Biometrika (2011), **xx**, x, *pp*. 1–8

Web-based Supplementary Material for "Functional Mixed Effects Spectral Analysis"

BY ROBERT T. KRAFTY

Department of Statistics, University of Pittsburgh, Pittsburgh, Pennsylvania, U.S.A. 15260

krafty@pitt.edu

MARTICA HALL

Department of Psychiatry, University of Pittsburgh, Pittsburgh, Pennsylvania, U.S.A. 15213

hallmh@upmc.edu

AND WENSHENG GUO

Division of Biostatistics, University of Pennsylvania, Philadelphia, Pennsylvania, U.S.A. 19104

wguo@mail.med.upenn.edu

1. PROOF OF THEOREM 1

Let $a_{jkm} = \int_{-1/2}^{1/2} A_0(\omega; U_{jk}) A_j(\omega; V_{jk}) e^{-2\pi i\omega m}\, d\omega$ so that X_{jkt} has the form $X_{jkt} = \sum_m a_{jkm} z_{jk\,t-m}$, $d^z_{jk\ell} = T^{-1/2} \sum_{t=1}^T z_{jkt} e^{-2\pi i \omega_\ell t}$ be the discreet Fourier transform of the unobserved white noise z_{jkt}, and $I^z_{jk\ell} = |d^z_{jk\ell}|^2$ be the corresponding periodogram. It is well known, i.e. Theorem 5.2.6 in Brillinger (2002), that under the conditions of Theorem 1, as $T \to \infty$, $I^z_{jk\ell}$ are asymptotically distributed as χ^2_{df}/df random variables with df=2 if $\omega_\ell \neq 0, 1/2$ and df=1 if $\omega_\ell = 0, 1/2$. There have been several approaches to the study of non-linear functions/functionals of the periodograms of white noise for use in a variety of applications. The following Lemma is taken as an immediate consequence of the Edgeworth expansion of Götze & Hipp (1983). Further details about the application of this expansion to periodograms can be found in Lemma A.1 of von Sachs (1994) and a discussion about its applicability under the unbounded $\log$ transformation when the absolute continuity of the distribution of I^z_{jkt} is assured by (RA2) can be found in Janas & von Sachs (1995).

LEMMA 1. *If (RA1)-(RA3) hold, then uniformly in* j, k, ℓ, ℓ'

(i) $E\left(\log I^z_{jk\ell}\right) = -\gamma_\ell + O\left(T^{-1}\right)$

(ii) $var(\log I^z_{jk\ell}) = \sigma^2_\ell + O\left(T^{-1}\right)$

(iii) $cov\left(\log I^z_{jk\ell}, \log I^z_{jk\ell'}\right) = O\left(T^{-1}\right)$ *when* $|\ell| \neq |\ell'|$.

Theorem 1 will follow from Lemma 1 after finding the decay of the remainder of the Bartlett's decomposition

$$\sup_{j,k,\ell} \mathrm{E}\left\{\left|\log I_{jk\ell} - \log f_{jk}(\omega_\ell; U_{jk}, V_{jk}) - \log I^z_{jk\ell}\right|^2\right\} = O(T^{-1}).$$

To show this decay, a first order Taylor's series expansion of $\log I_{jk\ell}$ around $\log f_{jk}(\omega_\ell; U_{jk}, V_{jk})$ can be taken where there exist $\eta_{jk\ell} \in [0,1]$ such that

$$\log I_{jk\ell} - \log f_{jk}(\omega_\ell; U_{jk}, V_{jk}) - \log I^z_{jk\ell} = \frac{R_{jk\ell}}{f_{jk}(\omega_\ell; U_{jk}, V_{jk}) I^z_{jk\ell} + \eta_{jk\ell} R_{jk\ell}}$$

for $R_{jk\ell} = I_{jk\ell} - f_{jk}(\omega_\ell; U_{jk}, V_{jk}) I^z_{jk\ell}$. The proof of Theorem 1 can then be completed by applying Lemmas 2 and 3 which respectively find that $|R_{jk\ell}| = O_p(T^{-1/2})$ and $|f_{jk}(\omega_\ell; U_{jk}, V_{jk}) I^z_{jk\ell} + \eta_{jkl} R_{jk\ell}|^{-2} = O_p(1)$.

LEMMA 2. *Under the assumptions of Theorem 1,* $\sup_{j,k,\ell} E|R_{jk\ell}|^2 = O(T^{-1})$ *as* $T \to \infty$.

Proof. Let $Q_{jk\ell m} = \sum_{t=1-m}^{T-m} z_{jkt} e^{-2\pi i \omega_\ell t} - \sum_{t=1}^{T} z_{jkt} e^{-2\pi i \omega_\ell t}$ so that $d_{jk\ell} = A_0(\omega_\ell; U_{jk}) A_j(\omega_\ell; V_{jk}) d^z_{jk\ell} + r_{jk\ell}$ where $r_{jk\ell} = T^{-1/2} \sum_m a_{jkm} e^{-2\pi i m \omega_\ell} Q_{jk\ell m}$. It then follows from Cauchy–Schwarz

$$E\left(|R_{jk\ell}|^2\right) \leq 2\left[E\left\{f_j^2(\omega_\ell;U_{jk},V_{jk})\right\}\right]^{1/2}\left[E\left\{(I^z_{jk\ell})^2\right\}\right]^{1/2}\left\{E\left(|r_{jk\ell}|^4\right)\right\}^{1/2} + 2E\left(|r_{jk\ell}|^4\right).$$

Since the fourth cummulant of Z_{jk} being bounded implies that $E(I^z_{jk\ell})^2$ is uniformly bounded and the finite fourth moments of the transfer functions in conjunction with their continuity and (RA5) implies $E\left\{f_j^2(\omega_\ell;U_{jk},V_{jk})\right\}$ is uniformly bounded, the lemma is completed once the decay of $r_{jk\ell}$ is found.

Define the random functions

$$\mathcal{A}_{jk}(\omega) = \int_\nu\int_\zeta\int_\xi A_0(\omega-\nu-\zeta;U_{jk})A_j(\omega-\nu-\zeta;V_{jk})A_0(\xi;V_{jk})A_j(\xi;V_{jk}) \times \overline{A_0(\zeta-\xi;U_{jk})A_j(\zeta-\xi;V_{jk})A_0(\zeta;U_{jk})A_j(\zeta;V_{jk})}\,d\nu\,d\zeta\,d\xi$$

so that $E\left\{\mathcal{A}_{jk}(\omega)\right\}$ has the Fourier coefficients $E\left(|a_{jkm}|^4\right)$ by Theorem 1.12 in Zygmund (2003). From the smoothness of h_p^0, h_q and (RA5), the second derivative of $E\left(\mathcal{A}_{jk}\right)$ is uniformly absolutely continuous and subsequently there exists a constant C_0 such that $E\left(|a_{jkm}|^4\right) \leq C_0|m|^{-2}$ for all j,k,m. Note that, by Theorem 10.3.1 in Brockwell & Davis (2006), $E|Q_{jk\ell m}|^4 \leq 6|m|^2 + 2\left[\sup_\omega E\left\{|Z_{jk}(\omega)|^4\right\}\right]|m|$. Consequently,

$$\mathrm{E}\left(|r_{jk\ell}|^4\right) \leq \left[T^{-1/2}\sum_m \left\{E\left(|a_{jkm}|^4\right)\right\}^{1/4}\left\{\mathrm{E}\left(|Q_{jk\ell m}|^4\right)\right\}^{1/4}\right]^4 = O\left(T^{-2}\right) \text{ uniformly in } j,k,\ell. \qquad \square$$

LEMMA 3. *Under the assumptions of Theorem 1,* $\sup_{j,k,\ell} E\left\{|f_{jk}(\omega_\ell;U_{jk},V_{jk})I^z_{jk\ell} + \eta_{jk\ell}R_{jk\ell}|^{-2}\right\} = O(1)$.

Proof. By Schwarz's inequality and the definition of $R_{jk\ell}$, $E\left\{\left|I_{jk\ell}/I^z_{jk\ell} - f_{jk}(\omega_\ell;U_{jk},V_{jk})\right|^2\right\} \leq E\left(|R_{jk\ell}|^2\right)\,E\left(|I^z_{jk\ell}|^{-2}\right)$. Lemma 2 found that $\sup_{j,k,\ell} E\left(|R_{jk}|^2\right) = o(1)$ and Lemma 5 in Fay et al. (2002) shows that $\sup_{j,k,\ell} E\left(|I^z_{jk\ell}|^{-2}\right) = O(1)$ under (RA1)-(RA3). Consequently, $\left|I_{jk\ell}/I^z_{jk\ell} - f_{jk}(\omega_\ell;U_{jk},V_{jk})\right| = o_p(1)$ and it suffices to prove Lemma 3 over the event

$$\Omega = \left\{ \sup_{j,k,\ell} \left| I_{jk\ell}/I^z_{jk\ell} - f_{jk}(\omega_\ell; U_{jk}, V_{jk}) \right| < \epsilon \right\}$$

where ϵ is defined in (RA4). Under Ω, $I_{jk\ell} \geq \{f_{jk}(\omega_\ell; U_{jk}, V_{jk}) - \epsilon\} I^z_{jk\ell}$ and

$$f_{jk}(\omega_\ell; U_{jk}, V_{jk}) I^z_{jk\ell} + \eta_{jk\ell} R_{jk\ell} \geq \{f_{jk}(\omega_\ell; U_{jk}, V_{jk}) - \epsilon\} I^z_{jk\ell}$$

so that $\sup_{j,k,\ell} E \left\{ \left| f_{jk}(\omega_\ell; U_{jk}, V_{jk}) I^z_{jk\ell} + \eta_{jk\ell} R_{jk\ell} \right|^{-2} \mid \Omega \right\} = O(1).$ □

2. PROOF OF THEOREM 2

Let $||\cdot||_\infty$ be the norm over $\mathbb{R}^P$ such that $||x||_\infty = \sup x_j$ where $x = (x_1, \ldots, x_P)^T$, define the induced operator norm on the space of $P \times P$ matrices where $||M||_\star = \sup_x ||Mx||_\infty / ||x||_\infty$, and let $\tilde{\Lambda} = \lambda^{-1}\Lambda$ and $\tilde{\Theta} = \theta^{-1}\Theta$. The proof will make use of the fact that, since $\beta \in \otimes^P W^m_{2,\text{per}}$, $\beta(\omega) = \sum_{m=-\infty}^{\infty} b_m \exp(2\pi i \omega m)$ where the Fourier coefficients satisfy $||b_m||_\infty = O(|m|^{-2})$ (Zygmund, 2003). The proof will also use that, since Γ_q is the covariance of a stochastic process with realizations almost surely in $W^2_{2,\text{per}}$, $\Gamma_q \in W^2_{2,\text{per}} \otimes W^2_{2,\text{per}}$ so that $\Gamma_q(\omega, \nu) = \sum_{m=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} g_{qms} e^{2\pi i m\omega + 2\pi i s\nu}$ where $|g_{qms}| = O(|m|^{-2}|s|^{-2})$.

Decomposing the mean square error $E\left\{ \left| \hat{\beta}_p(\omega) - \beta_p(\omega) \right|^2 \right\} = \left| E\left\{ \hat{\beta}_p(\omega) \right\} - \beta_p(\omega) \right|^2 + \text{var}\left\{ \hat{\beta}_p(\omega) \right\}$, we will consider the bias and variance terms separately. By the Woodburry Formula

$$\left\{ U^T U + \lambda n (2\pi m)^4 \tilde{\Lambda} \right\}^{-1} = (U^T U)^{-1} + \lambda (2\pi m)^4 \left\{ n^{-1} U^T U + \lambda (2\pi m)^4 \tilde{\Lambda} \right\}^{-1} \tilde{\Lambda} (U^T U)^{-1} \tag{1}$$

so that $\sup_p |E\left\{ \hat{\beta}_p(\omega) \right\} - \beta_p(\omega)| \leq ||B_1(\omega)||_\infty + ||B_2(\omega)||_\infty + O(T^{-1})$ where

$$B_1(\omega) = T^{-1} \sum_{\ell=1-L}^{L} \sum_{m=1-L}^{L} \beta(\omega_l) e^{2\pi i m(\omega - \omega_\ell)} - \beta(\omega)$$

$$B_2(\omega) = T^{-1} \sum_{\ell=1-L}^{L} \sum_{m=1-L}^{L} \lambda (2\pi m)^4 \left\{ n^{-1} \tilde{\Lambda}^{-1} U^T U + \lambda (2\pi m)^4 \right\}^{-1} \beta(\omega_\ell) e^{2\pi i m(\omega - \omega_\ell)}$$

and the $O(T^{-1})$ term is the error in the approximation of the mean of Y_ℓ with $U\beta(\omega_\ell)$ obtained in Theorem 1. Approximating averaging with integration and using the decay of the Fourier coefficients of β we find that

$$\begin{aligned}\sup_\omega ||B_1(\omega)||_\infty &= \sup_\omega || \sum_{m=1-L}^{L} \int \beta(\nu) e^{2\pi i m(\omega-\nu)}\, d\nu - \beta(\omega)||_\infty + O(T^{-1}) \\ &\leq \sum_{|m|>L} ||b_m||_\infty + O(T^{-1}) \\ &= O(T^{-1})\end{aligned}$$

$$\begin{aligned}\sup_\omega ||B_2(\omega)||_\infty &= \sup_\omega || \sum_{m=1-L}^{L} \lambda(2\pi m)^4 \left\{ n^{-1}\tilde{\Lambda}^{-1}U^TU + \lambda(2\pi m)^4 \right\}^{-1} b_m e^{2\pi i m\omega}||_\infty + O(T^{-1}) \\ &\leq \sum_{m=-\infty}^{\infty} \frac{\lambda(2\pi m)^4}{D_1 + \lambda C(2\pi m)^4} ||b_m||_\infty + O(T^{-1})\end{aligned}$$

where D_1 is defined in (RA6). By Schwarz's inequality, $2D_1^{1/2}C^{1/2}\lambda^{1/2}(2\pi m)^2 \leq D_1 + \lambda C(2\pi m)^4$, so that

$$\begin{aligned}\sup_\omega ||B_2(\omega)||_\infty &\leq \lambda^{1/2} 2\pi^2 D_1^{-1/2} C^{-1/2} \sum_{m=-\infty}^{\infty} |m|^2 ||b_m||_\infty + O(T^{-1}) \\ &= O(\lambda^{1/2}) + O(T^{-1})\end{aligned}$$

and it then follows that $\sup_{p,\omega} \left| E\left\{\hat{\beta}_p(\omega)\right\} - \beta_p(\omega)\right|^2 = O(\lambda) + O(T^{-1})$.

To compute the variance term, note that $\sup_p \operatorname{var}\left\{\hat{\beta}_p(\omega)\right\} \leq ||V_\epsilon(\omega)||_\star + ||V_\alpha(\omega)||_\star + O(T^{-1})$ where

$$\begin{aligned}V_\epsilon(\omega) = \frac{1}{nT^2} &\sum_{\ell,r=1-L}^{L} \sigma_\ell^2 \delta_{|\ell||r|} \sum_{m,s=1-L}^{L} \left\{ n^{-1}U^TU + \lambda(2\pi m)^4\tilde{\Lambda}\right\}^{-1} \left(n^{-1}U^TU\right) \\ &\times \left\{ n^{-1}U^TU + \lambda(2\pi s)^4\tilde{\Lambda}\right\}^{-1} e^{2\pi i(m+s)\omega} e^{-2\pi i(m\omega_\ell + s\omega_r)} \\ V_\alpha(\omega) = \frac{1}{nT^2} &\sum_{\ell,r=1-L}^{L} \sum_{m,s=1-L}^{L} \left\{ n^{-1}U^TU + \lambda(2\pi m)^4\tilde{\Lambda}\right\}^{-1} \left\{ n^{-1} \sum_{j=1}^{N} U_j^T V_j \Gamma(\omega_l, \omega_r) V_j^T U_j \right\} \\ &\times \left\{ n^{-1}U^TU + \lambda(2\pi s)^4\tilde{\Lambda}\right\}^{-1} e^{2\pi i m(\omega-\omega_\ell) + 2\pi i s(\omega-\omega_r)}.\end{aligned}$$

To find the decay of V_ϵ, let $S_{ms} = \sigma_0^2 + \sigma_{1/2}^2 e^{\pi i(m+s)} + \sum_{|\ell|=1}^{L-1} \sigma_\ell^2 e^{2\pi i m\omega_\ell}$ and note that $|S_{ms}| \leq \sigma_0^2 T$ so that an application of Cauchy–Schwarz leads to

$$\begin{aligned} ||V_\epsilon(\omega)||_\star &\leq \frac{1}{nT^2} \sup_{m,s} |S_{m,s}| \, ||n^{-1}U^TU||_\star \sum_{m=1-L}^{L} \left|\left| \left\{ n^{-1}U^TU + \lambda(2\pi m)^4 \tilde{\Lambda} \right\}^{-1} \right|\right|_\star^2 \\ &\leq \frac{\sigma_0^2 D_2}{nT} \sum_{m=-\infty}^{\infty} \left\{ D_1 + \lambda C(2\pi m)^4 \right\}^{-2}. \end{aligned}$$

Since $2\sum_{m=-\infty}^{\infty} \left\{ D_1 + \lambda C(2\pi m)^4 \right\}^{-2} \leq C^{-1/4}\pi^{-1}\lambda^{-1/4} \int_{-\infty}^{\infty} (D_1 + \nu^4)^{-2}\, d\nu$, it follows that $\sup_\omega ||V_\epsilon(\omega)||_\star = O(N^{-1}T^{-1}\lambda^{-1/4})$. If we define $g_{mr} = \text{diag}(g_{1mr}, \ldots, g_{Qmr})$, then

$$\begin{aligned} V_\alpha(\omega) = n^{-1} &\sum_{m,s=1-L}^{L} \left\{ n^{-1}U^TU + \lambda(2\pi m)^4 \tilde{\Lambda} \right\}^{-1} \left(n^{-1} \sum_{j=1}^{N} U_j^T V_j g_{ms} V_j^T U_j \right) \\ &\times \left\{ n^{-1}U^TU + \lambda(2\pi s)^4 \tilde{\Lambda} \right\}^{-1} e^{2\pi i(m+s)\omega} + O(T^{-1}). \end{aligned}$$

Since $|| \left\{ n^{-1}U^TU + \lambda(2\pi r)^4 \tilde{\Lambda} \right\}^{-1} ||_\star = O(1)$ as $\lambda \to 0$ and the summability of the Fourier coefficients of Γ assure that there exists a constant C_0 such that $\sum_{m,r=1-L}^{L} ||n^{-1} \sum_{j=1}^{N} U_j^T V_j g_{mr} V_j^T U_j||_\star \leq C_0$ whenever U_j, V_j satisfy (RA6) and (RA7), it follows that $\sup_\omega ||V_\alpha(\omega)||_\star = O(N^{-1} + T^{-1})$.

Consequently, $\sup_{p,\omega} E\left\{ |\hat{\beta}_p(\omega) - \beta_p(\omega)|^2 \right\} = O(\lambda + N^{-1}\, T^{-1}\lambda^{-1/4} + T^{-1} + N^{-1})$ and Theorem 2 follows.

3. Proof of Theorem 3

Decomposing the mean squared error

$$\sup_{q,\omega,\nu} \mathrm{E}\left\{ \left| \hat{\Gamma}_q(\omega,\nu) - \Gamma_q(\omega,\nu) \right|^2 \right\} = \sup_{q,\omega,\nu} \left| \mathrm{E}\left\{ \hat{\Gamma}_q(\omega,\nu) \right\} - \Gamma_q(\omega,\nu) \right|^2 + \sup_{q,\omega,\nu} \mathrm{E}\left\{ \left| \hat{\Gamma}_q(\omega,\nu) - \mathrm{E}\hat{\Gamma}_q(\omega,\nu) \right|^2 \right\}$$

we will sketch the proof for the convergence of the mean and bias terms separately assuming that $\lambda \sim N^{-4/5}T^{-4/5}$. To investigate the bias term, apply the Woodbury Formula (1) so that

$$\sup_{q,\omega,\nu}\left|\mathrm{E}\left\{\hat{\Gamma}_q(\omega,\nu)\right\} - \Gamma_q(\omega,\nu)\right| \leq \sum_{k,\ell=1,2}\sup_{j,\omega,\nu}||B_{\alpha jk\ell}(\omega,\nu)||_\star + \sup_{j,\omega,\nu}||B_{\epsilon j}(\omega,\nu)||_\star$$
$$+O(N^{-2/5}T^{-2/5} + N^{-1/2} + T^{-1/2})$$

where

$$B_{\alpha j11}(\omega,\nu) = \frac{1}{T^2}\sum_{\ell,m,r,s=1-L}^{L}\Gamma(\omega_\ell,\nu_r)e^{2\pi im(\omega-\omega_\ell)+2\pi is(\nu-\omega_r)} - \Gamma(\omega,\nu)$$
$$B_{\alpha j12}(\omega,\nu) = \frac{1}{T^2}\sum_{\ell,m,r,s=1-L}^{L}\theta(2\pi s)^4\Gamma(\omega_\ell,\nu_r)\left\{n_jV_j^TV_j + \theta(2\pi s)^4\tilde{\Theta}\right\}^{-1}\tilde{\Theta}e^{2\pi im(\omega-\omega_\ell)+2\pi is(\nu-\omega_r)}$$
$$B_{\alpha j21}(\omega,\nu) = B_{\alpha j12}(\nu,\omega)$$
$$B_{\alpha j22}(\omega,\nu) = \frac{1}{T^2}\sum_{\ell,m,r,s=1-L}^{L}\theta^2(2\pi m)^4(2\pi s)^4\tilde{\Theta}\left\{n_jV_j^TV_j + \theta(2\pi m)^4\tilde{\Theta}\right\}^{-1}\Gamma(\omega_\ell,\nu_r)$$
$$\times\left[n_jV_j^TV_j + \theta(2\pi s)^4\tilde{\Theta}\right]^{-1}\tilde{\Theta}e^{2\pi im(\omega-\omega_\ell)+2\pi is(\nu-\omega_r)}$$
$$B_{\epsilon j}(\omega,\nu) = \frac{1}{T^2}\sum_{\ell,r=1-L}^{L}\sigma_\ell^2\delta_{|\ell||r|}\sum_{m,s=1-L}^{L}\left\{V_j^TV_j + \theta n_j(2\pi m)^4\tilde{\Theta}\right\}^{-1}V_j^TV_j$$
$$\times\left\{V_j^TV_j + \theta n_j(2\pi s)^4\tilde{\Theta}\right\}^{-1}e^{2\pi i(m+s)\omega}e^{2\pi im(\omega-\omega_\ell)+2\pi is(\nu-\omega_r)}.$$

Using similar algebra to that used in the proof of Theorem 2 it can be found that

$$\sup_{j,\omega,\nu}||B_{\alpha j11}(\omega,\nu)||_\star \leq \sum_{|m|,|s|>L}||g_{ms}||_\star + O(T^{-1}) = O(T^{-1})$$
$$\sup_{j,\omega,\nu}||B_{\alpha j12}(\omega,\nu)||_\star \leq \sum_{m,s=-\infty}^{\infty}\frac{\theta(2\pi s)^4}{n_-D_3 + \theta(2\pi s)^4}||g_{ms}||_\star + O(T^{-1}) = O(\theta^{1/2} + T^{-1})$$

$$\sup_{j,\omega,\nu}||B_{\alpha j22}(\omega,\nu)||_\star \leq \sum_{m,s=-\infty}^{\infty}\frac{\theta^2(2\pi s)^4(2\pi m)^4}{\left\{n_-D_3 + \theta(2\pi m)^4\right\}\left\{n_-D_3 + \theta(2\pi s)^4\right\}}||g_{ms}||_\star + O(T^{-1}) = O(\theta + T^{-1})$$
$$\sup_{j,\omega,\nu}||B_{\epsilon j}(\omega,\nu)||_\star \leq \frac{\sigma_0^2D_3}{T}\sum_{m=-\infty}^{\infty}\left\{D_3 + \theta n_jC(2\pi m)^4\right\}^{-2} = O(T^{-1}\theta^{-1/4})$$

and the supremum squared bias is $O\left(\theta + T^{-2}\theta^{-1/2} + N^{-4/5}T^{-4/5} + N^{-1} + T^{-1}\right)$.

To investigate the variance term, let $\mathcal{E}_{j\ell mrs} = \left\{ Y_{j\ell}^{*} Y_{jr}^{*T} - \mathrm{E} Y_{j\ell}^{*} Y_{jr}^{*T} \right\} e^{-2\pi i (m\omega_\ell + s\omega_r)}$ so that

$$\begin{aligned}
\sup_{q,\omega,\nu} E\left\{ \left| \hat{\Gamma}_q(\omega,\nu) - \mathrm{E}\Gamma_q(\omega,\nu) \right|^2 \right\} &\le \sup_{\omega,\nu} \mathrm{E}\left[\left|\left| \hat{\Gamma}(\omega,\nu) - \mathrm{E}\left\{ \hat{\Gamma}(\omega,\nu) \right\} \right|\right|_\star^2 \right] \\
&\le N^{-1} \sup_{j=1,\ldots,N} \left[\left|\left| V_j^T V_j \right|\right|_\star^2 \left\{ \sum_{m=-\infty}^{\infty} \left|\left| V_j^T V_j + \theta n_j (2\pi m)^4 \tilde{\Theta} \right|\right|_\star^{-2} \right\}^2 \right. \\
&\quad \left. \times \sum_{\ell,rm,s=\infty}^{\infty} \mathrm{E}\left|\left| \mathcal{E}_{jms} \right|\right|_\star^2 / T^4 \right] + O(N^{-4/5} T^{-4/5} + N^{-1} + T^{-1}) \\
&= N^{-1} D_3^2 \left[\sum_{m=-\infty}^{\infty} \left\{ D_3 + \theta n_j (2\pi m)^4 C \right\}^{-2} \right]^2 \\
&\quad \times \left[\tau_8 \sup_{\omega, U_{jk}, V_{jk}} \mathrm{E}\left\{ \log f_{jk}^4(\omega; U_{jk}, V_{jk}) \right\} \right] + O(N^{-4/5} T^{-4/5} + N^{-1} + T^{-1}) \\
&= O\left(N^{-1} \theta^{-1/2} + N^{-4/5} T^{-4/5} + N^{-1} + T^{-1} \right).
\end{aligned}$$

Combining the results from the bias and variance terms, $\sup_{q,\omega,\nu} \mathrm{E}\left\{ \left| \hat{\Gamma}_q(\omega,\nu) - \Gamma_q(\omega,\nu) \right|^2 \right\} = O\left\{ \theta + (N^{-1} + T^{-2})\theta^{-1/2} + N^{-4/5} T^{-4/5} + N^{-1} + T^{-1} \right\}$ and Theorem 3 follows.

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