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Supplemental Information

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SUPPLEMENTARY MATERIAL

Differential geometry of the sphere

In this section, we use following parametrization of a sphere:

$$\vec{r} = \begin{pmatrix} R_S \sin\theta \cos\phi \\ R_S \sin\theta \sin\phi \\ R_S \cos\theta \end{pmatrix} \tag{1}$$

With R_S being the radius of the sphere, θ the polar angle and ϕ the azimuthal angle. The principal curvatures are easily visible as the inverse of the sphere radius, therefore gaussian curvature is:

$$K = \frac{1}{R_S^2} \tag{2}$$

Mean curvature is given by:

$$H = \frac{1}{R_S} \tag{3}$$

The area element of the sphere, as known, is:

$$\mathrm{d}A = R_S^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi \tag{4}$$

Differential geometry of the torus

Analogous to (1) we parametrize the torus in the following way:

$$\vec{r} = \begin{pmatrix} \left(R_T - R_B \sin\phi_1\right) \cos\phi_2 \\ \left(R_T - R_B \sin\phi_1\right) \sin\phi_2 \\ R_B \cos\phi_1 \end{pmatrix} \tag{5}$$

For mean and gaussian curvatures, the first and second fundamental forms are needed. With:

$$\mathbf{I}_{ij} = \frac{\partial \vec{r}}{\partial \phi_i} \cdot \frac{\partial \vec{r}}{\partial \phi_j} \tag{6}$$

The first fundamental form is given by:

$$\mathbf{I} = \begin{pmatrix} R_B^2 & 0 \\ 0 & \left(R_T - R_B \sin\phi_1\right)^2 \end{pmatrix} \tag{7}$$

The second fundamental form,

$$\mathbf{II}_{ij} = \frac{\partial \vec{n}}{\partial \phi_i} \cdot \frac{\partial \vec{r}}{\partial \phi_j} \tag{8}$$

is given by:

$$\mathbf{II} = \begin{pmatrix} -R_B & 0 \\ 0 & \left(R_T - R_B \sin\phi_1\right)\sin\phi_1 \end{pmatrix} \tag{9}$$

The area element $\mathrm{d}A$ of the toroid geometry is given by:

$$\mathrm{d}A = \sqrt{\det \mathbf{I}} = R_B \left(R_T - R_B \sin\phi_1\right) \mathrm{d}\phi_1 \mathrm{d}\phi_2 \tag{10}$$

Gaussian curvature, as the product of both principal curvatures, is given as the ratio of the determinants of both the first and second fundamental form:

$$K = \kappa_1 \kappa_2 = \frac{\det \mathbf{II}}{\det \mathbf{I}} = \frac{-\sin\phi_1}{R_B\left(R_T - R_B \sin\phi_1\right)} \tag{11}$$

As one of the principal curvatures of the torus in the direction of the x-axis is just the inverse minor torus radius, the second principal curvature can be read of the gaussian curvature. Using this, the mean curvature is given by:

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\left(\frac{1}{R_B} - \frac{\sin\phi_1}{R_T - R_B \sin\phi_1}\right) \tag{12}$$

Bending energy of surface coating membrane

The approximate bending energy for a thin membrane sheet is given by Helfrich (2):

$$\begin{aligned} E_{\text{bend}} &= \int_M \mathrm{d}A \left\{\frac{1}{2}\kappa(2H - c_0)^2 + \bar{\kappa}K\right\} \\ &= \int_M \mathrm{d}A \left\{2\kappa H^2 - 2\kappa H c_0 + \frac{1}{2}\kappa c_0^2 + \bar{\kappa}K\right\} \\ &= \int_M \mathrm{d}A \{E_H - E_{HC} + E_C + E_K\} \end{aligned} \tag{13}$$

On the torus section, this integral has to be evaluated around the rotational axis, thus from $\phi_2 = 0$ to $\phi_2 = 2\pi$ and along the x-axis from $\phi_1 = \alpha$ to $\phi_1 = \pi - \alpha$. Given the mean and gaussian curvatures, the terms involved in this integral are:

$$E_H = 2\pi \int_{\alpha}^{\pi-\alpha} \mathrm{d}\phi_1 \left\{\frac{\kappa\left(R_T - 2R_B \sin\phi_1\right)^2}{2R_B\left(R_T - R_B \sin\phi_1\right)}\right\} \tag{14}$$

$$= 2\pi\kappa \left[2\cos\phi_1 - \frac{R_T^2 \arctan\left\{\frac{R_B - R_T \tan\frac{\phi_1}{2}}{W}\right\}}{R_B W}\right]_{\alpha}^{\pi-\alpha} \tag{15}$$

With $W = \sqrt{R_T^2 - R_B^2}$. Using a small angle approximation(3), thus $\arctan x \approx x$, $\tan x \approx x$ and $R_B^2 \ll R_T^2$ so that $W \approx R_T$, we obtain:

$$\begin{aligned} E_H &\approx 2\pi\kappa \left[\frac{R_T \phi_1}{2R_B} + 2\cos\phi_1 - 1\right]_{\alpha}^{\pi-\alpha} &(16) \\ &\approx \pi\kappa \left(\frac{R_T}{R_B}(\pi - 2\alpha) - 8\cos\alpha\right) &(17) \end{aligned}$$

The second term is:

$$\begin{aligned} E_{HC} &= 2\pi \int_{\alpha}^{\pi-\alpha} c_0 \kappa \left(2R_B \sin\phi_1 - R_T\right) \mathrm{d}\phi_1 &(18) \\ &= 2\pi c_0 \kappa \left[R_T(\pi - 2\alpha) + 4R_B \cos\alpha\right] &(19) \end{aligned}$$

The third term:

$$
\begin{aligned}
E_C &= 2\pi \int_{\alpha}^{\pi-\alpha} \frac{1}{2}\kappa R_B c_0^2 (R_T - R_B \sin\phi_1)\mathrm{d}\phi_1 &(20)\\
&= \pi\kappa c_0^2 R_B (R_T(\pi - 2\alpha) - 2R_B \cos\alpha) &(21)
\end{aligned}
$$

Lastly, the term including gaussian curvature:

$$
\begin{aligned}
E_K &= -\bar{\kappa} \int_{\alpha}^{\pi-\alpha} \sin\phi_1 \mathrm{d}\phi_1 &(22)\\
&= -4\pi\bar{\kappa}\cos\alpha &(23)
\end{aligned}
$$

Analogously, this terms for the sphere, where integration is carried out for $\theta = [0, 2\pi]$ and $\phi = [0, \alpha]$, are given as:

$$
\begin{aligned}
E_H^{(S)} &= 4\pi\kappa(\cos\alpha + 1) &(24)\\
E_{HC}^{(S)} &= -4\pi R_S c_0 \kappa(\cos\alpha + 1) &(25)\\
E_C^{(S)} &= \pi R_S^2 c_0^2 \kappa(\cos\alpha + 1) &(26)\\
E_K^{(S)} &= 2\pi\bar{\kappa}(\cos\alpha + 1) &(27)
\end{aligned}
$$

The total energy change upon fusion is now given as the difference of the bending energy of a membrane on two spheres and the bending energy on two spheres, lacking a sphere cap corresponding to the delaminated membrane are within the contact zone and the bending energy of the toroid contact zone itself, leading to the expression given in Eq. 18 in the main text.

Supporting References

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