results based on the visit time model that includes BMI as an additional covariate. In Tables F.8 and F.9, we present the results with the visit time model with the covariate chosen as the discretized initial PBB variable.

[Table 6 about here.]

[Table 7 about here.]

[Table 8 about here.]

[Table 9 about here.]

Web Appendix F: Histogram of Visit Gap Times

The histogram of the gap times between adjacent visits is provided in Figure F.1.

[Figure 1 about here.]

References

Alexander, K. S. (1984). Probability inequalities for empirical processes and a law of the iterated logarithm. *The Annals of Probability* **12,** 1041–1067.

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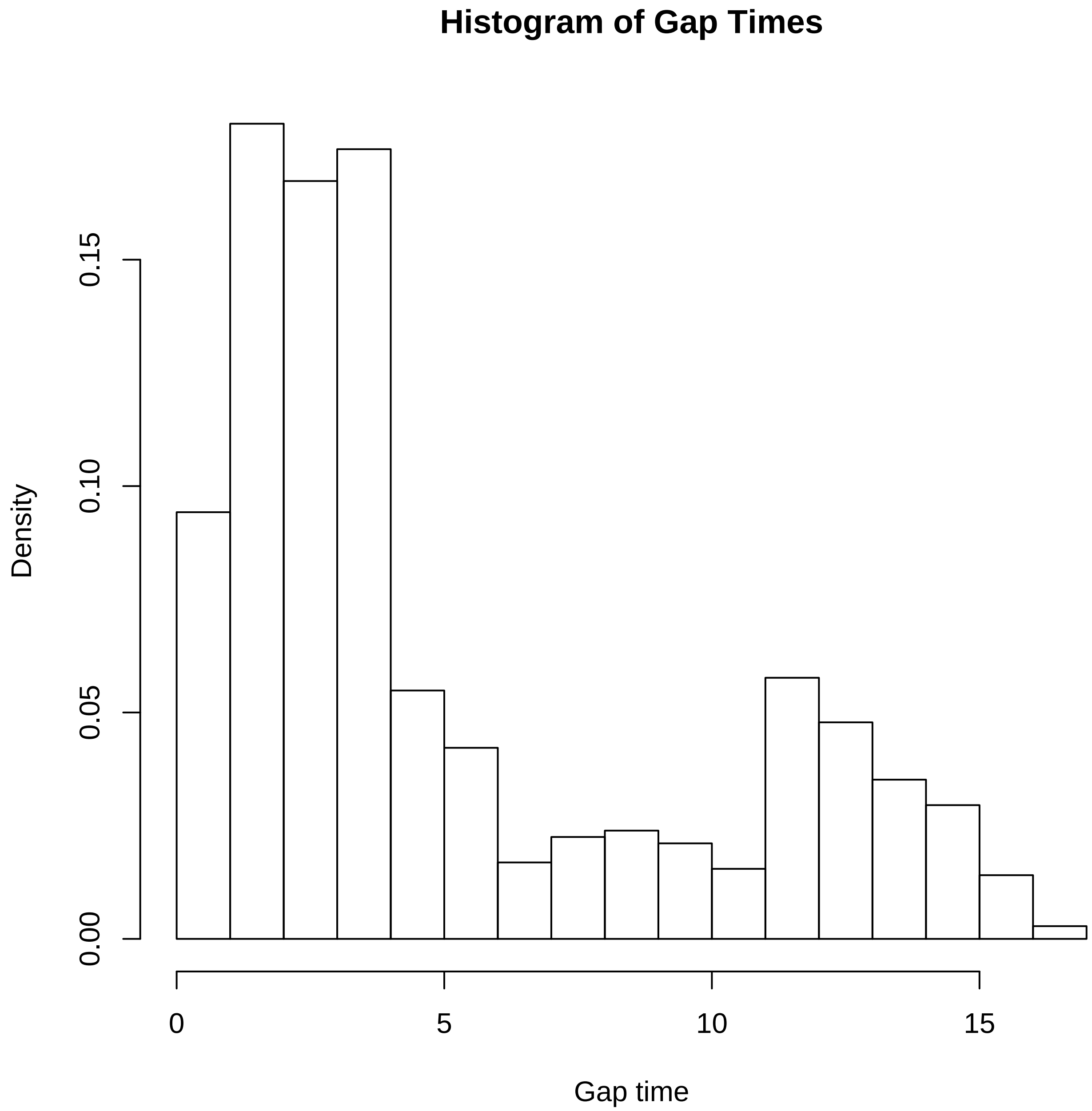

Figure F.1: Distribution of the gap times between adjacent visits.

Table F.1: Simulation studies that compared the proposed method and the naive approach: EmpSD – empirical standard deviation; AvgSD – the average of standard deviation estimates; Cov95 – the coverage rate of a 95% confidence interval.

		Naive		Proposed					
						Bootstrapping		Sample-based	
Effect	True	Bias	EmpSD	Bias	EmpSD	AvgSD	Cov95	AvgSD	Cov95
				Case 1					
				$\tau = 0.85$					
Intercept	5.018	-0.063	0.171	-0.003	0.162	0.174	0.96	0.246	0.94
Z_1	-0.482	0.124	0.340	-0.007	0.307	0.325	0.96	0.471	0.94
Z_2	1.518	0.142	0.220	0.002	0.197	0.205	0.94	0.204	0.93
t	-1	0.024	0.031	-0.001	0.029	0.033	0.98	0.037	0.97
				$\tau = 0.9$					
Intercept	5.141	-0.061	0.189	-0.004	0.173	0.190	0.96	1.637	0.97
Z_1	-0.359	0.126	0.383	-0.004	0.330	0.354	0.96	1.209	0.96
Z_2	1.641	0.144	0.250	0.002	0.219	0.225	0.94	1.721	0.97
t	-1	0.024	0.034	-0.001	0.031	0.036	0.98	0.056	0.98
				$\tau = 0.95$					
Intercept	5.322	-0.057	0.234	-0.005	0.208	0.230	0.96	2.340	0.98
Z_1	-0.178	0.115	0.489	-0.007	0.396	0.418	0.96	1.915	0.96
Z_2	1.822	0.134	0.303	-0.0006	0.260	0.267	0.94	2.073	0.96
t	-1	0.025	0.043	-0.0007	0.039	0.043	0.99	0.071	0.98
				Case 2					
				$\tau = 0.85$					
Intercept	5.003	-0.063	0.248	0.0003	0.215	0.266	0.94	0.347	0.93
Z_1	-0.497	0.155	0.502	-0.002	0.401	0.507	0.93	0.653	0.94
Z_2	1.503	0.187	0.305	0.002	0.249	0.322	0.93	0.263	0.93
t	-1	0.037	0.043	0.0002	0.040	0.048	0.93	0.051	0.96
				$\tau = 0.9$					
Intercept	5.170	-0.062	0.304	0.003	0.257	0.316	0.92	1.480	0.98
Z_1	-0.330	0.174	0.618	-0.003	0.476	0.602	0.92	1.472	0.96
Z_2	1.670	0.219	0.380	0.004	0.291	0.392	0.92	1.546	0.95
t	-1	0.038	0.050	0.0001	0.045	0.057	0.95	0.074	0.98
				$\tau = 0.95$					
Intercept	5.438	-0.047	0.440	0.010	0.339	0.403	0.92	2.209	0.97
Z_1	-0.062	0.196	0.894	-0.007	0.619	0.751	0.91	2.119	0.97
Z_2	1.938	0.248	0.549	0.017	0.389	0.495	0.91	2.048	0.97
t	-1	0.037	0.067	-0.006	0.058	0.073	0.96	0.099	0.98

Table F.2: Simulation studies ($n = 400$) that compared the proposed method and the naive approach: EmpSD – empirical standard deviation; AvgSD – the average of standard deviation estimates; Cov95 – the coverage rate of a 95% confidence interval.

		Naive		Proposed					
						Bootstrapping		Sample-based	
Effect	True	Bias	EmpSD	Bias	EmpSD	AvgSD	Cov95	AvgSD	Cov95
Case 1									
$\tau = 0.25$									
Intercept	4.163	-0.075	0.135	-0.008	0.115	0.119	0.96	0.129	0.95
Z_1	-1.337	0.146	0.253	0.018	0.217	0.225	0.95	0.239	0.95
Z_2	0.663	0.121	0.169	0.004	0.127	0.132	0.95	0.137	0.95
t	-1	0.033	0.040	4e-4	0.025	0.028	0.97	0.035	0.98
$\tau = 0.5$									
Intercept	4.5	-0.067	0.125	-0.006	0.102	0.106	0.95	0.111	0.94
Z_1	-1	0.139	0.242	0.019	0.194	0.200	0.95	0.206	0.94
Z_2	1	0.124	0.170	-0.004	0.114	0.120	0.95	0.121	0.95
t	-1	0.028	0.034	-5e-4	0.020	0.022	0.97	0.024	0.97
$\tau = 0.75$									
Intercept	4.837	-0.061	0.125	-0.003	0.106	0.110	0.95	0.114	0.95
Z_1	-0.663	0.127	0.247	0.008	0.200	0.207	0.96	0.208	0.94
Z_2	1.337	0.135	0.189	-0.004	0.127	0.129	0.95	0.129	0.93
t	-1	0.025	0.032	-0.001	0.019	0.021	0.97	0.023	0.96
Case 2									
$\tau = 0.25$									
Intercept	4.134	-0.024	0.091	0.005	0.084	0.088	0.95	0.094	0.95
Z_1	-1.366	0.054	0.171	-6e-4	0.154	0.157	0.95	0.165	0.94
Z_2	0.634	0.056	0.112	5e-4	0.087	0.092	0.96	0.094	0.95
t	-1	0.021	0.029	-7e-4	0.020	0.022	0.98	0.025	0.96
$\tau = 0.5$									
Intercept	4.418	-0.042	0.110	0.006	0.097	0.101	0.95	0.104	0.95
Z_1	-1.082	0.109	0.214	-0.002	0.176	0.182	0.94	0.187	0.93
Z_2	0.918	0.092	0.152	0.002	0.106	0.109	0.96	0.109	0.94
t	-1	0.026	0.032	-0.001	0.019	0.021	0.97	0.023	0.96
$\tau = 0.75$									
Intercept	4.777	-0.053	0.144	0.007	0.124	0.133	0.96	0.137	0.95
Z_1	-0.723	0.136	0.295	-0.001	0.230	0.237	0.96	0.240	0.94
Z_2	1.277	0.139	0.220	-0.005	0.144	0.145	0.95	0.143	0.93
t	-1	0.034	0.040	-0.001	0.023	0.025	0.97	0.027	0.96

Table F.3: Robustness study with $n = 200$ based on 1000 replications: Scenario S1

	Case 1					Case 2				
		Naive		Proposed			Naive		Proposed	
Effect	True	Bias	SD	Bias	SD	True	Bias	SD	Bias	SD
			$\tau = 0.25$					$\tau = 0.25$		
Intercept	4.163	-0.068	0.159	-0.054	0.159	4.134	-0.034	0.017	-0.023	0.016
Z_1	-1.337	0.122	0.315	0.093	0.317	-1.366	0.070	0.061	0.049	0.056
Z_2	0.663	0.125	0.185	0.095	0.185	0.634	0.070	0.024	0.048	0.021
t	-1	0.035	0.032	0.028	0.032	-1	0.023	0.001	0.016	0.001
			$\tau = 0.5$					$\tau = 0.5$		
Intercept	4.500	-0.062	0.153	-0.047	0.153	4.418	-0.049	0.029	-0.035	0.026
Z_1	-1	0.119	0.302	0.088	0.300	-1.082	0.107	0.106	0.076	0.093
Z_2	1	0.132	0.182	0.100	0.177	0.918	0.109	0.044	0.076	0.034
t	-1	0.029	0.028	0.022	0.028	-1	0.027	0.002	0.019	0.001
			$\tau = 0.75$					$\tau = 0.75$		
Intercept	4.837	-0.060	0.171	-0.045	0.163	4.777	-0.060	0.050	-0.042	0.044
Z_1	-0.663	0.122	0.347	0.089	0.331	-0.723	0.145	0.203	0.102	0.167
Z_2	1.337	0.134	0.218	0.100	0.205	1.277	0.156	0.095	0.104	0.068
t	-1	0.025	0.030	0.019	0.029	-1	0.032	0.002	0.023	0.002

Table F.4: Robustness Study with $n = 200$ based on 1000 replications: Scenario S2

	Case 1					Case 2				
		Naive		Proposed			Naive		Proposed	
Effect	True	Bias	SD	Bias	SD	True	Bias	SD	Bias	SD
					$a_1 = 0.5$					
		$\tau = 0.25$					$\tau = 0.25$			
Intercept	4.163	-0.065	0.148	-0.012	0.151	4.134	-0.026	0.015	0.004	0.014
Z_1	-1.337	0.131	0.291	0.020	0.302	-1.366	0.067	0.053	0.005	0.046
Z_2	0.663	0.127	0.170	0.013	0.182	0.634	0.070	0.021	0.003	0.016
t	-1	0.030	0.029	0.003	0.031	-1	0.020	0.001	0.003	6e-4
		$\tau = 0.5$					$\tau = 0.5$			
Intercept	4.5	-0.057	0.145	-0.005	0.145	4.418	-0.038	0.023	0.004	0.020
Z_1	-1	0.121	0.278	0.003	0.277	-1.082	0.102	0.089	0.003	0.067
Z_2	1	0.136	0.173	0.011	0.169	0.918	0.106	0.038	0.005	0.022
t	-1	0.025	0.026	0.001	0.027	-1	0.025	0.001	0.003	6e-4
		$\tau = 0.75$					$\tau = 0.75$			
Intercept	4.837	-0.050	0.159	-0.002	0.152	4.777	-0.058	0.044	-0.007	0.034
Z_1	-0.663	0.121	0.311	0.004	0.290	-0.723	0.153	0.184	0.021	0.119
Z_2	1.337	0.134	0.197	0.005	0.181	1.277	0.161	0.084	010	0.040
t	-1	0.022	0.027	0.001	0.026	-1	0.031	0.002	0.003	0.001
					$a_1 = 1.0$					
		$\tau = 0.25$					$\tau = 0.25$			
Intercept	4.163	-0.062	0.149	-0.022	0.151	4.134	-0.035	0.017	-0.008	0.015
Z_1	-1.337	0.108	0.297	0.027	0.311	-1.366	0.075	0.061	0.025	0.051
Z_2	0.663	0.133	0.177	0.057	0.179	0.634	0.068	0.022	0.023	0.017
t	-1	0.033	0.031	0.014	0.033	-1	0.022	0.001	0.008	8e-4
		$\tau = 0.5$					$\tau = 0.5$			
Intercept	4.5	-0.057	0.147	-0.021	0.147	4.418	-0.045	0.025	-0.015	0.021
Z_1	-1	0.108	0.288	0.029	0.289	-1.082	0.101	0.096	0.034	0.074
Z_2	1	0.140	0.178	0.056	0.173	0.918	0.107	0.040	0.036	0.026
t	-1	0.027	0.027	0.012	0.028	-1	0.027	0.002	0.010	8e-4
		$\tau = 0.75$					$\tau = 0.75$			
Intercept	4.837	-0.056	0.163	-0.022	0.155	4.777	-0.060	0.048	-0.019	0.037
Z_1	-0.663	0.120	0.315	0.039	0.296	-0.723	0.137	0.191	0.042	0.130
Z_2	1.337	0.137	0.207	0.050	0.191	1.277	0.163	0.093	0.052	0.051
t	-1	0.024	0.028	0.010	0.027	-1	0.034	0.002	0.011	0.001

Table F.5: Robustness Study with $n = 200$ based on 1000 replications: Scenario S3

	Case 1					Case 2				
		Naive		Proposed			Naive		Proposed	
Effect	True	Bias	SD	Bias	SD	True	Bias	SD	Bias	SD
			$\tau = 0.25$					$\tau = 0.25$		
Intercept	4.163	-0.023	0.131	0.032	0.137	4.134	-0.002	0.012	0.020	0.011
Z_1	-1.337	0.152	0.260	-0.061	0.275	-1.366	0.069	0.048	-0.040	0.038
Z_2	0.663	0.156	0.158	-0.060	0.177	0.634	0.091	0.022	-0.020	0.013
t	-1	0.015	0.026	-0.014	0.031	-1	0.010	6e-4	-0.008	6e-4
			$\tau = 0.5$					$\tau = 0.5$		
Intercept	4.5	-0.020	0.130	0.026	0.127	4.418	-0.007	0.018	0.018	0.014
Z_1	-1	0.151	0.259	-0.059	0.249	-1.082	0.116	0.085	-0.044	0.052
Z_2	1	0.157	0.158	-0.068	0.162	0.918	0.128	0.038	-0.033	0.018
t	-1	0.011	0.023	-0.011	0.024	-1	0.012	7e-4	0.027	6e-4
			$\tau = 0.75$					$\tau = 0.75$		
Intercept	4.837	-0.020	0.148	0.022	0.135	4.777	-0.001	0.040	0.024	0.029
Z_1	-0.663	0.151	0.298	-0.056	0.259	-0.723	0.161	0.174	-0.053	0.092
Z_2	1.337	0.155	0.183	-0.074	0.169	1.277	0.170	0.079	-0.059	0.034
t	-1	0.011	0.025	-0.010	0.023	-1	0.014	0.001	-0.012	9e-4

Table F.6: Parameter estimates of the proportional intensity model from the sensitivity analysis (Case A) for PBB study

Coeff	Estimate	exp(Estimate)	p-value
α_1	0.059	1.06	0.16
α_2	0.584	1.79	< 0.001
α_3	0.028	1.03	0.66
BMI	0.013	1.01	0.08

Table F.7: Parameter estimates and 95% confidence interval from the sensitivity analysis (Case A) for PBB study

Quantile	Estimate	95% CI
	Intercept	
25th	0.182	(0.150, 0.215)
50th	0.755	(0.429, 1.080)
75th	1.435	(1.253, 1.617)
85th	2.057	(1.657, 2.457)
90th	2.845	(2.214, 3.476)
95th	4.038	(3.376, 4.701)
	Time	
25th	5e-15	(-0.008, 0.008)
50th	-0.001	(-0.018, 0.015)
75th	-6e-14	(-0.011, 0.011)
85th	-8e-4	(-0.026, 0.024)
90th	-0.027	(-0.057, 0.004)
95th	-0.060	(-0.114, -0.006)

Table F.8: Parameter estimates of the proportional intensity model from the sensitivity analysis (Case B) for PBB study

Year	Effect	coeff	exp(coeff)	p value
before 1981	log (Initial PBB)$\in (1,3]$	0.057	1.06	0.61
	log(Initial PBB) > 3	0.206	1.23	0.24
1982 - 1989	log (Initial PBB)$\in (1,3]$	1.649	5.20	<0.001
	log (Initial PBB) > 3	2.859	17.44	<0.001
1990 - 1993	log (Initial PBB)$\in (1,3]$	0.147	1.16	0.36
	log (Initial PBB) > 3	0.053	1.05	0.85

Table F.9: Parameter estimates and 95% confidence interval from the sensitivity analysis (Case B) for PBB study

Quantile	Estimate	95% CI
	Intercept	
25th	0.182	(0.169, 0.196)
50th	0.576	(0.131, 1.021)
75th	1.455	(1.269, 1.641)
85th	2.094	(1.696, 2.492)
90th	2.840	(2.220, 3.459)
95th	3.998	(3.383, 4.613)
	Time	
25th	1e-16	(-0.003, 0.003)
50th	0.004	(-0.018, 0.026)
75th	-0.004	(-0.018, 0.011)
85th	-0.010	(-0.035, 0.014)
90th	-0.027	(-0.057, 0.004)
95th	-0.053	(-0.118, 0.011)