Appendix A: Details of Scaling in Organisms

This appendix gives the full derivation of the scaling equations. We begin with the total network resistance discussed in Section 3.1, and its subsequent effect on the energy time product scaling. Assume that $D_l = 3$ for 3 dimensional organisms and that $\lambda^{-H} = N^{-1}$. Using these values and simplifying, equation 6 is transformed.

$$R = \frac{8\mu l_0}{\pi r_0^4} N^{-1} \sum_{i=0}^{H} \lambda^{i(\frac{4}{3} - \frac{4}{D_r})} \tag{17}$$

Let the summand $S = \sum_{i=0}^{H} \lambda^{i(\frac{4}{3} - \frac{4}{D_r})}$. $R \propto N^{-1} S$. How S scales with N is dependent on the exponent $\frac{4}{3} - \frac{4}{D_r}$, and reduces to three different cases:

Case 1: $D_r = 3$: In this case the exponent is equal to 0, and the $S = H + 1 \propto \log(N)$, and $R \propto \frac{\log(N)}{N}$, because $\log(N)$ in this case grows much more slowly than N, it is reasonable to conclude that $R \propto N^{-1}$. In this case the energy time product scales as $l_0 + u_0^{-1} N^{2 - \frac{2}{D_r}}$.

Case 2: $D_r < 3$: Here (and in subsequent cases) we can use the geometric series to calculate the exact value of S. In particular

$$\begin{aligned} S &= \frac{(1 - \lambda^{(\frac{4}{3} - \frac{4}{D_r})(H+1)})}{1 - \lambda^{\frac{4}{3} - \frac{4}{D_r}}} \\ &= \frac{1 - \left(\lambda^{H}\right)^{(\frac{4}{3} - \frac{4}{D_r})} \lambda^{(\frac{4}{3} - \frac{4}{D_r}))}}{1 - \lambda^{\frac{4}{3} - \frac{4}{D_r}}} \\ &= \frac{1 - N^{(\frac{4}{3} - \frac{4}{D_r})} \lambda^{(\frac{4}{3} - \frac{4}{D_r}))}}{1 - \lambda^{\frac{4}{3} - \frac{4}{D_r}}} \end{aligned}$$

If we let $c = \lambda^{(\frac{4}{3} - \frac{4}{D_r})}$ we see that

$$S = \frac{1 - c N^{(\frac{4}{3} - \frac{4}{D_r})}}{1 - c} \tag{18}$$

Because $\frac{4}{3} - \frac{4}{D_r} < 0$ is negative in this case and N is large in practice, $cN^{(\frac{4}{3}-\frac{4}{D_r})}$ is small, and S is proportional to a constant ($S \approx \frac{1}{1-c}$). This implies that $R \propto N^{-1}$. Once again, equation 4 scales as $\propto l_0 + u_0^{-1} N^{2-\frac{2}{D_r}}$.

Case 3: $D_r > 3$: In this case the exponent in S is positive, meaning that S scales directly with N. Note that $c > 1$ in this case and we can write

$$S = \frac{cN^{(\frac{4}{3}-\frac{4}{D_r})} - 1}{c-1} \tag{19}$$

This means that $S \propto N^{(\frac{4}{3}-\frac{4}{D_r})}$. This implies that $R \propto N^{-1}S \propto N^{(\frac{1}{3}-\frac{4}{D_r})}$. This means that resistance still scales inversely with size, but at a faster rate than if $D_r \leq 3$. This implies the energy time produce scales as $\propto N^{\frac{4}{3}-\frac{4}{D_r}} + N^{2-\frac{2}{D_r}}$. Note that this results in a positive scaling of R with N if $D_r > 12$.

6.1 Length, Velocity and Mass Scaling

We determine the scaling of l_0, u_0 and M following the method presented in Banavar et al. [3]. Specifically, we assume that $M \propto V \propto V_{net}$, where V is the volume of the organism, V_{net} is the volume of the network transporting oxygen molecules, and

$$u_0 \propto l_0 \propto \left(\frac{V}{N}\right)^{\frac{1}{3}} \tag{20}$$

We calculate V_{net} as:

$$
\begin{aligned}
V_{net} &= \sum_{i=0}^{H} V_i = \sum s^{-1} l_i r_i^2 n_i \\
&= \sum_{i=0}^{H} {l_0}^{-1} l_0 \lambda^{\frac{i}{D_l}} r_0^2 \lambda^{\frac{2i}{D_r}} \lambda^{H-i} \\
&= \lambda^H r_0^2 \sum_{i=0}^{H} \lambda^{i\left(\frac{i}{D_l}+\frac{2}{D_r}-1\right)} \\
&\propto N^{\frac{1}{D_l}+\frac{2}{D_r}}
\end{aligned}
$$

Note that $s \propto l_0$ is the linear distance between oxygen molecules that will be delivered to the same capillary, allowing narrower vessels when oxygen travels at higher velocity (see [3] Figure 3 for further explanation). The calculation assumes that $\frac{1}{D_l} + \frac{2}{D_r} - 1 > 0$ which will always be the case with $2 < D_r < 3$, and $D_l = 3$.

Therefore $M \propto N^{\frac{1}{D_l}+\frac{2}{D_r}}$, and $N \propto M^{\frac{1}{\frac{1}{3}+\frac{2}{D_r}}}$, where $D_l = 3$. Additionally

$$
\begin{aligned}
u_0 \propto l_0 &\propto \left(\frac{V}{N}\right)^{\frac{1}{3}} \\
&\propto \left(N^{\frac{1}{3}+\frac{2}{D_r}-1}\right)^{\frac{1}{3}} \\
&\propto N^{\frac{2}{3D_r}-\frac{2}{9}}
\end{aligned}
$$

Combining these results with E_{sys} and T_{sys} in Table 1, we derive how metabolic rate, B (measured in power), scales with the mass of an organism.

$$
\begin{aligned}
B = \frac{E_{sys}}{T_{sys}} =& \frac{RQ^2}{N} + Q \\
\propto& u_0^2 l_0 N^{-2} N^{\frac{4}{D_r}} + u_0 N^{\frac{2}{D_r}} \\
\propto& N^{\frac{6}{D_r} - \frac{8}{3}} + N^{\frac{8}{3D_r} - \frac{2}{9}} \\
\propto& M^{\frac{\frac{6}{D_r} - \frac{8}{3}}{\frac{2}{D_r} + \frac{1}{3}}} + M^{\frac{\frac{8}{3D_r} - \frac{2}{9}}{\frac{2}{D_r} + \frac{1}{3}}} \\
\propto& M^{\frac{18 - 8D_r}{6 + D_r}} + M^{\frac{24 - 2D_r}{18 - 3D_r}}
\end{aligned}
$$

Appendix B: Details of Scaling in Electronics

In this section we give a detailed analysis of the derivation of the scaling of the network capacitance and network latency discussed in Section 3.3.

7.1 Capacitance

Recall that $D_l = 2$ for 2 dimensional computer chips and that $\lambda^{-H} = N^{-1}$. We can then calculate capacitance as:

$$C \propto N^{(1-\frac{1}{D_l}} \sum_{i=0}^{H} \lambda^{i\left(\frac{1}{D_l}+\frac{1}{D_w}-1\right)} \tag{21}$$

Similar to how we handled organsims we are interested in whether the exponent $\frac{1}{D_l} + \frac{1}{D_w} - 1$ is positive or negative.

Let the summand $S = \sum_{i=0}^{H} \lambda^{i(\frac{1}{D_l}+\frac{1}{D_w}-1)}$. $C \propto N^{1-\frac{1}{D_l}} S$.

Case 1: $D_r = \frac{D_l}{D_l-1}$: In this case the exponent is equal to 0, and the $S = H + 1 \propto \log(N)$, and $C \propto \log(N) N^{1-\frac{1}{D_l}}$, because $\log(N)$ in this case grows much more slowly than $N^{1-\frac{1}{D_l}}$ and we know $D_l = 2$ for 2 dimensional chips, it is reasonable to conclude that $C \propto N^{\frac{1}{2}}$

Case 2: $D_r > \frac{D_l}{D_l-1}$: Here (and in subsequent cases) we can use the geometric series to calculate the exact value of S, using a similar approach to 6. In this case the exponent is negative and S is a small constant, leaving $C \propto N^{\frac{1}{2}}$

Case 3: $D_r < \frac{D_l}{D_l-1}$: In this case the exponent in S is positive, meaning that S scales directly with N. Now the summand contributes an $N^{\frac{1}{D_l}+\frac{1}{D_w}-1}$ and $C \propto N^{\frac{1}{D_w}}$.

7.2 Network Delay

Recall that we wish to determine the network latency L which is defined as:

$$T_{net} \propto \max_i L_i \tag{22}$$

with

$$L_i \propto RC = \frac{\rho \epsilon l i^2}{r_i^2} = \frac{\rho \epsilon l_0^2}{r_0^2} \lambda^{i\left(\frac{2}{D_l} - \frac{2}{D_r}\right)} \tag{23}$$

L_i will scale differently depending on the relative values of D_r and D_l.

Case 1: $D_r > D_l$: In this case the fraction in the exponent is greater than 0 and the latency will be highest when $i = H$, resulting in $L \propto N^{\frac{2}{D_l} - \frac{2}{D_r}}$.

Case 2: $D_r < D_l$: In this case the exponent is negative and the highest latency occurs at the bottom of the network $i = 0$, leaving $L \propto \frac{l_0^2}{r_0^2} \propto N^0$

Case 3: $D_r = D_l$: In this case the exponent is 0 and there is equal latency at all levels and $L \propto N^0$.

7.3 Chip Data

We obtain data on chip power usage and throughput from several third party sources. For power output we consulted two web archives of over 523 chips listing power consumption and transistor count [48, 30]. When possible the figures were cross checked with data directly from the manufacturer. For throughput data, we consulted a combination of third party sources, starting with a list published on wikipedia [57]. Each source for the data was consulted independently and verified before inclusion in the dataset. Additional throughput data was obtained from benchmarks performed by online technology publication Tom's Hardware [24]. The data can be found in the supplementary information.