BIOMETRICS 0, 1–9 DOI: 0.0.0.0
September 2015

Web-based Supplementary Material for "Survival Impact Index and Ultrahigh-dimensional Model-free Screening with Survival Outcomes"

Jialiang Li[1,2,3*], **Qi Zheng**[4,**], **Limin Peng** [4,***], **and Zhipeng Huang**[1,3,5****]

[1]Department of Statistics and Applied Probability, National University of Singapore, Singapore 117546, Singapore

[2]Duke-NUS Graduate Medical School, Singapore

[3]Singapore Eye Research Institute, Singapore

[4]Department of Biostatistics and Bioinformatics, Emory University, Atlanta, GA, 30321, U.S.A

[5]McDermott Center for Human Growth and Development, UT Southwestern Medical Center, Dallas, TX 75390, U.S.A

**email:* stalj@nus.edu.sg

***email:* qzheng9@emory.edu

****email:* lpeng@emory.edu

*****email:* zhipeng.huang@utsouthwestern.edu

This paper has been submitted for consideration for publication in *Biometrics*

Proofs of Corollaries

Proof of Corollary 1: $1 - Pr\left(\mathcal{M}_* \subset \hat{\mathcal{M}}\right) = Pr\left(\exists j \in \mathcal{M}_*, \hat{\xi}_j < bn^{-\alpha}\right) = 1 - Pr\Big(\exists j \in \mathcal{M}_*, |\hat{\xi}_j - \xi_j| \geq (c_0 - b)n^{-\alpha}\Big) \leq c_3 \left|\mathcal{M}_*\right| \exp\left(-nc_4(c_0-b)^2 n^{-2\alpha} - c_5 \log((c_0-b)n^{-\alpha})\right)$, which converges to 0, if $\alpha < (1-c)/2$ and $p = o(\exp(n^c))$. The sure screening property has been established. □

Proof of Corollary 2: Let $\pi_n := c_3 \exp\left(-nc_4\nu_n^2 - c_5 \log \nu_n\right)$. We have

$$
\begin{aligned}
p\pi_n &= c_3 \exp\left(-nc_4\nu_n^2 - c_5 \log \nu_n + \log p\right) \\
&\leq c_3 \exp\left(-nc_4 \frac{\log p + c_5 \log n/2}{c_4 n} - \frac{c_5}{2}(\log(\log p + c_5 \log n/2) - \log n - \log c_4) + \log p\right) \\
&= \frac{c_3 c_4}{(\log p + c_5 \log n/2)^{\frac{c_5}{2}}} < 1.
\end{aligned}
$$

Since $\binom{p-|\mathcal{M}_*|}{k} \leq p^k$, then we have $Pr\left(|\hat{\mathcal{M}} \setminus \mathcal{M}_*| = k\right) \leq p^k \pi_n^k$, which implies

$$
\begin{aligned}
Pr\left(|\hat{\mathcal{M}} \setminus \mathcal{M}_*| < n^c\right) &\geq 1 - \sum_{k=n^c}^{p} (\pi_n p)^k \\
&\geq 1 - \sum_{k=n^c}^{p} \left(c_3 \exp\left(-nc_4\nu_n^2 - c_5 \log \nu_n + \log p\right)\right)^k \\
&\geq 1 - 2\left(c_3 \exp\left(-nc_4\nu_n^2 - c_5 \log \nu_n + \log p\right)\right)^{n^c} \\
&\geq 1 - 2c_3 p \exp\left(-nc_4\nu_n^2 - c_5 \log \nu_n\right),
\end{aligned}
$$

where the last inequality follows from $p\pi_n < 1$ established above. This completes the proof of Corollary 2. □

Lemmas

LEMMA 1: *For any* $0 < \epsilon < 1$,

$$
Pr\left(\sup_{0 \leq x \leq x_0} \left|\hat{S}_j(x) - S_j(x)\right| > \epsilon\right) \leq 12 \exp\left(-n\epsilon^2/9 - \log \epsilon\right). \quad (1)
$$

Proof: If $|\hat{S}_j(\eta_{k-1}^{(j)}) - S_j(\eta_{k-1}^{(j)})| < \epsilon/3$ and $|\hat{S}_j(\eta_k^{(j)}) - S_j(\eta_k^{(j)})| < \epsilon/3$, then for $\eta_{k-1}^{(j)} \leq x \leq \eta_k^{(j)}$,

$$
\hat{S}_j(x) - S_j(x) \leq \hat{S}_j(\eta_{k-1}^{(j)}) - S_j(\eta_{k-1}^{(j)}) + S_j(\eta_{k-1}^{(j)}) - S_j(x) \leq 2\epsilon/3,
$$

The other direction can be shown by the same arguments. Thus, we have $|\hat{S}_j(x) - S_j(x)| < \epsilon$

for any $\eta_{k-1}^{(j)} \le x \le \eta_k^{(j)}$. Hence, if $\sup_{0 \le x \le x_0} |\hat{S}_j(x) - S_j(x)| > \epsilon$, then for some $0 \le k \le L(\epsilon)$, $|\hat{S}_j(\eta_k^{(j)}) - S_j(\eta_k^{(j)})| \ge \epsilon/3$. Thus,

$$\begin{aligned} & Pr\left(\sup_{0 \le x \le x_0} \left|\hat{S}_j(x) - S_j(x)\right| > \epsilon\right) \le L(\epsilon) Pr\left(\left|\hat{S}_j(\eta_k^{(j)}) - S_j(\eta_k^{(j)})\right| > \epsilon/3\right) \\ \le \quad & 2L(\epsilon)\exp\left(\frac{-1}{18}\frac{n^2\epsilon^2}{n/4 + n\epsilon/9}\right) \le 12\exp\left(-n\epsilon^2/9 - \log\epsilon\right) \end{aligned}$$

where the second inequality follows from the fact Var $[1\{X_{ji} > x\} - S_j(x)] \le 1/4$ and Bernstein's inequality. This completes the proof of (1). Moreover, we have

$$\begin{aligned} & Pr\left(\sup_{1 \le k \le L(\epsilon)} \left|\hat{S}_j\left(\eta_{k-1}^{(j)}\right) - \hat{S}_j\left(\eta_k^{(j)}\right)\right| > \epsilon\right) \\ \le \quad & Pr\left(\sup_{1 \le k \le L(\epsilon)} \left|\hat{S}_j\left(\eta_{k-1}^{(j)}\right) - S_j\left(\eta_{k-1}^{(j)}\right)\right| + \left|\hat{S}_j\left(\eta_k^{(j)}\right) - S_j\left(\eta_k^{(j)}\right)\right| > 2\epsilon/3\right) \\ \le \quad & L(\epsilon) Pr\left(\left|\hat{S}_j(\eta_k^{(j)}) - S_j(\eta_k^{(j)})\right| > \epsilon/3\right) \\ \le \quad & 12\exp\left(-n\epsilon^2/9 - \log\epsilon\right). \end{aligned} \tag{2}$$

□

LEMMA 2: *Given $x \in [0, x_0]$, for any $24/(n\tau\gamma^4) < \epsilon < \tau/2$, we have*

$$\begin{aligned} & Pr\left(\sup_{t \in \mathcal{T}} |\hat{S}_j(t;x) - S_j(t;x)| > \epsilon\right) \\ \le \quad & 12\exp\left(-n\epsilon^2/9 - \log\epsilon\right) + c_1\exp\left(-n\tau\gamma^6\epsilon^2/(2c_2) - \log\epsilon\right). \end{aligned} \tag{3}$$

Proof: By Foldes and Rejto (1981), we obtain

$$Pr\left(\sup_{t \in \mathcal{T}} |\hat{S}_j(t;x) - S_j(t;x)| > \epsilon | M_{jn}(x) = k\right) \le c_1\exp(-k\gamma^6\epsilon^2/c_2 - \log\epsilon), \tag{4}$$

for $12/(k\gamma^4) < \epsilon < 1$. Here, as defined in Appendix, $M_{jn}(x) = \sum_{i=1}^n 1\{X_{ji} > x\}$.

If $24/(n\tau\gamma^4) < \epsilon < \tau/2$, then

$$
\begin{aligned}
& Pr\left(\sup_{t\in\mathcal{T}} |\hat{S}_j(t;x) - S_j(t;x)| > \epsilon\right) \\
= & \sum_{k=0}^{n} Pr\left(\sup_{t\in\mathcal{T}} |\hat{S}_j(t;x) - S_j(t;x)| > \epsilon | M_{jn}(x) = k\right) Pr\left(M_{jn}(x) = k\right) \\
\le & Pr\left(M_{jn}(x) < nS_j(x) - n\epsilon\right) \\
+ & \sum_{k\ge nS_j(x)-n\epsilon}^{n} Pr\left(\sup_{t\in\mathcal{T}} |\hat{S}_j(t;x) - S_j(t;x)| > \epsilon | M_{jn}(x) = k\right) Pr\left(M_{jn}(x) = k\right) \\
\le & Pr\left(|\hat{S}_j(x) - S_j(x)| \ge \epsilon\right) \\
& \quad +\frac{c_1}{\epsilon}\exp\left(-(nS_j(x) - n\epsilon)\gamma^6\epsilon^2/c_2\right) Pr\left(M_{jn}(x) > nS_j(x) - n\epsilon\right) \\
= & 12\exp\left(-n\epsilon^2/9 - \log\epsilon\right) + c_1\exp\left(-n\tau\gamma^6\epsilon^2/(2c_2) - \log\epsilon\right),
\end{aligned}
\tag{5}
$$

where the second inequality follows from (4) and the fact that for all $k \ge nS_j(x) - n\epsilon$,

$$
\frac{12}{k\gamma^4} \le \frac{12}{(nS_j(x) - n\epsilon)\gamma^4} \le \frac{24}{nS_j(x)\gamma^4} \le \frac{24}{n\tau\gamma^4} < \epsilon,
$$

and the last inequality follows from (1) and $2\epsilon < \tau \le S_j(x)$. This completes the proof of Lemma 2. □

LEMMA 3: *For any* $0 < \epsilon < \lambda$,

$$
Pr\left(\sup_{(t,x)\in\mathcal{T}\times\mathcal{X}} \max_{1\le i\le n} |1\{Y_i \le t\} n \log V_{ji}(x)| > \frac{2}{\lambda - \epsilon}\right) \le 2\exp\left(-n\epsilon^2/4\right). \tag{6}
$$

Proof:

$$
\begin{aligned}
& Pr\left(\sup_{(t,x)\in\mathcal{T}\times\mathcal{X}}\max_{1\le i\le n}|1\{Y_i\le t\}n\log V_{ji}(x)|>\frac{2}{\lambda-\epsilon}\right)\\
&= Pr\left(\sup_{(t,x)\in\mathcal{T}\times\mathcal{X}}\max_{1\le i\le n}\left|1\{Y_i\le t\}n\log\left(\frac{\sum_{k=1}^n 1\left\{Y_k>Y_i,X_{jk}>x\right\}}{\sum_{k=1}^n 1\left\{Y_k>Y_i,X_{jk}>x\right\}+1}\right)\right|>\frac{2}{\lambda-\epsilon}\right)\\
&\le Pr\left(\sup_{(t,x)\in\mathcal{T}\times\mathcal{X}} n\left|\log\left(1-\frac{1}{\sum_{k=1}^n 1\left\{Y_k>t,X_{jk}>x\right\}+1}\right)\right|>\frac{2}{\lambda-\epsilon}\right)\\
&\le Pr\left(\sup_{(t,x)\in\mathcal{T}\times\mathcal{X}}\frac{2n}{\sum_{k=1}^n 1\left\{Y_k>t,X_{jk}>x\right\}+1}>\frac{2}{\lambda-\epsilon}\right)\\
&\le Pr\left(\frac{2n}{\sum_{k=1}^n 1\left\{Y_k>t_0,X_{jk}>x_0\right\}+1}>\frac{2}{\lambda-\epsilon}\right)\\
&\le Pr\left(\left|\frac{\sum_{l=1}^n 1\left\{Y_l>t_0,X_{jl}>x_0\right\}+1}{2n}-\frac{\lambda}{2}\right|>\frac{\epsilon}{2}\right)\\
&\le 2\exp\left(-n\epsilon^2/4\right),
\end{aligned}
$$

where the last inequality follow from Bernstein's inequality. This completes the proof of Lemma 3. □

LEMMA 4: *Let* $\left\{\eta_k^{(j)}, 0\le k\le L(\epsilon)\right\}$ *be the same grid points as constructed in Lemma* 1 *given* $0<\epsilon<1$. *Then*

$$
\begin{aligned}
& Pr\left(\sup_{t\in\mathcal{T},1\le k\le L(\epsilon),\eta_{k-1}^{(j)}<x<\eta_k^{(j)}}\max_{1\le i\le n} n\left[1\{Y_i\le t\}\log\left(\frac{V_{ji}(x)}{V_{ji}\left(\eta_{k-1}^{(j)}\right)}\right)\right]>\frac{8\epsilon}{\lambda^2}\right)\\
\le\ & 12\exp\left(-n\epsilon^2/9-\log\epsilon\right)+2\exp\left(-n\lambda^2/16\right). \qquad (7)
\end{aligned}
$$

Proof:

$$
\begin{aligned}
&Pr\left(\sup_{t\in\mathcal{T},1\le k\le L(\epsilon),\eta_{k-1}^{(j)}<x<\eta_k^{(j)}}\max_{1\le i\le n} n\left[1\{Y_i\le t\}\log\left(\frac{V_{ji}(x)}{V_{ji}\left(\eta_{k-1}^{(j)}\right)}\right)\right]>\frac{8\epsilon}{\lambda^2}\right)\\
\le\; &Pr\Bigg(\sup_{t\in\mathcal{T},1\le k\le L(\epsilon),\eta_{k-1}^{(j)}<x<\eta_k^{(j)}}\max_{1\le i\le n} 2n\Bigg[1\{Y_i\le t\}\\
&\quad\times\frac{\sum_{l=1}^n 1\left\{Y_l>Y_i,\eta_{k-1}^{(j)}<X_{jl}<x\right\}}{\left(\sum_{l=1}^n 1\left\{Y_l>Y_i,X_{jl}>x\right\}\right)\left(\sum_{l=1}^n 1\left\{Y_l>Y_i,X_{jl}>\eta_{k-1}^{(j)}\right\}\right)}\Bigg]>\frac{8\epsilon}{\lambda^2}\Bigg)\\
\le\; &Pr\left(\sup_{1\le k\le L(\epsilon)} 2n\frac{\sum_{l=1}^n 1\left\{\eta_{k-1}^{(j)}<X_{jl}<\eta_k^{(j)}\right\}}{\left(\sum_{l=1}^n 1\left\{Y_l>t_0,X_{jl}>x_0\right\}\right)^2}>\frac{8\epsilon}{\lambda^2}\right)\\
\le\; &Pr\left(\sup_{1\le k\le L(\epsilon)}\left|\hat{S}_j\left(\eta_{k-1}^{(j)}\right)-\hat{S}_j\left(\eta_k^{(j)}\right)\right|>\epsilon\right)+Pr\left(\frac{2n^2}{\left(\sum_{l=1}^n 1\left\{Y_l>t_0,X_{jl}>x_0\right\}\right)^2}>\frac{8}{\lambda^2}\right)\\
\le\; &12\exp\left(-n\epsilon^2/9-\log\epsilon\right)+2\exp\left(-n\lambda^2/16\right),
\end{aligned}
$$

where the last inequality follows from (2) and the proof of Lemma 3. This completes the proof of Lemma 4. □

A useful result

By Foldes and Rejto (1981), for some universal constants c_1 and c_2 , and $12/(n\gamma^4)<\epsilon<1$.

$$
Pr\left(\sup_{t\in\mathcal{T}}|\hat{S}(t)-S(t)|>\epsilon\right)\le c_1\exp(-n\gamma^6\epsilon^2/c_2-\log\epsilon), \tag{8}
$$

Hadamard differentiability of $\int_{\Omega^c(\kappa_j)} W_\xi(t,x)|\cdot|dtdx$

For any convergent sequences $u_n \to 0$ there exists an N, such that for all $n > N$, $|u_n h_n(t;x)| \le \kappa_j$ uniformly in $\mathcal{T} \times \mathcal{X}$. Then for $n > N$,

$$\begin{aligned}
&\int_{\Omega^c(\kappa_j)} W_\xi(t,x)|S_j(t;x) - S(t) + u_n h_n(t,x)|dtdx - \int_{\Omega^c(\kappa_j)} W_\xi(t,x)|S_j(t;x) - S(t)|dtdx \\
&= \int_{\Omega^c(\kappa_j)} W_\xi(t,x)\text{sgn}\left(S_j(t;x) - S(t)\right)\left[S_j(t;x) - S(t) + u_n h_n(t,x)\right]dtdx \\
&\quad - \int_{\Omega^c(\kappa_j)} W_\xi(t,x)\text{sgn}(S_j(t;x) - S(t))\left[S_j(t;x) - S(t)\right]dtdx \\
&= \int_{\Omega^c(\kappa_j)} W_\xi(t,x)\text{sgn}(S_j(t;x) - S(t))u_n h_n(t,x)dtdx.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\lim_{n\to\infty} u_n^{-1}\Bigg(\int_{\Omega^c(\kappa_j)} W_\xi(t,x)|S_j(t;x) - S(t) + u_n h_n(t,x)|dtdx \\
&\qquad\qquad - \int_{\Omega^c(\kappa_j)} W_\xi(t,x)|S_j(t;x) - S(t)|dtdx\Bigg) \\
&= \int_{\Omega^c(\kappa_j)} W_\xi(t,x)\text{sgn}(S_j(t;x) - S(t))h(t,x)dtdx,
\end{aligned}$$

which implies $\int_{\Omega^c(\kappa_j)} W_\xi(t,x)|\cdot|dtdx$ is Hadamard-differentiable at $S_j(\cdot;\cdot) - S(\cdot)$.

Detailed arguments of Theorem 2

Step 1: We consider $\sup_{t\in\mathcal{T},0\le k\le L(\epsilon)}\left|\hat{S}_j\left(t;\eta_k^{(j)}\right) - S_j\left(t;\eta_k^{(j)}\right)\right|$ first. By Lemma 2, we obtain that for any $24/(n\tau\gamma^4) < \epsilon < \tau/2$,

$$\begin{aligned}
&Pr\left(\sup_{t\in\mathcal{T},0\le k\le L(\epsilon)}\left|\hat{S}_j\left(t;\eta_k^{(j)}\right) - S_j\left(t;\eta_k^{(j)}\right)\right| > \epsilon\right) \\
\le\ & L(\epsilon)Pr\left(\sup_{t\in\mathcal{T}}\left|\hat{S}_j\left(t;\eta_k^{(j)}\right) - S_j\left(t;\eta_k^{(j)}\right)\right| > \epsilon\right) \\
\le\ & 72\exp\left(-n\epsilon^2/9 - 2\log\epsilon\right) + 6c_1\exp\left(-n\tau\gamma^6\epsilon^2/(2c_2) - 2\log\epsilon\right). \qquad (9)
\end{aligned}$$

Step 2: Next, we consider $\sup_{t\in\mathcal{T},1\le k\le L(\epsilon),\eta_{k-1}<x<\eta_k}|S_j(t;x)-S_j(t;\eta_{k-1})|$. If $\eta_{k-1}^{(j)}<x<\eta_k^{(j)}$,

$$\begin{aligned}\left|S_j(t;x)-S_j(t;\eta_{k-1}^{(j)})\right| &= \left|\frac{Pr(T>t,X_j>x)}{Pr(X_j>x)}-\frac{Pr\left(T>t,X_j>\eta_{k-1}^{(j)}\right)}{Pr\left(X_j>\eta_{k-1}^{(j)}\right)}\right| \\ &\le \frac{Pr(T>t,X_j>x)\left|Pr\left(X_j>\eta_{k-1}^{(j)}\right)-Pr\left(X_j>x\right)\right|}{Pr\left(X_j>x\right)Pr\left(X_j>\eta_{k-1}^{(j)}\right)} \\ &+ \frac{Pr(X_j>x)\left|Pr(T>t,X_j>x)-Pr\left(T>t,X_j>\eta_{k-1}^{(j)}\right)\right|}{Pr\left(X_j>x\right)Pr\left(X_j>\eta_{k-1}^{(j)}\right)} \\ &\le \frac{\epsilon/3}{\tau}+\frac{\epsilon/3}{\tau}=\frac{2\epsilon}{3\tau}.\end{aligned}$$

Therefore, we obtain

$$\sup_{t\in\mathcal{T},1\le k\le L(\epsilon),\eta_{k-1}^{(j)}<x<\eta_k^{(j)}}\left|S_j(t;x)-S_j(t;\eta_{k-1}^{(j)})\right|<\frac{2\epsilon}{3\tau}. \tag{10}$$

Step 3: Then we deal with $\sup_{t\in\mathcal{T},1\le k\le L(\epsilon),\eta_{k-1}^{(j)}<x<\eta_k^{(j)}}\left|\hat{S}_j(t;x)-\hat{S}_j\left(t;\eta_{k-1}^{(j)}\right)\right|$. For any $0<u,v\le 1$, we have $|u-v|\le|\log u-\log v|$. Therefore,

$$\begin{aligned}&Pr\left(\sup_{t\in\mathcal{T},1\le k\le L(\epsilon),\eta_{k-1}^{(j)}<x<\eta_k^{(j)}}\left|\hat{S}_j(t;x)-\hat{S}_j\left(t;\eta_{k-1}^{(j)}\right)\right|>\frac{4\epsilon}{\lambda}+\frac{8\epsilon}{\lambda^2}\right)\\ &\le Pr\left(\sup_{t\in\mathcal{T},1\le k\le L(\epsilon),\eta_{k-1}^{(j)}<x<\eta_k^{(j)}}\left|\log\hat{S}_j(t;x)-\log\hat{S}_j\left(t;\eta_{k-1}^{(j)}\right)\right|>\frac{4\epsilon}{\lambda}+\frac{8\epsilon}{\lambda^2}\right)\\ &\le Pr\left(\sup_{t\in\mathcal{T},1\le k\le L(\epsilon),\eta_{k-1}^{(j)}<x<\eta_k^{(j)}}\left|\sum_{i=1}^n\log V_{ji}(x)\right.\right.\\ &\qquad\left.\left.\times\left(1\left\{\delta_i=1,Y_i\le t,X_{ji}>x\right\}-1\left\{\delta_i=1,Y_i\le t,X_{ji}>\eta_{k-1}^{(j)}\right\}\right)\right|>\frac{4\epsilon}{\lambda}\right)\\ &+Pr\left(\sup_{t\in\mathcal{T},1\le k\le L(\epsilon),\eta_{k-1}^{(j)}<x<\eta_k^{(j)}}\sum_{i=1}^n 1\left\{\delta_i=1,Y_i\le t,X_{ji}>\eta_{k-1}^{(j)}\right\}\right.\\ &\qquad\left.\times\log\left(\frac{V_{ji}(x)}{V_{ji}\left(\eta_{k-1}^{(j)}\right)}\right)>\frac{8\epsilon}{\lambda^2}\right)\end{aligned}$$

where the last inequality follows from the decomposition:

$$
\begin{aligned}
&\log \hat{S}_j(t;x) - \log \hat{S}_j\left(t; \eta_{k-1}^{(j)}\right) \\
&= \sum_{i=1}^{n} \left[1\left\{\delta_i = 1, Y_i \le t, X_{ji} > x\right\} - 1\left\{\delta_i = 1, Y_i \le t, X_{ji} > \eta_{k-1}^{(j)}\right\}\right] \log V_{ji}(x) \\
&+ \sum_{i=1}^{n} 1\left\{\delta_i = 1, Y_i \le t, X_{ji} > \eta_{k-1}^{(j)}\right\} \left[\log V_{ji}(x) - \log V_{ji}\left(\eta_{k-1}^{(j)}\right)\right].
\end{aligned}
$$

Then we have,

$$
\begin{aligned}
&Pr\left(\sup_{t\in\mathcal{T}, 1\le k\le L(\epsilon), \eta_{k-1}^{(j)} < x < \eta_k^{(j)}} \left| \sum_{i=1}^{n} \log V_{ji}(x) \right.\right. \\
&\qquad \left.\left. \times \left(1\left\{\delta_i = 1, Y_i \le t, X_{ji} > x\right\} - 1\left\{\delta_i = 1, Y_i \le t, X_{ji} > \eta_{k-1}^{(j)}\right\}\right)\right| > \frac{4\epsilon}{\lambda}\right) \\
&\le Pr\left(\sup_{(t,x)\in\mathcal{T}\times\mathcal{X}} \max_{1\le i\le n} \left|1\left\{Y_i \le t\right\} n \log V_{ji}(x)\right| \right. \\
&\qquad \left. \times \sup_{t\in\mathcal{T}, 1\le k\le L(\epsilon)} \frac{\sum_{i=1}^{n} 1\left\{\delta_i = 1, Y_i \le t, \eta_{k-1}^{(j)} < X_{ji} < \eta_k^{(j)}\right\}}{n} > \frac{4\epsilon}{\lambda}\right) \\
&\le Pr\left(\sup_{(t,x)\in\mathcal{T}\times\mathcal{X}} \max_{1\le i\le n} \left|1\left\{Y_i \le t\right\} n \log V_{ji}(x)\right| \sup_{1\le k\le L(\epsilon)} \left[\hat{S}_j\left(\eta_{k-1}^{(j)}\right) - \hat{S}_j\left(\eta_k^{(j)}\right)\right] > \frac{4\epsilon}{\lambda}\right) \\
&\le Pr\left(\sup_{(t,x)\in\mathcal{T}\times\mathcal{X}} \max_{1\le i\le n} \left|1\left\{Y_i \le t\right\} n \log V_{ji}(x)\right| > \frac{4}{\lambda}\right) \\
&+ Pr\left(\sup_{t\in\mathcal{T}, 1\le k\le L(\epsilon)} \left[\hat{S}_j\left(\eta_{k-1}^{(j)}\right) - \hat{S}_j\left(\eta_k^{(j)}\right)\right] > \epsilon\right) \\
&\le 2\exp\left(-n\lambda^2/16\right) + 12\exp\left(-n\epsilon^2/9 - \log\epsilon\right),
\end{aligned}
$$

following from Lemma 3 and (2). On the other hand, by Lemma 4.

$$
\begin{aligned}
&Pr\left(\sup_{t\in\mathcal{T}, 1\le k\le L(\epsilon), \eta_{k-1}^{(j)} < x < \eta_k^{(j)}} \sum_{i=1}^{n} 1\left\{\delta_i = 1, Y_i \le t, X_{ji} > \eta_{k-1}^{(j)}\right\} \right. \\
&\qquad \left. \times \log\left(\frac{V_{ji}(x)}{V_{ji}\left(\eta_{k-1}^{(j)}\right)}\right) > \frac{8\epsilon}{\lambda^2}\right) \\
&\le Pr\left(\sup_{t\in\mathcal{T}, 1\le k\le L(\epsilon), \eta_{k-1}^{(j)} < x < \eta_k^{(j)}} \max_{1\le i\le n} n \left[1\{Y_i \le t\} \log\left(\frac{V_{ji}(x)}{V_{ji}\left(\eta_{k-1}^{(j)}\right)}\right)\right] > \frac{8\epsilon}{\lambda^2}\right) \\
&\le 12\exp\left(-n\epsilon^2/9 - \log\epsilon\right) + 2\exp\left(-n\lambda^2/16\right).
\end{aligned}
$$

Thus,

$$\begin{aligned} & Pr\left(\sup_{t\in\mathcal{T},1\le k\le L(\epsilon),\eta_{k-1}^{(j)}<x<\eta_k^{(j)}}\left|\hat{S}_j(t;x)-\hat{S}_j\left(t;\eta_{k-1}^{(j)}\right)\right| > \frac{4\epsilon}{\lambda}+\frac{8\epsilon}{\lambda^2}\right) \\ \le\ & 24\exp\left(-n\epsilon^2/9-\log\epsilon\right)+4\exp\left(-n\lambda^2/16\right). \end{aligned} \tag{11}$$

Combining (9), (10) and (11) yields,

$$\begin{aligned} & Pr\left(\sup_{t\in\mathcal{T},0\le x\le x_0}|\hat{S}_j(t;x)-S_j(t;x)| > \epsilon+\frac{2\epsilon}{3\lambda}+\frac{4\epsilon}{\lambda}+\frac{8\epsilon}{\lambda^2}\right) \\ \le\ & 72\exp\left(-n\epsilon^2/9-2\log\epsilon\right)+6c_1\exp\left(-n\tau\delta^6\epsilon^2/(2c_2)-2\log\epsilon\right) \\ & +24\exp\left(-n\epsilon^2/9-\log\epsilon\right)+4\exp\left(-n\lambda^2/16\right), \end{aligned} \tag{12}$$

for any $24/(n\tau\gamma^4)<\epsilon<\lambda\wedge\tau/2$. This and (8) together imply

$$\begin{aligned} & Pr\left(\sup_{t\in\mathcal{T},0\le x\le x_0}\left|\hat{S}_j(t;x)-\hat{S}(t)-(S_j(t;x)-S(t))\right| > \epsilon\right) \\ \le\ & c_3\exp\left(-nc_4\epsilon^2-c_5\log\epsilon\right), \end{aligned} \tag{13}$$

for any $c_6n^{-1}<\epsilon<1$, where c_3, c_4, c_5, c_6 are some constant depending only on τ, γ, and λ, and n is sufficiently large.

Since

$$|\hat{\xi}_j-\xi_j|\le\int_0^{x_0}\int_0^{t_0}W_\xi(t,x)\left|\hat{S}_j(t;x)-\hat{S}(t)-S_j(t;x)+S(t)\right|dtdx,$$

then by (13),

$$\begin{aligned} & Pr\left(|\hat{\xi}_j-\xi_j|>\epsilon\right) \\ \le\ & Pr\left(\int_0^{x_0}\int_0^{t_0}W_\xi(t,x)\left|\hat{S}_j(t;x)-\hat{S}(t)-S_j(t;x)+S(t)\right|dtdx>\epsilon\right) \\ \le\ & Pr\left(\sup_{t\in\mathcal{T},0\le x\le x_0}\left|\hat{S}_j(t;x)-\hat{S}(t)-(S_j(t;x)-S(t))\right|>\epsilon\right) \\ \le\ & c_3\exp\left(-nc_4\epsilon^2-c_5\log\epsilon\right). \end{aligned} \tag{14}$$

and hence

$$Pr\left(\max_{1\le j\le p}|\hat{\xi}_j-\xi_j|>\epsilon\right)\le c_3\exp\left(-nc_4\epsilon^2-c_5\log\epsilon+\log p\right). \tag{15}$$

References

Foldes, A. and Rejto, L. (1981). Strong uniform consistency for nonparametric survival curve estimators from randomly censored data. *Annals of Statistics* **9,** 122–129.