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Web-based Supplementary Materials for "A New Flexible Dependence Measure for Semi-competing Risks"

Jing Yang and Limin Peng*

Department of Biostatistics and Bioinformatics, Emory University, Atlanta, Georgia, U.S.A.

**email:* lpeng@sph.emory.edu

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Web Appendix A: Theoretical Proofs and Arguments

Define

$$\boldsymbol{S}_n(\boldsymbol{b},\tau,t_0) = n^{-1/2}\sum_{i=1}^{n}\frac{I(L_i^*\leqslant t_0)I(Y_i^*>t_0)\eta_i^*}{\hat{G}(Y_i^*-L_i^*)}\boldsymbol{A}_i^*(t_0)\{I[\log(Y_i^*-t_0)\leqslant \boldsymbol{A}_i^{*T}(t_0)\boldsymbol{b}]-\tau\},$$

$$\boldsymbol{S}_n^G(\boldsymbol{b},\tau,t_0) = n^{-1/2}\sum_{i=1}^{n}\frac{I(L_i^*\leqslant t_0)I(Y_i^*>t_0)\eta_i^*}{G(Y_i^*-L_i^*)}\boldsymbol{A}_i^*(t_0)\{I[\log(Y_i^*-t_0)\leqslant \boldsymbol{A}_i^{*T}(t_0)\boldsymbol{b}]-\tau\},$$

$$\begin{aligned}
\boldsymbol{\mu}(\boldsymbol{b},\tau,t_0) &= n^{-1/2}E\{\boldsymbol{S}_n^G(\boldsymbol{b},\tau,t_0)\}\\
&= c(t_0)E[I(T_2>t_0)\tilde{\boldsymbol{A}}(t_0)\{I[\log(T_2-t_0)\leqslant \tilde{\boldsymbol{A}}^T(t_0)\boldsymbol{b}]-\tau\}]\\
&= c(t_0)E\{\tilde{\boldsymbol{A}}(t_0)[P(T_2>t_0,\log(T_2-t_0)\leqslant \tilde{\boldsymbol{A}}^T(t_0)\boldsymbol{b}|\tilde{\boldsymbol{A}}^T(t_0))-\tau P(T_2>t_0|\tilde{\boldsymbol{A}}(t_0))]\}\\
&= c(t_0)E\{\tilde{\boldsymbol{A}}(t_0)[P(T_2\leqslant t_0+\exp(\tilde{\boldsymbol{A}}^T(t_0)\boldsymbol{b})|\tilde{\boldsymbol{A}}(t_0))-P(T_2\leqslant t_0|\tilde{\boldsymbol{A}}(t_0))-\tau P(T_2>t_0|\tilde{\boldsymbol{A}}(t_0))]\}.
\end{aligned}$$

For brevity, we use $\sup_{\boldsymbol{b}}$, $\sup_\tau$ and $\sup_{t_0}$ to denote supremum taken over $\boldsymbol{b}\in R^2$, $\tau\in[\tau_L,\tau_U]$ and $t_0\in[t_L,t_U]$, respectively.

1.1 *Proof for* $E\{\boldsymbol{S}_n^G(\boldsymbol{\beta}_0(\tau,t_0),\tau,t_0)\}=0$

Given the independence between D and (T_1,T_2,L), it is easy to show that the distributions of D and D^* are equivalent, and D^* is also independent of (T_1^*,T_2^*,L^*). Note that $I(Y^*>t_0)\eta^*\boldsymbol{A}^*(t_0)=I(T_2^*>t_0,T_2^*<C^*)\tilde{\boldsymbol{A}}^*(t_0)$. Thus, we have

$$\begin{aligned}
&E\left\{\frac{I(L^*\leqslant t_0)I(Y^*>t_0)\eta^*}{G(Y^*-L^*)}\boldsymbol{A}^*(t_0)\{I[\log(Y^*-t_0)\leqslant \boldsymbol{A}^{*T}(t_0)\boldsymbol{b}]-\tau\}\right\}\\
&= E\left\{\frac{I(L^*\leqslant t_0)I(T_2^*>t_0,T_2^*<C^*)}{G(T_2^*-L^*)}\tilde{\boldsymbol{A}}^*(t_0)\{I[\log(T_2^*-t_0)\leqslant \tilde{\boldsymbol{A}}^{*T}(t_0)\boldsymbol{b}]-\tau\}\right\}\\
&= E\left\{\frac{I(L^*\leqslant t_0)I(T_2^*>t_0)\tilde{\boldsymbol{A}}^*(t_0)\{I[\log(T_2^*-t_0)\leqslant \tilde{\boldsymbol{A}}^{*T}(t_0)\boldsymbol{b}]-\tau\}}{G(T_2^*-L^*)}E[I(T_2^*-L^*<D^*)|T_1^*,T_2^*,L^*]\right\}\\
&= E\left\{I(L^*\leqslant t_0)I(T_2^*>t_0)\tilde{\boldsymbol{A}}^*(t_0)\{I[\log(T_2^*-t_0)\leqslant \tilde{\boldsymbol{A}}^{*T}(t_0)\boldsymbol{b}]-\tau\}\times\frac{G(T_2^*-L^*)}{G(T_2^*-L^*)}\right\}\\
&= c(t_0)E\left\{I(T_2>t_0)\tilde{\boldsymbol{A}}(t_0)\{I[\log(T_2-t_0)\leqslant \tilde{\boldsymbol{A}}^T(t_0)\boldsymbol{b}]-\tau\}\right\}
\end{aligned}$$

and

$$
\begin{aligned}
&E\left\{I(T_2>t_0)\tilde{\boldsymbol{A}}(t_0)I[\log(T_2-t_0)\leqslant \tilde{\boldsymbol{A}}^T(t_0)\boldsymbol{b}]\right\}\\
&\quad = E\left\{\tilde{\boldsymbol{A}}(t_0)P[T_2>t_0,\log(T_2-t_0)\leqslant \tilde{\boldsymbol{A}}^T(t_0)\boldsymbol{b}|\tilde{\boldsymbol{A}}^T(t_0)]\right\}\\
&\quad = E\left\{\tilde{\boldsymbol{A}}(t_0)P[\log(T_2-t_0)\leqslant \tilde{\boldsymbol{A}}^T(t_0)\boldsymbol{b}|T_2>t_0,\tilde{\boldsymbol{A}}(t_0)]P[T_2>t_0|\tilde{\boldsymbol{A}}(t_0)]\right\}\\
&\quad = \tau E\{I(T_2>t_0)\tilde{\boldsymbol{A}}(t_0)\},
\end{aligned}
$$

where $G(t)=P(D>t)$, $\alpha=P(Y\geqslant L)$ and $c(t_0)=P(L\leqslant t_0)/\alpha$.

Therefore, we have $E\{\boldsymbol{S}_n^G(\boldsymbol{\beta}_0(\tau,t_0),\tau,t_0)\}=0$.

1.2 *Proof of Theorem 3.1*

By condition C1, we have $\sup_{t<\nu}|\hat{G}(t)-G(t)|=o(n^{-1/2+r})$, a.s., for every $r>0$. This implies that

$$\sup_{\boldsymbol{b},\tau,t_0}\|n^{-1/2}\boldsymbol{S}_n(\boldsymbol{b},\tau,t_0)-n^{-1/2}\boldsymbol{S}_n^G(\boldsymbol{b},\tau,t_0)\|=o(n^{-1/2+r}),\quad a.s.$$

Define $\mathcal{F}=\left\{\frac{I(L_i^*\leqslant t_0)I(Y_i^*>t_0)\eta_i^*}{G(Y_i^*-L_i^*)}\boldsymbol{A}_i^*(t_0)\{I[\log(Y_i^*-t_0)\leqslant \boldsymbol{A}_i^{*T}(t_0)\boldsymbol{b}]-\tau\},\boldsymbol{b}\in R^2,\tau\in[\tau_L,\tau_U],t_0\in[t_L,t_U]\right\}$. The function class $\mathcal{F}$ is Donsker and thus Glivenko-Cantelli because the class indicator functions is Donsker and both $\boldsymbol{A}_i^*(t_0)$ and $G(Y_i^*-L_i^*)$ is uniformly bounded (Van der Vaart and Wellner, 1996). Then $\sup_{\boldsymbol{b},\tau,t_0}\|n^{-1/2}\boldsymbol{S}_n^G(\boldsymbol{b},\tau,t_0)-\boldsymbol{\mu}(\boldsymbol{b},\tau,t_0)\|=o(1),a.s.$ by the Glivenko-Cantelli Theorem and thus $\sup_{\boldsymbol{b},\tau,t_0}\|n^{-1/2}\boldsymbol{S}_n(\boldsymbol{b},\tau,t_0)-\boldsymbol{\mu}(\boldsymbol{b},\tau,t_0)\|=o(1),a.s.$. This, coupled with the fact that $\boldsymbol{\mu}\{\boldsymbol{\beta}_0(\tau,t_0),\tau,t_0\}=0$ and $n^{-1/2}\boldsymbol{S}_n(\hat{\boldsymbol{\beta}}(\tau,t_0),\tau,t_0)=o(1),a.s.$, implies that

$$\sup_{\tau,t_0}\|\boldsymbol{\mu}\{\hat{\boldsymbol{\beta}}(\tau,t_0),\tau,t_0\}-\boldsymbol{\mu}\{\boldsymbol{\beta}_{\mathbf{0}}(\tau,t_0),\tau,t_0\}\|=o(1),\quad a.s.$$

Following the same line of Peng and Fine (2009), we can show that Condition C3 and the monotonicity of $\boldsymbol{\mu}(\boldsymbol{b},\tau,t_0)$ in $\boldsymbol{b}$ imply

$$\inf_{\boldsymbol{b}\notin B(\rho_0),\tau,t_0}\|\boldsymbol{\mu}\{\boldsymbol{b},\tau,t_0\}-\boldsymbol{\mu}\{\boldsymbol{\beta}_{\mathbf{0}}(\tau,t_0),\tau,t_0\}\|\geqslant c_0\rho_0.$$

Consequently, $\{\hat{\boldsymbol{\beta}}(\tau, t_0) : \tau \in [\tau_L, \tau_U], t_0 \in [t_L, t_U]\} \subseteq B(\rho_0)$ for large enough n with probability 1. Applying Taylor expansion to $\boldsymbol{\mu}\{\hat{\boldsymbol{\beta}}(\tau, t_0), \tau, t_0\}$ around $\boldsymbol{\beta}_0(\tau, t_0)$ gives

$$\begin{aligned}
&\sup_{\tau, t_0} \|\hat{\boldsymbol{\beta}}(\tau, t_0) - \boldsymbol{\beta}_0(\tau, t_0)\| \\
&\quad = \sup_{\tau, t_0} \|H\{\breve{\boldsymbol{\beta}}(\tau, t_0), t_0\}^{-1}[\boldsymbol{\mu}\{\hat{\boldsymbol{\beta}}(\tau, t_0), \tau, t_0\} - \boldsymbol{\mu}\{\boldsymbol{\beta}_{\mathbf{0}}(\tau, t_0), \tau, t_0\}]\| \\
&\quad \leqslant c_0^{-1} \sup_{\tau, t_0} \|\boldsymbol{\mu}\{\hat{\boldsymbol{\beta}}(\tau, t_0), \tau, t_0\} - \boldsymbol{\mu}\{\boldsymbol{\beta}_{\mathbf{0}}(\tau, t_0), \tau, t_0\}\|
\end{aligned}$$

where $\breve{\boldsymbol{\beta}}(\tau, t_0)$ lies between $\hat{\boldsymbol{\beta}}(\tau, t_0)$ and $\boldsymbol{\beta}_{\mathbf{0}}(\tau, t_0)$ and is therefore within $B(\rho_0)$ for large enough n. The uniform consistency of $\hat{\boldsymbol{\beta}}(\tau, t_0)$ to $\boldsymbol{\beta}_{\mathbf{0}}(\tau, t_0)$ for $\tau \in [\tau_L, \tau_U], t_0 \in [t_L, t_U]$ then follows.

1.3 *Proof of Theorem 3.2*

From Pepe (1991), $\sup_{t \in [0, \nu)} \|n^{1/2}[\hat{G}(t) - G(t)] - n^{-1/2} \sum_{i=1}^n G(t) \int_0^t y(s)^{-1} dM_i^G(s)\| \to 0$. Using similar empirical process arguments for $\mathcal{F}$, we can show that $n^{-1} \sum_{i=1}^n \boldsymbol{A}_i^*(t_0) Y_i(t) I(L_i^* \leqslant t_0) I(Y_i^* > t_0) \eta_i^* \{I[\log(Y_i^* - t_0) \leqslant \boldsymbol{A}_i^{*T}(t_0)\boldsymbol{b}] - \tau\} G(Y_i^* - L_i^*)^{-1}$ converges to $\boldsymbol{w}(\boldsymbol{b}, \tau, t_0, t)$ uniformly in $\boldsymbol{b}, \tau, t_0$ and t.

Let $\approx$ denote asymptotic equivalence uniformly in $\tau \in [\tau_L, \tau_U]$ and $t_0 \in [t_L, t_U]$. Simple algebraic manipulations show that

$$
\begin{aligned}
&\boldsymbol{S}_n\{\boldsymbol{\beta}_0(\tau,t_0),\tau,t_0\} \\
&= \boldsymbol{S}_n^G\{\boldsymbol{\beta}_0(\tau,t_0),\tau,t_0\} + [\boldsymbol{S}_n\{\boldsymbol{\beta}_0(\tau,t_0),\tau,t_0\} - \boldsymbol{S}_n^G\{\boldsymbol{\beta}_0(\tau,t_0),\tau,t_0\}] \\
&= n^{-1/2}\sum_{i=1}^n \boldsymbol{\xi}_{1,i}(\tau,t_0) - n^{-1/2}\sum_{i=1}^n \boldsymbol{A}_i^*(t_0)\frac{\hat{G}(Y_i^*-L_i^*)-G(Y_i^*-L_i^*)}{\hat{G}(Y_i^*-L_i^*)G(Y_i^*-L_i^*)} I(L_i^* \leqslant t_0)I(Y_i^*>t_0)\eta_i^* \\
&\quad\times\{I[\log(Y_i^*-t_0) \leqslant \boldsymbol{A}_i^{*T}(t_0)\boldsymbol{\beta}_0(\tau,t_0)] - \tau\} \\
&\approx n^{-1/2}\sum_{i=1}^n \boldsymbol{\xi}_{1,i}(\tau,t_0) - n^{-1}\sum_{i=1}^n \boldsymbol{A}_i^*(t_0)\frac{n^{-1/2}\sum_{j=1}^n\int_0^\infty Y_i(s)y(s)^{-1}dM_j^G(s)}{G(Y_i^*-L_i^*)} I(L_i^* \leqslant t_0)I(Y_i^*>t_0)\eta_i^* \\
&\quad\times\{I[\log(Y_i^*-t_0) \leqslant \boldsymbol{A}_i^{*T}(t_0)\boldsymbol{\beta}_0(\tau,t_0)] - \tau\} \\
&= n^{-1/2}\sum_{i=1}^n \boldsymbol{\xi}_{1,i}(\tau,t_0) \\
&\quad - n^{-1/2}\sum_{i=1}^n\int_0^\infty\left\{\sum_{j=1}^n \frac{\boldsymbol{A}_j^*(t_0)Y_j(s)I(L_j^*\leqslant t_0)I(Y_j^*>t_0)\eta_j^*\{I[\log(Y_j^*-t_0)\leqslant \boldsymbol{A}_j^{*T}(t_0)\boldsymbol{\beta}_0(\tau,t_0)]-\tau\}}{nG(Y_j^*-L_j^*)}\right\} \\
&\quad\times\frac{dM_i^G(s)}{y(s)} \\
&\approx n^{-1/2}\sum_{i=1}^n \boldsymbol{\xi}_{1,i}(\tau,t_0) - n^{-1/2}\sum_{i=1}^n\int_0^\infty \boldsymbol{w}(\boldsymbol{\beta}_0(\tau,t_0),\tau,t_0,s)\frac{dM_i^G(s)}{y(s)} \\
&= n^{-1/2}\sum_{i=1}^n\{\boldsymbol{\xi}_{1,i}(\tau,t_0) - \boldsymbol{\xi}_{2,i}(\tau,t_0)\}.
\end{aligned}
$$

We claim that $\mathcal{F}^* = \{\boldsymbol{\xi}_{1,i}(\tau,t_0), \tau \in [\tau_L,\tau_U], t_0 \in [t_L,t_U]\}$ and $\mathcal{F}^{**} = \{\boldsymbol{\xi}_{2,i}(\tau,t_0), \tau \in [\tau_L,\tau_U], t_0 \in [t_L,t_U]\}$ are Donsker classess by using similar arguments of Peng and Fine (2009). As a result of the Donsker theorem, $\boldsymbol{S}_n\{\boldsymbol{\beta}_0(\tau,t_0),\tau,t_0\}$ converges weakly to a mean zero Gaussian process with covariance matrix $\boldsymbol{\Sigma}(\tau',t_0',\tau,t_0) = E\{\boldsymbol{\zeta}_1(\tau',t_0')\boldsymbol{\zeta}_1(\tau,t_0)^T\}$, where $\boldsymbol{\zeta}_i(\tau,t_0) = \boldsymbol{\xi}_{1,i}(\tau,t_0) - \boldsymbol{\xi}_{2,i}(\tau,t_0)$, $i=1,\ldots,n$.

Next, we establish the asymptotic linearity of $\boldsymbol{S}_n^G(\boldsymbol{b},\tau,t_0)$ in the vicinity of $\boldsymbol{b} = \boldsymbol{\beta}_0(\tau,t_0)$; that is, for any positive sequence of $\{d_n\}_{n=1}^\infty$ such that $d_n \to 0$,

$$
\sup_{\boldsymbol{b},\boldsymbol{b}'\in B(\rho_0),\|\boldsymbol{b}-\boldsymbol{b}'\|\leqslant d_n,t_0} \|\{\boldsymbol{S}_n^G(\boldsymbol{b},\tau,t_0) - \boldsymbol{S}_n^G(\boldsymbol{b}',\tau,t_0)\} - n^{1/2}\{\boldsymbol{\mu}(\boldsymbol{b},\tau,t_0) - \boldsymbol{\mu}(\boldsymbol{b}',\tau,t_0)\}\| = o(1), a.s. \tag{1}
$$

Its proof greatly resembles the lines of Alexander (1984) and Lai and Ying (1988). The key is to show

$$Var(I(L_i^* \leqslant t_0)I(Y_i^* > t_0)\eta_i^* G(Y_i^* - L_i^*)^{-1}\boldsymbol{A}_i^*(t_0)\{I[\log(Y_i^* - t_0) \leqslant \boldsymbol{A}_i^{*T}(t_0)\boldsymbol{b}] \\ - I[\log(Y_i^* - t_0) \leqslant \boldsymbol{A}_i^{*T}(t_0)\boldsymbol{b}']\}) \leqslant G_0\|\boldsymbol{b} - \boldsymbol{b}'\|.$$

This follows from the uniform boundedness of $f(t|\tilde{\boldsymbol{A}}(t_0))$ and boundedness of $B(\rho_0)$ and $G(t)$.

It follows from (1) that

$$\begin{aligned}
&\boldsymbol{S}_n(\hat{\boldsymbol{\beta}}(\tau, t_0), \tau, t_0) - \boldsymbol{S}_n(\boldsymbol{\beta}_0(\tau, t_0), \tau, t_0) \\
&= n^{-1/2}\sum_{i=1}^{n} I(L_i^* \leqslant t_0)I(Y_i^* > t_0)\eta_i^* G(Y_i^* - L_i^*)^{-1}\boldsymbol{A}_i^*(t_0)\{I[\log(Y_i^* - t_0) \leqslant \boldsymbol{A}_i^{*T}(t_0)\hat{\boldsymbol{\beta}}(\tau, t_0)] \\
&- I[\log(Y_i^* - t_0) \leqslant \boldsymbol{A}_i^{*T}(t_0)\boldsymbol{\beta}_0(\tau, t_0)]\} \\
&+ n^{-1/2}\sum_{i=1}^{n} I(L_i^* \leqslant t_0)I(Y_i^* > t_0)\eta_i^* \boldsymbol{A}_i^*(t_0)\{I[\log(Y_i^* - t_0) \leqslant \boldsymbol{A}_i^{*T}(t_0)\hat{\boldsymbol{\beta}}(\tau, t_0)] \\
&- I[\log(Y_i^* - t_0) \leqslant \boldsymbol{A}_i^{*T}(t_0)\boldsymbol{\beta}_0(\tau, t_0)]\}\{\hat{G}(Y_i^* - L_i^*)^{-1} - G(Y_i^* - L_i^*)^{-1}\} \\
&\approx n^{1/2}[\boldsymbol{\mu}\{\hat{\boldsymbol{\beta}}(\tau, t_0), \tau, t_0\} - \boldsymbol{\mu}\{\boldsymbol{\beta_0}(\tau, t_0), \tau, t_0\}].
\end{aligned}$$

Taylor expansion of $\boldsymbol{\mu}(\boldsymbol{b})$ around $\boldsymbol{b} = \boldsymbol{\beta}_0(\tau, t_0)$, along with the fact that $\hat{\boldsymbol{\beta}}_0(\tau, t_0)$ uniformly converges to $\boldsymbol{\beta}_0(\tau, t_0)$, gives that

$$\boldsymbol{S}_n(\hat{\boldsymbol{\beta}}(\tau, t_0), \tau, t_0) - \boldsymbol{S}_n(\boldsymbol{\beta}_0(\tau, t_0), \tau, t_0) \approx \boldsymbol{H}\{\boldsymbol{\beta_0}(\tau, t_0), t_0\}n^{1/2}\{\hat{\boldsymbol{\beta}}(\tau, t_0) - \boldsymbol{\beta}_0(\tau, t_0)\}.$$

This implies

$$n^{1/2}\{\hat{\boldsymbol{\beta}}(\tau, t_0) - \boldsymbol{\beta}_0(\tau, t_0)\} \approx -\boldsymbol{H}\{\boldsymbol{\beta_0}(\tau, t_0), t_0\}^{-1}\boldsymbol{S}_n(\boldsymbol{\beta}_0(\tau, t_0), \tau, t_0)$$

and then $n^{1/2}\{\hat{\boldsymbol{\beta}}(\tau, t_0) - \boldsymbol{\beta}_0(\tau, t_0)\}$ converges weakly to a mean zero Gaussian process with covariance matrix

$$\boldsymbol{H}\{\boldsymbol{\beta}_0(\tau', t_0'), t_0'\}^{-1}E\{\boldsymbol{\zeta}(\tau', t_0')\boldsymbol{\zeta}(\tau, t_0)^T\}\boldsymbol{H}\{\boldsymbol{\beta}_0(\tau, t_0), t_0\}^{-T}.$$

1.4 *The justification for the proposed Covariance Estimate*

Denote $\boldsymbol{b}_{n,j}(\tau,t_0)=\boldsymbol{S}_n^{-1}\{\boldsymbol{e}_{n,j}(\tau,t_0),\tau,t_0\}, j=1,2$. It is implied from the proof of Theorem 3.1 that $\{\boldsymbol{b}_{n,j}(\tau,t_0),\tau\in[\tau_L,\tau_U],t_0\in[t_L,t_U]\}$ is within $B(\rho_0)$ with probability 1 for large enough n, and thus $\sup_{\tau,t_0}\|\boldsymbol{b}_{n,j}(\tau,t_0)-\boldsymbol{\beta}_0(\tau,t_0)\|\to 0$, a.s., $j=1,2$. Using arguments similar to proof of weak convergence, we can show that

$$\boldsymbol{S}_n(\boldsymbol{b}_{n,j}(\tau,t_0),\tau,t_0)-\boldsymbol{S}_n(\boldsymbol{\beta}_0(\tau,t_0),\tau,t_0)\approx\boldsymbol{H}\{\boldsymbol{\beta_0}(\tau,t_0),t_0\}n^{1/2}\{\boldsymbol{b}_{n,j}(\tau,t_0)-\boldsymbol{\beta}_0(\tau,t_0)\}.$$

The definitions of $\boldsymbol{D}_n(\tau,t_0)$ and $\boldsymbol{E}_n(\tau,t_0)$ imply $\boldsymbol{H}^{-1}\{\boldsymbol{\beta}_0(\tau,t_0),t_0\}\approx\sqrt{n}\boldsymbol{D}_n(\tau,t_0)\boldsymbol{E}_n^{-1}(\tau,t_0)$. It follows immediately that

$$n\boldsymbol{D}_n(\tau',t_0')\boldsymbol{E}_n^{-1}(\tau',t_0')\hat{\boldsymbol{\Sigma}}(\tau',t_0',\tau,t_0)\boldsymbol{E}_n^{-1}(\tau,t_0)\boldsymbol{D}_n^T(\tau,t_0)$$

is a consistent estimate for $\boldsymbol{\Phi}(\tau',t_0',\tau,t_0)=\boldsymbol{H}\{\boldsymbol{\beta}_0(\tau',t_0'),t_0'\}^{-1}\boldsymbol{\Sigma}(\tau',t_0',\tau,t_0)\boldsymbol{H}\{\boldsymbol{\beta}_0(\tau,t_0),t_0\}^{-T}$, which is the asymptotic covariance matrix of $\sqrt{n}\{\hat{\boldsymbol{\beta}}(\tau,t_0)-\boldsymbol{\beta}_0(\tau,t_0)\}$.

1.5 *The justification for the proposed estimating equation in Section 4*

Recall that $\boldsymbol{K}^*(t_0)=(1,I(X^*>t_0),\tilde{\boldsymbol{Z}}^{*T},\tilde{\boldsymbol{Z}}^{*T}I(X^*>t_0))^T$ and the estimating equation is

$$\boldsymbol{S}_n(\boldsymbol{r},\tau,t_0)=0,$$

where

$$\boldsymbol{S}_n(\boldsymbol{r},\tau,t_0)=n^{-1/2}\sum_{i=1}^n\frac{I(L_i^*\leqslant t_0)I(Y_i^*>t_0)\eta_i^*}{\hat{G}(Y_i^*-L_i^*)}\boldsymbol{K}_i^*(t_0)\{I[\log(Y_i^*-t_0)\leqslant\boldsymbol{K}_i^{*T}(t_0)\boldsymbol{r}]-\tau\}.$$

Similar to the one-sample case, we further define $\tilde{\boldsymbol{K}}^*(t_0)=(1,I(T_1^*>t_0),\tilde{\boldsymbol{Z}}^{*T},\tilde{\boldsymbol{Z}}^{*T}I(T_1^*>t_0))^T$ and $\tilde{\boldsymbol{K}}(t_0)=(1,I(T_1>t_0),\tilde{\boldsymbol{Z}}^T,\tilde{\boldsymbol{Z}}^TI(T_1>t_0))^T$. Denote

$$E\left\{\frac{I(L^*\leqslant t_0)I(Y^*>t_0)\eta^*}{G(Y^*-L^*)}\boldsymbol{K}^*(t_0)\{I[\log(Y^*-t_0)\leqslant\boldsymbol{K}^{*T}(t_0)\boldsymbol{r}]-\tau\}\right\}$$

by (I). The key justification for the proposed estimating equation in Section 4 is to show $(I)=0$.

First, it is easy to see that

$$(I) = E\left\{\frac{I(L^* \leqslant t_0)I(T_2^* > t_0, T_2^* < C^*)}{G(T_2^* - L^*)}\tilde{\boldsymbol{K}}^*(t_0)\{I[\log(T_2^* - t_0) \leqslant \tilde{\boldsymbol{K}}^{*T}(t_0)\boldsymbol{r}] - \tau\}\right\}.$$

Note that $\{L \leqslant Y\} \subseteq \{L \leqslant t_0, t_0 < T_2 \leqslant t_0 + u, T_2 < C, T_1 > t_0\}$. This implies

$$f_{(L^*,C^*,T_1^*,T_2^*,\tilde{\boldsymbol{Z}}^*)}(l, c, t_1, t_2, \bar{\boldsymbol{z}}) = \frac{1}{\alpha} f_{(L,C,T_1,T_2,\tilde{\boldsymbol{Z}}^*)}(l, c, t_1, t_2, \bar{\boldsymbol{z}})$$

in the region of $\{(l, c, t_1, t_2) : l \leqslant t_0, t_0 < t_2 \leqslant t_0 + u, t_2 < c, t_1 > t_0\}$. Thus,

$$\begin{aligned}
(I) &= \frac{1}{\alpha}E\left\{\frac{I(L \leqslant t_0)I(T_2 > t_0, T_2 < C)}{G(T_2 - L)}\tilde{\boldsymbol{K}}(t_0)\{I[\log(T_2 - t_0) \leqslant \tilde{\boldsymbol{K}}^T(t_0)\boldsymbol{r}] - \tau\}\right\} \\
&= \frac{1}{\alpha}E\left\{\frac{I(L \leqslant t_0)I(T_2 > t_0)\tilde{\boldsymbol{K}}(t_0)\{I[\log(T_2 - t_0) \leqslant \tilde{\boldsymbol{K}}^T(t_0)\boldsymbol{r}] - \tau\}}{G(T_2 - L)}E[I(T_2 - L < D)|T_1, T_2, L, \tilde{\boldsymbol{Z}}]\right\} \\
&= \frac{1}{\alpha}E\left\{I(L \leqslant t_0)I(T_2 > t_0)\tilde{\boldsymbol{K}}(t_0)\{I[\log(T_2 - t_0) \leqslant \tilde{\boldsymbol{K}}^T(t_0)\boldsymbol{r}] - \tau\} \times \frac{G(T_2 - L)}{G(T_2 - L)}\right\} \\
&= \frac{1}{\alpha}E\left\{I(L \leqslant t_0)I(T_2 > t_0)\tilde{\boldsymbol{K}}(t_0)\{I[\log(T_2 - t_0) \leqslant \tilde{\boldsymbol{K}}^T(t_0)\boldsymbol{r}] - \tau\}\right\},
\end{aligned}$$

where the third equality above uses the assumption of $D \perp (T_1, T_2, L, \tilde{\boldsymbol{Z}})$. Under the assumption of $L \perp T_2 | (T_1, \tilde{\boldsymbol{Z}})$, we have

$$\begin{aligned}
&E\left\{I(L \leqslant t_0)I(T_2 > t_0)\tilde{\boldsymbol{K}}(t_0)I[\log(T_2 - t_0) \leqslant \tilde{\boldsymbol{K}}^T(t_0)\boldsymbol{r}]\right\} \\
&= E\left\{\tilde{\boldsymbol{K}}(t_0)P[L \leqslant t_0, T_2 > t_0, \log(T_2 - t_0) \leqslant \tilde{\boldsymbol{K}}^T(t_0)\boldsymbol{r}|\tilde{\boldsymbol{K}}^T(t_0)]\right\} \\
&= E\left\{\tilde{\boldsymbol{K}}(t_0)P[L \leqslant t_0|\tilde{\boldsymbol{K}}^T(t_0)] \times P[T_2 > t_0, \log(T_2 - t_0) \leqslant \tilde{\boldsymbol{K}}^T(t_0)\boldsymbol{r}|\tilde{\boldsymbol{K}}^T(t_0)]\right\} \\
&= E\left\{\tilde{\boldsymbol{K}}(t_0)P[L \leqslant t_0|\tilde{\boldsymbol{K}}^T(t_0)] \times P[\log(T_2 - t_0) \leqslant \tilde{\boldsymbol{K}}^T(t_0)\boldsymbol{r}|T_2 > t_0, \tilde{\boldsymbol{K}}(t_0)] \times P[T_2 > t_0|\tilde{\boldsymbol{K}}(t_0)]\right\} \\
&= \tau E\left\{\tilde{\boldsymbol{K}}(t_0)P[L \leqslant t_0|\tilde{\boldsymbol{K}}^T(t_0)] \times P[T_2 > t_0|\tilde{\boldsymbol{K}}(t_0)]\right\} \\
&= \tau E\left\{\tilde{\boldsymbol{K}}(t_0)P[L \leqslant t_0, T_2 > t_0|\tilde{\boldsymbol{K}}(t_0)]\right\} \\
&= \tau E\left\{I(L \leqslant t_0)I(T_2 > t_0)\tilde{\boldsymbol{K}}(t_0)\right\}.
\end{aligned}$$

It then follows that $(I) = 0$.

Web Appendix B: Additional Results from Numerical Studies

2.1 *A numerical example comparing the proposed method with the copula modeling approach*

We conduct simulation studies to compare the proposed method with the classic copula modeling approach. Specifically, we generate T_1 and T_2 as follows:

0. Set $n = 0$.
1. Generate a $Bernoulli(0.5)$ random variate W
2. If $W = 0$, then keep generating $\log(T_1)$ from $N(0, 0.4)$ distribution and T_2 as $\log(T_2) = 0.2 + 0.4\log(T_1) + \epsilon_1$ until $T_2 > 1.4$, where ϵ_1 is a $N(0, 0.3)$ random error. Take the (T_1, T_2) with $T_2 > 1.4$.
   If $W = 1$, then keep generating T_1 from $N(0, 0.3)$ distribution and T_2 as $\log(T_2) = 0.3 - 1.4\log(T_1) + \epsilon_2$ until $T_2 \leqslant 1.4$, where ϵ_2 is a $N(0, 0.2)$ random error. Take the (T_1, T_2) with $T_2 \leqslant 1.4$.
3. Increase n by 1.
4. Go back to steps 1-3 unless n equals the specified sample size.

By this data generate scheme, we have T_1 and T_2 are negatively associated when T_2 is small (i.e. $T_2 \leqslant 1.4$) but are positively associated when T_2 is large (i.e. $T_2 > 1.4$). In addition, we generate C with $\log C$ following $Unif(-0.2, 2)$ distribution. Figure S1 presents the scatter plots for (T_1, T_2) and $(T_1 \wedge T_2, T_2)$ based on one randomly selected simulated dataset of size 1000.

We first apply Fine et al. (2001)'s method to this simulated dataset, assuming a Clayton's copula model for the dependence structure between T_1 and T_2. We obtain a copula parameter estimate, 0.96, with 95% CI, $(0.83, 1.08)$. This result suggests the independence between T_1 and T_2 (in the upper wedge), which clearly contradicts with the true relationship between T_1 and T_2. We examine 499 other simulated datasets, based on 467 of which, the application of Fine et al. (2001)'s method leads to the same conclusion that T_1 and T_2 are independent (in the upper wedge).

[Figure 1 about here.]

We then apply the proposed method to the same simulated dataset shown in Figure S1. The true $LCQRR(\tau; t_0)$ and the estimated $LCQRR(\tau; t_0)$ (along with the corresponding 95% confidence intervals) are presented in Figure S2. We can see that the proposed method allow us to effectively utilize the observed data of (X, Y) to identify the positive dependence between T_1 and T_2 associated with large T_2. With small t_0 (e.g. $t_0 < 1$) and small τ (e.g. $\tau = 0.25$), our measure $LCQRR(\tau; t_0)$ can also partially capture the negative dependence associated with small T_2.

[Figure 2 about here.]

As suggested by this example, existing modeling approaches that assume constant dependence between T_1 and T_2 may only reveal an overall average dependence, which can lead to misleading conclusions when the constant dependence assumption is violated. In contrast, the proposed new dependence measure entails a sensible approach to uncovering interesting dynamic patterns in the dependence structure without involving strong model assumptions.

2.2 *Additional results for numerical studies in Section 5*

Figure S3 presents the simulation results for Scenario 2 on the estimation of $LCQRR(\tau; t_0)$.

[Figure 3 about here.]

Table 1 reports the simulation results on the second stage inference for $LCQRR(\tau; t_0)$ over τ.

[Table 1 about here.]

Figure S4 plots the estimated $LCQRR(\tau; t_0)$ along with 95% pointwise confidence intervals for fixed t_0 values based on the Denmark diabetes registry data.

[Figure 4 about here.]

Figure S5 plots the estimated $\boldsymbol{\gamma}_0^{(4)}(\tau, t_0)$ along with 95% pointwise confidence intervals for fixed t_0 values based on the Denmark diabetes registry data.

[Figure 5 about here.]

2.3 *Sample code*

A sample code for implementing the proposed method can be found at
http://web1.sph.emory.edu/users/lpeng/Rpackage.html.

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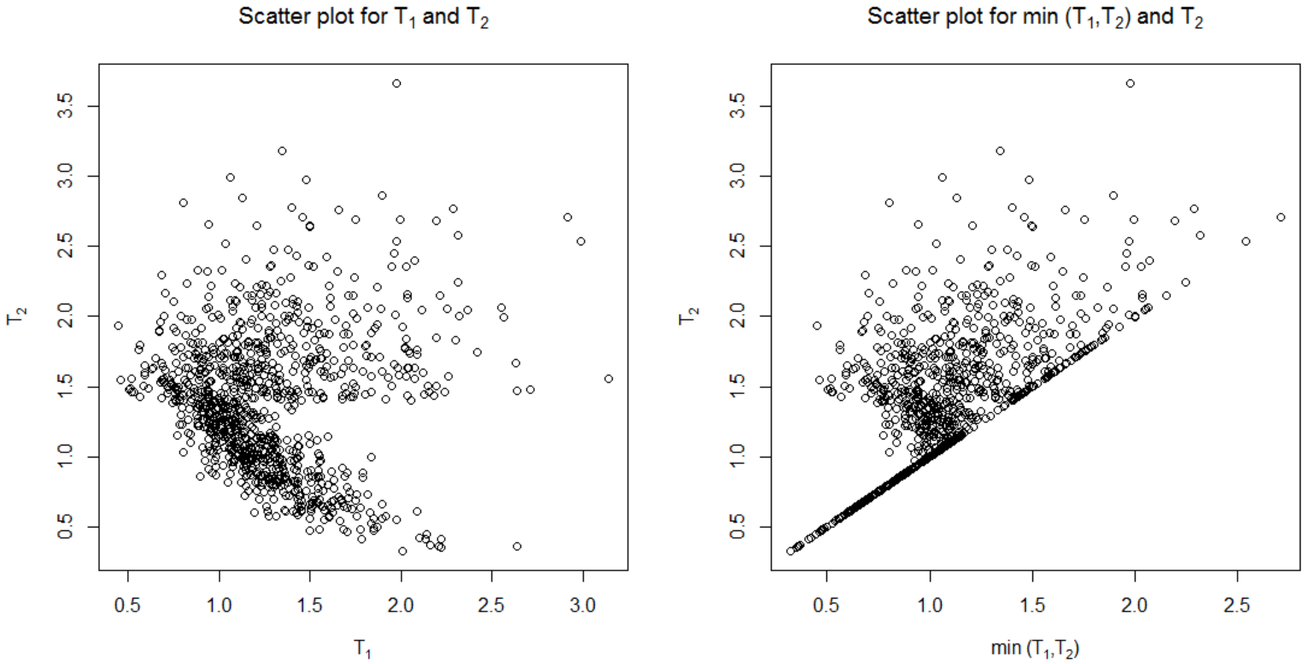

Figure S1. A simulated data example: scatter plots for (T_1, T_2) and $(T_1 \wedge T_2, T_2)$ based on one simulated dataset of size 1000.

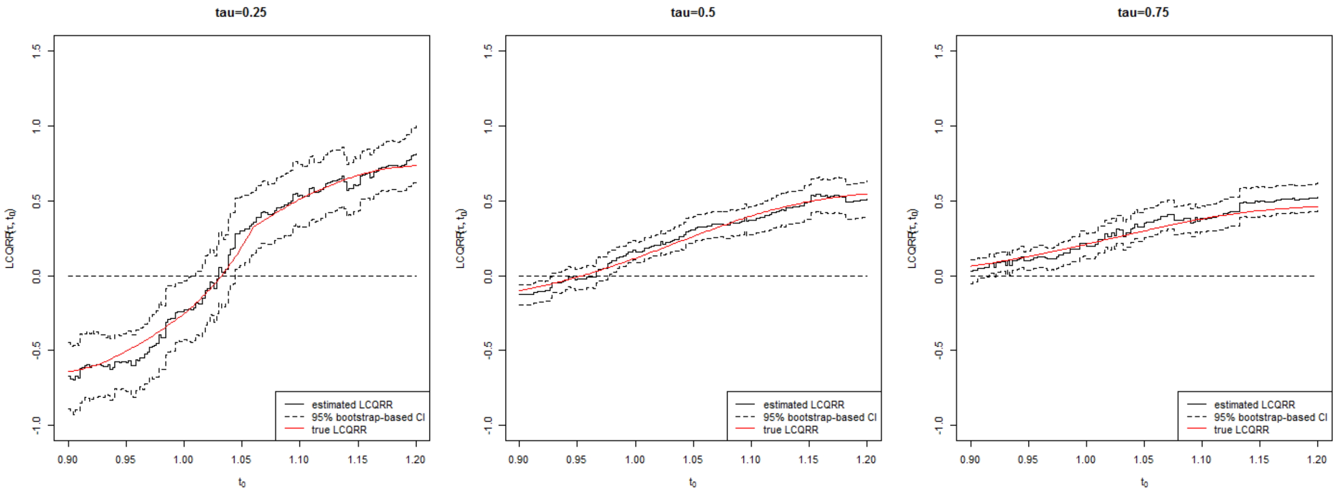

Figure S2. A simulated data example: estimated $LCQRR(\tau; t_0)$ (black line), true value of $LCQRR(\tau; t_0)$ (red line) and 95% Wald-type boostrapping CI (dashed line).

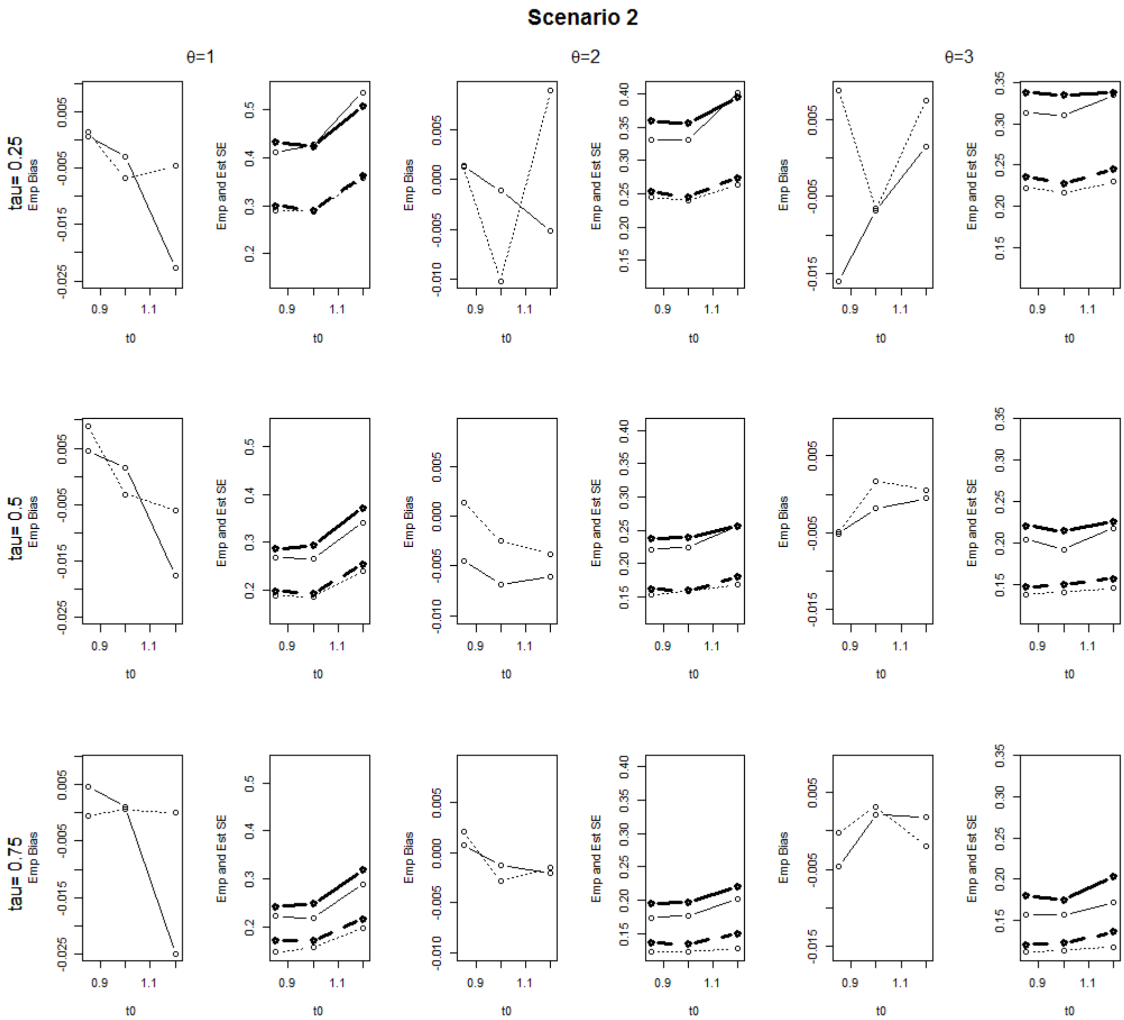

Figure S3. Simulation results for Scenario 2: Empirical bias (EmpBias), empirical standard error (EmpSE) and average estimated standard error (EstSE) of the proposed estimator of $LCQRR$. EmpBias for $n = 200$ and that for $n = 400$ are plotted in solid lines and dotted lines respectively. EmpSE and EstSE for $n = 200$ are plotted in solid lines and bold solid lines respectively. EmpSE and EstSE for $n = 200$ are plotted in dotted lines and bold dashed lines respectively.

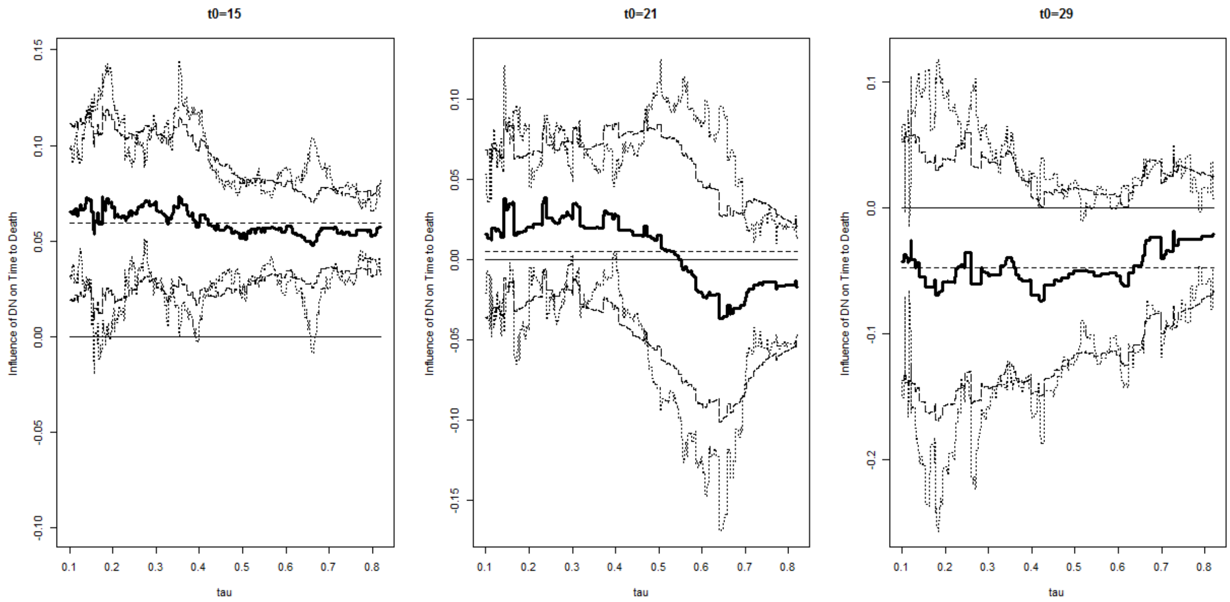

Figure S4. Denmark Diabetes Registry Study: Estimated $LCQRR(\tau; t_0)$ (bold solid lines), corresponding 95% pointwise confidence intervals (dotted lines), 95% pointwise Wald-type bootstrapping confidence intervals (long-dashed lines), and overall influence of DN over τ (horizontal dashed lines)

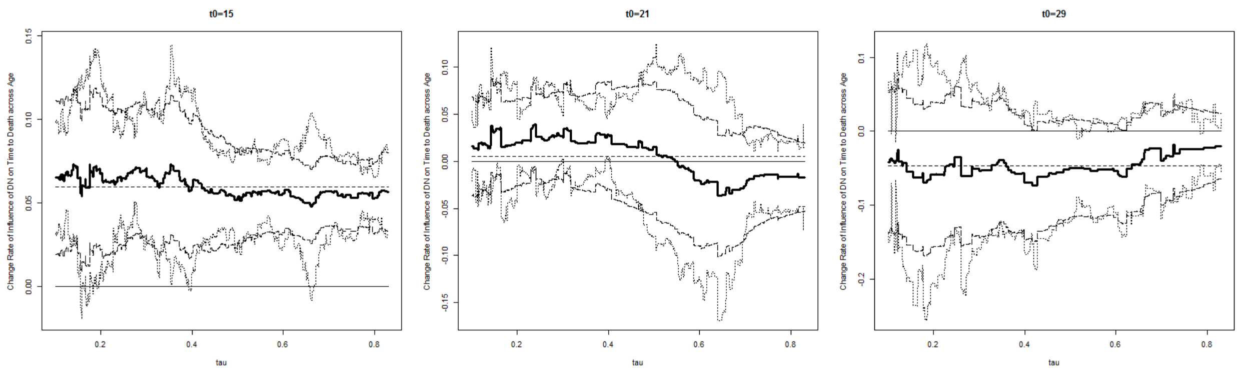

Figure S5. Denmark Diabetes Registry Study: Estimated $\boldsymbol{\gamma}_0^{(4)}(\tau, t_0)$ (bold solid lines), the corresponding 95% pointwise confidence intervals (dotted lines) and 95% pointwise Wald-type bootstrapping confidence intervals (long-dashed lines), and the overall influence of DN across time (horizontal dashed lines).

Table 1

Empirical biases, empirical standard errors and average standard errors estimates of $\hat{\Omega}_{t_0}$ and empirical rejection rates for H_{03} and H_{04}.

			$\hat{\Omega}_{t_0}$			H_{03}	H_{04}
θ	t_0	n	EmpBias	EmpSE	EstSE	EmpRR	EmpRR
			Scenario 1 $\tau \in [0.1, 0.87]$				
1	0.55	200	0.006	0.193	0.195	0.060	0.041
		400	0.004	0.128	0.134	0.043	0.047
	0.84	200	0.006	0.197	0.201	0.052	0.036
		400	0.001	0.144	0.141	0.054	0.046
	1.10	200	0.005	0.262	0.247	0.061	0.050
		400	0.002	0.175	0.174	0.052	0.054
2	0.55	200	0.018	0.205	0.211	0.575	0.051
		400	0.007	0.139	0.144	0.893	0.069
	0.84	200	0.018	0.221	0.219	0.791	0.092
		400	0.006	0.153	0.152	0.982	0.139
	1.10	200	-0.005	0.269	0.274	0.697	0.053
		400	-0.004	0.191	0.193	0.955	0.091
3	0.55	200	0.017	0.220	0.216	0.918	0.072
		400	0.001	0.146	0.149	0.999	0.154
	0.84	200	-0.005	0.223	0.225	0.984	0.126
		400	0.006	0.156	0.157	1.000	0.292
	1.10	200	0.000	0.302	0.310	0.908	0.051
		400	-0.001	0.214	0.216	0.997	0.114
			Scenario 2 $\tau \in [0.1, 0.9]$				
1	0.85	200	0.003	0.244	0.236	0.066	0.045
		400	0.003	0.167	0.164	0.052	0.047
	1.00	200	0.004	0.242	0.232	0.061	0.045
		400	0.007	0.161	0.164	0.052	0.047
	1.20	200	-0.009	0.303	0.279	0.075	0.064
		400	-0.003	0.215	0.203	0.073	0.060
2	0.85	200	0.002	0.204	0.198	0.859	0.092
		400	0.005	0.138	0.139	0.992	0.180
	1.00	200	-0.012	0.188	0.194	0.929	0.157
		400	0.003	0.136	0.137	1.000	0.314
	1.20	200	-0.007	0.220	0.213	0.938	0.271
		400	0.004	0.156	0.151	0.997	0.426
3	0.85	200	0.010	0.185	0.182	0.998	0.236
		400	0.003	0.129	0.128	1.000	0.492
	1.00	200	0.003	0.178	0.179	1.000	0.363
		400	-0.002	0.126	0.126	1.000	0.664
	1.20	200	-0.007	0.187	0.190	1.000	0.492
		400	0.001	0.136	0.134	1.000	0.837