Web-based Supplementary Materials for "Alternative Measures of Between-Study Heterogeneity in Meta-Analysis: Reducing the Impact of Outlying Studies" by

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Web Appendix A: Sensitivity analysis

Since the weighted median in Q_m is discontinuous due to the indicator function (Parzen et al., 1994) in Equation (2) in the main text, the approach in Horowitz (1998) is applied to approximate the indicator function $\mathbb{I}(t > 0)$ by a smooth function $J(t)$ in the following simulations and case studies. For example, $J(t)$ can be the scaled expit function $J_\epsilon(t) = 1/[1+\exp(-t/\epsilon)]$, where ϵ is a pre-specified small constant, say 10^{-4}. This section presents sensitivity analysis on the choice of ϵ. We use the data of the case study in Section 6.1 in the main text. Web Table 1 presents the results based on $B = 10000$ resampling iterations.

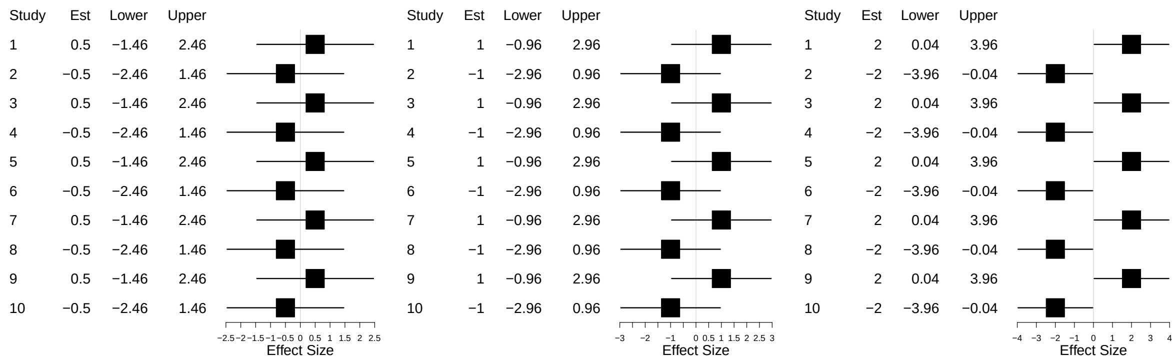
Web Table 1: Sensitivity analysis on the choice of ϵ.

ϵ	Q_m	p-value	$\hat{\tau}_m$ (95% CI)	H_m (95% CI)	I_m^2 (95% CI)
10^{-2}	31.340	0.006	0.298 (0, 0.561)	1.354 (1, 1.884)	0.455 (0, 0.718)
10^{-3}	31.273	0.006	0.296 (0, 0.563)	1.352 (1, 1.886)	0.453 (0, 0.719)
10^{-4}	31.259	0.006	0.296 (0, 0.563)	1.351 (1, 1.886)	0.452 (0, 0.719)
10^{-5}	31.259	0.006	0.296 (0, 0.563)	1.351 (1, 1.886)	0.452 (0, 0.719)

Web Appendix B: Artificial meta-analyses

This section illustrates that I_r^2 and I_m^2 can be larger than I^2 and provide useful information on assessing heterogeneity. Three artificial meta-analyses were created; each contains ten studies with the same within-study variance 1. The observed effect sizes in half of the studies are $y_i = b$, and those in another half are $y_i = -b$, where b was set to 0.5, 1, and 2. Web Figure 1 presents the corresponding forest plots. Note that in these meta-analyses, the condition $w_i(y_i - \bar{\mu})^2 = C$ is satisfied, so the equality in $I_r^2 \leq I^2 + (1 - 2/\pi)(1 - I^2)$ holds.

Web Figure 1a shows the meta-analysis with $b = 0.5$. Since the observed effect size of each study is contained in the 95% CIs of all other studies, the collected studies are considered homogeneous; all of I^2, I_r^2, and I_m^2 are calculated as 0. For the meta-analysis with $b = 1.0$ shown in Web Figure 1b, five studies report the effect size -1, lying outside the 95% CIs $(-0.96, 2.96)$ of the other five studies. Despite this, the 95% CIs of the total ten studies overlap in a large region, i.e., $(-0.96, 0.96)$. Therefore, the between-study heterogeneity is moderate, but may not be substantial. The three heterogeneity measures are calculated as $I^2 = 0.10$, $I_r^2 = 0.43$, and $I_m^2 = 0.36$; I^2 may indicate homogeneity but both I_r^2 and I_m^2 imply moderate heterogeneity. Web Figure 1c shows the meta-analysis with $b = 2$. The 95% CIs of five studies do not overlap with those in the other five studies; therefore, these studies are clearly heterogeneous. The heterogeneity measures are calculated as $I^2 = 0.78$, $I_r^2 = 0.86$, and $I_m^2 = 0.84$; all suggest considerable heterogeneity.



(a) $I^2 = I_r^2 = I_m^2 = 0$. (b) $I^2 = 0.10$, $I_r^2 = 0.43$, $I_m^2 = 0.36$. (c) $I^2 = 0.78$, $I_r^2 = 0.86$, $I_m^2 = 0.84$.

Web Figure 1: Forest plots of the three artificial meta-analyses. The column "Est" contains the observed effect size in each study; the columns "Lower" and "Upper" contain the lower and upper bounds of the corresponding 95% CI.

Web Appendix C: Proofs

The proofs will frequently use the property about the mean of folded normal distribution: if $X \sim N(\mu, \sigma^2)$, then $\mathrm{E}\,|X| = \sigma\sqrt{\frac{2}{\pi}}e^{-\mu^2/(2\sigma^2)} + \mu(1 - 2\Phi(-\mu/\sigma))$, where $\Phi(\cdot)$ is the cumulative density function of standard normal distribution. Let $\lfloor x \rfloor$ be the largest integer less than or equal to x.

Proof of Proposition 1. Note that $\bar{\mu} = \frac{\sum_{i=1}^n w_i y_i/n}{\sum_{i=1}^n w_i/n} \xrightarrow{P} \frac{\mathrm{E}[w_1 y_1]}{\mathrm{E}[w_1]} = \mu$, we have

$$\begin{aligned}
Q/n &= \frac{1}{n}\sum_{i=1}^n w_i[(y_i - \mu) - (\bar{\mu} - \mu)]^2 \\
&= \frac{1}{n}\sum_{i=1}^n w_i(y_i - \mu)^2 - 2(\bar{\mu} - \mu)\cdot\frac{1}{n}\sum_{i=1}^n w_i(y_i - \mu) + (\bar{\mu} - \mu)^2\cdot\frac{1}{n}\sum_{i=1}^n w_i \\
&\xrightarrow{P} \mathrm{E}[w_1(y_1 - \mu)^2] = 1.
\end{aligned}$$

Therefore, $I^2 = 1 - \frac{1}{Q/(n-1)} \xrightarrow{P} 0$.

For Q_r, applying the triangle inequality $|x| - |y| \leq |x - y|$, we have

$$\begin{aligned}
\sqrt{w_i}|y_i - \bar{\mu}| - \sqrt{w_i}|y_i - \mu| &\leq \sqrt{w_i}|\bar{\mu} - \mu|; \\
\sqrt{w_i}|y_i - \mu| - \sqrt{w_i}|y_i - \bar{\mu}| &\leq \sqrt{w_i}|\bar{\mu} - \mu|.
\end{aligned}$$

Averaging each of the above two inequalities for $i = 1, \ldots, n$, we have

$$\left| Q_r/n - \frac{1}{n} \sum_{i=1}^{n} \sqrt{w_i} |y_i - \mu| \right| \leq |\bar{\mu} - \mu| \cdot \frac{1}{n} \sum_{i=1}^{n} \sqrt{w_i} \xrightarrow{P} 0.$$

Furthermore,

$$\frac{1}{n} \sum_{i=1}^{n} \sqrt{w_i} |y_i - \mu| \xrightarrow{P} \mathrm{E}[|\sqrt{w_1}(y_1 - \mu)|] = \sqrt{2/\pi}.$$

Therefore, $Q_r/n \xrightarrow{P} \sqrt{2/\pi}$, and $I_r^2 = 1 - \frac{n-1}{n} \cdot \frac{2/\pi}{(Q_r/n)^2} \xrightarrow{P} 0$.

For Q_m, by the theory of M-estimation (Huber and Ronchetti, 2009), the weighted median $\hat{\mu}_m \xrightarrow{P} \mu$. Similarly applying the triangle inequality, we have

$$\left| Q_m/n - \frac{1}{n} \sum_{i=1}^{n} \sqrt{w_i} |y_i - \mu| \right| \leq |\hat{\mu}_m - \mu| \cdot \frac{1}{n} \sum_{i=1}^{n} \sqrt{w_i} \xrightarrow{P} 0.$$

Hence, $Q_m/n \xrightarrow{P} \mathrm{E}[|\sqrt{w_1}(y_1 - \mu)|] = \sqrt{2/\pi}$ and $I_m^2 = 1 - \frac{2/\pi}{(Q_m/n)^2} \xrightarrow{P} 0$. □

Proof of Proposition 2. Now, the weights w_i have a common value $w = 1/\sigma^2$. Under the random-effects setting, the weighted average and weighted median still converge to the true overall effect size μ in probability. Similarly to the derivations in Proposition 1, $Q/n \xrightarrow{P} \mathrm{E}[w(y_1-\mu)^2] = (\sigma^2+\tau^2)/\sigma^2$; both Q_r/n and Q_m/n converge to $\mathrm{E}[|\sqrt{w}(y_1-\mu)|] = \sqrt{\frac{2}{\pi}}\sqrt{(\sigma^2 + \tau^2)/\sigma^2}$. Hence, $I^2 = 1 - \frac{1}{Q/(n-1)} \xrightarrow{P} I_0^2$, $I_r^2 = 1 - \frac{n-1}{n} \cdot \frac{2/\pi}{(Q_r/n)^2} \xrightarrow{P} I_0^2$, and $I_m^2 = 1 - \frac{2/\pi}{(Q_m^2/n)^2} \overset{P}{\xrightarrow{P}} I_0^2$, where $I_0^2 = \tau^2/(\sigma^2 + \tau^2)$. □

Proof of Proposition 3. Without loss of generality, let $y_i = z_i + C$ for $i = 1, \ldots, \lfloor n\eta \rfloor$ and $y_i = z_i$ for $i = \lfloor n\eta \rfloor + 1, \ldots, n$, where $z_i \overset{\text{iid}}{\sim} N(\mu, \sigma^2 + \tau^2)$. Denote the weights $w_i = w = 1/\sigma^2$.

Note that $\bar{\mu} = \frac{\sum_{i=1}^{n} y_i}{n} = \frac{\lfloor n\eta \rfloor}{n} \cdot \frac{\sum_{i=1}^{\lfloor n\eta \rfloor}(z_i+C)}{\lfloor n\eta \rfloor} + \frac{n-\lfloor n\eta \rfloor}{n} \cdot \frac{\sum_{i=n-\lfloor n\eta \rfloor+1}^{n} z_i}{n-\lfloor n\eta \rfloor} \xrightarrow{P} \eta(\mu+C)+(1-\eta)\mu = \mu + \eta C$. Therefore,

$$\begin{aligned} Q/n &= \frac{w}{n} \sum_{i=1}^{n} [(y_i - \mu - \eta C) - (\bar{\mu} - \mu - \eta C)]^2 \\ &= \frac{w}{n} \sum_{i=1}^{n} (y_i - \mu - \eta C)^2 - 2w(\bar{\mu} - \mu - \eta C) \cdot \frac{1}{n} \sum_{i=1}^{n} (y_i - \mu - \eta C) + w(\bar{\mu} - \mu - \eta C)^2. \end{aligned}$$

The last two terms on the right hand side converge to 0 in probability. For the first term,

$$\begin{aligned} &\frac{w}{n} \sum_{i=1}^{n} (y_i - \mu - \eta C)^2 \\ &= w\frac{\lfloor n\eta \rfloor}{n} \cdot \frac{\sum_{i=1}^{\lfloor n\eta \rfloor}(z_i - \mu + (1-\eta)C)^2}{\lfloor n\eta \rfloor} + w\frac{n - \lfloor n\eta \rfloor}{n} \cdot \frac{\sum_{i=n-\lfloor n\eta \rfloor+1}^{n}(z_i - \mu - \eta C)^2}{n - \lfloor n\eta \rfloor} \\ &\xrightarrow{P} w\eta\, \mathrm{E}[(z_1 - \mu + (1-\eta)C)^2] + w(1-\eta)\, \mathrm{E}[(z_1 - \mu - \eta C)^2] \\ &= \eta[\sigma^2 + \tau^2 + (1-\eta)^2 C^2]/\sigma^2 + (1-\eta)(\sigma^2 + \tau^2 + \eta^2 C^2)/\sigma^2 \\ &= (\sigma^2 + \tau^2)/\sigma^2 + \eta(1-\eta)C^2/\sigma^2 \\ &= (1 - I_0^2)^{-1} + r_1 r_2, \end{aligned}$$

where $I_0^2 = \tau^2/(\sigma^2+\tau^2)$, $r_1 = (1-\eta)C/\sigma$, and $r_2 = \eta C/\sigma$. Therefore, $Q/n \xrightarrow{P} (1-I_0^2)^{-1} + r_1 r_2$ and

$$I^2 = 1 - \frac{1}{Q/(n-1)} \xrightarrow{P} 1 - [(1-I_0^2)^{-1} + r_1 r_2]^{-1}.$$

To derive the asymptotic value of I_r^2, we apply the triangle inequality again as in the proof of Proposition 1, and obtain

$$\left| Q_r/n - \frac{\sqrt{w}}{n}\sum_{i=1}^n |y_i - \mu - \eta C| \right| \leq \sqrt{w}|\bar{\mu} - \mu - \eta C| \xrightarrow{P} 0.$$

Note that

$$\begin{aligned}
&\frac{\sqrt{w}}{n}\sum_{i=1}^n |y_i - \mu - \eta C| \\
&= \sqrt{w}\frac{\lfloor n\eta \rfloor}{n} \cdot \frac{\sum_{i=1}^{\lfloor n\eta \rfloor} |z_i - \mu + (1-\eta)C|}{\lfloor n\eta \rfloor} + \sqrt{w}\frac{n - \lfloor n\eta \rfloor}{n} \cdot \frac{\sum_{i=n-\lfloor n\eta \rfloor+1}^{n} |z_i - \mu - \eta C|}{n - \lfloor n\eta \rfloor} \\
&\xrightarrow{P} \sqrt{w}\eta \, \mathrm{E}[|z_1 - \mu + (1-\eta)C|] + \sqrt{w}(1-\eta)\, \mathrm{E}[|z_1 - \mu - \eta C|] \\
&= \frac{\eta}{\sigma}\left[\sqrt{\sigma^2+\tau^2}\sqrt{\frac{2}{\pi}}\exp\left(-\frac{(1-\eta)^2C^2}{2(\sigma^2+\tau^2)}\right) + (1-\eta)C\left(1 - 2\Phi\left(-\frac{(1-\eta)C}{\sqrt{\sigma^2+\tau^2}}\right)\right)\right] \\
&\quad + \frac{1-\eta}{\sigma}\left[\sqrt{\sigma^2+\tau^2}\sqrt{\frac{2}{\pi}}\exp\left(-\frac{\eta^2C^2}{2(\sigma^2+\tau^2)}\right) - \eta C\left(1 - 2\Phi\left(\frac{\eta C}{\sqrt{\sigma^2+\tau^2}}\right)\right)\right] \\
&= \eta\left[\sqrt{\frac{2}{\pi}}(1-I_0^2)^{-1/2}\exp\left(-\frac{1}{2}r_1^2(1-I_0^2)\right) + r_1\left(1 - 2\Phi\left(-r_1(1-I_0^2)^{1/2}\right)\right)\right] \\
&\quad + (1-\eta)\left[\sqrt{\frac{2}{\pi}}(1-I_0^2)^{-1/2}\exp\left(-\frac{1}{2}r_2^2(1-I_0^2)\right) - r_2\left(1 - 2\Phi\left(r_2(1-I_0^2)^{1/2}\right)\right)\right].
\end{aligned}$$

Therefore, Q_r/n also converges to the value above in probability, and

$$\begin{aligned}
I_r^2 &= 1 - \frac{n-1}{n} \cdot \frac{2/\pi}{(Q_r/n)^2} \\
&\xrightarrow{P} 1 - \left\{\eta\left[(1-I_0^2)^{-1/2}\exp\left(-\frac{1}{2}r_1^2(1-I_0^2)\right) + \sqrt{\frac{\pi}{2}}r_1\left(1 - 2\Phi\left(-r_1(1-I_0^2)^{1/2}\right)\right)\right]\right. \\
&\quad \left. + (1-\eta)\left[(1-I_0^2)^{-1/2}\exp\left(-\frac{1}{2}r_2^2(1-I_0^2)\right) - \sqrt{\frac{\pi}{2}}r_2\left(1 - 2\Phi\left(r_2(1-I_0^2)^{1/2}\right)\right)\right]\right\}^{-2}.
\end{aligned}$$

Finally, we derive the asymptotic value of I_m^2. The weighted median $\hat{\mu}_m$ is defined as the solution to $\sum_{i=1}^n \psi(\theta) = 0$, where $\psi(\theta) = w[\mathbb{I}(\theta \geq y_i) - 0.5]$. Equivalently, $\hat{\mu}_m$ is the solution to $\sum_{i=1}^n \tilde{\psi}(\theta) = 0$, where $\tilde{\psi}(\theta) = \mathbb{I}(\theta \geq y_i) - 0.5$ as we assume that the weights are equal. By the theory of M-estimation (Huber and Ronchetti, 2009), $\hat{\mu}_m \xrightarrow{P} \mu_0$, where

μ_0 is the solution to $\mathrm{E}[\tilde{\psi}(\theta)] = 0$. Specifically,

$$\begin{aligned}
\mathrm{E}[\tilde{\psi}(\theta)] &= \Pr(\theta \geq y_i) - 0.5 \\
&= \Pr(\theta \geq y_i, 1 \leq i \leq \lfloor n\eta \rfloor) + \Pr(\theta \geq y_i, \lfloor n\eta \rfloor + 1 \leq i \leq n) - 0.5 \\
&= \eta \Pr(z_i \leq \theta - C) + (1-\eta) \Pr(z_i \leq \theta) - 0.5 \\
&= \eta \Phi\left(\frac{\theta - \mu - C}{\sqrt{\sigma^2 + \tau^2}}\right) + (1-\eta)\Phi\left(\frac{\theta - \mu}{\sqrt{\sigma^2+\tau^2}}\right) - 0.5.
\end{aligned}$$

Therefore, μ_0 satisfied the following equation:

$$\eta\Phi\left(-\frac{\mu + C - \mu_0}{\sqrt{\sigma^2+\tau^2}}\right) + (1-\eta)\Phi\left(\frac{\mu_0 - \mu}{\sqrt{\sigma^2+\tau^2}}\right) = 0.5. \tag{1}$$

Applying the triangle inequality as in the proof of Proposition 1, we have

$$\left| Q_m/n - \frac{\sqrt{w}}{n}\sum_{i=1}^{n} |y_i - \mu_0| \right| \leq \sqrt{w}|\hat{\mu}_m - \mu_0| \xrightarrow{P} 0.$$

Note that

$$\begin{aligned}
&\frac{\sqrt{w}}{n}\sum_{i=1}^{n}|y_i - \mu_0| \\
&= \sqrt{w}\frac{\lfloor n\eta \rfloor}{n} \cdot \frac{\sum_{i=1}^{\lfloor n\eta \rfloor}|z_i - \mu_0 + C|}{\lfloor n\eta \rfloor} + \sqrt{w}\frac{n - \lfloor n\eta \rfloor}{n} \cdot \frac{\sum_{i=n-\lfloor n\eta \rfloor+1}^{n}|z_i - \mu_0|}{n - \lfloor n\eta \rfloor} \\
&\xrightarrow{P} \sqrt{w}\eta\, \mathrm{E}[|z_1 - \mu_0 + C|] + \sqrt{w}(1-\eta)\,\mathrm{E}[|z_1 - \mu_0|] \\
&= \frac{\eta}{\sigma}\left[\sqrt{\sigma^2+\tau^2}\sqrt{\frac{2}{\pi}}\exp\left(-\frac{(\mu - \mu_0 + C)^2}{2(\sigma^2+\tau^2)}\right) + (\mu - \mu_0 + C)\left(1 - 2\Phi\left(-\frac{\mu - \mu_0 + C}{\sqrt{\sigma^2+\tau^2}}\right)\right)\right] \\
&\quad + \frac{1-\eta}{\sigma}\left[\sqrt{\sigma^2+\tau^2}\sqrt{\frac{2}{\pi}}\exp\left(-\frac{(\mu-\mu_0)^2}{2(\sigma^2+\tau^2)}\right) + (\mu - \mu_0)\left(1 - 2\Phi\left(-\frac{\mu - \mu_0}{\sqrt{\sigma^2+\tau^2}}\right)\right)\right] \\
&= \eta\left[\sqrt{\frac{2}{\pi}}(1 - I_0^2)^{-1/2}\exp\left(-\frac{1}{2}s_1^2(1-I_0^2)\right) + s_1\left(1 - 2\Phi\left(-s_1(1-I_0^2)^{1/2}\right)\right)\right] \\
&\quad + (1-\eta)\left[\sqrt{\frac{2}{\pi}}(1-I_0^2)^{-1/2}\exp\left(-\frac{1}{2}s_2^2(1-I_0^2)\right) - s_2\left(1 - 2\Phi\left(s_2(1-I_0^2)^{1/2}\right)\right)\right],
\end{aligned}$$

where $s_1 = (\mu + C - \mu_0)/\sigma$ and $s_2 = (\mu_0 - \mu)/\sigma$. Therefore,

$$\begin{aligned}
I_m^2 &= 1 - \frac{2/\pi}{(Q_m/n)^2} \\
&\xrightarrow{P} 1 - \left\{\eta\left[(1-I_0^2)^{-1/2}\exp\left(-\frac{1}{2}s_1^2(1-I_0^2)\right) + \sqrt{\frac{\pi}{2}}s_1\left(1 - 2\Phi\left(-s_1(1-I_0^2)^{1/2}\right)\right)\right]\right. \\
&\quad \left. + (1-\eta)\left[(1-I_0^2)^{-1/2}\exp\left(-\frac{1}{2}s_2^2(1-I_0^2)\right) - \sqrt{\frac{\pi}{2}}s_2\left(1 - 2\Phi\left(s_2(1-I_0^2)^{1/2}\right)\right)\right]\right\}^{-2}.
\end{aligned}$$

Notice that $s_2 = C/\sigma - s_1$ and Equation (1) can be rewritten as

$$\eta\Phi\left(-s_1(1-I_0^2)^{1/2}\right) + (1-\eta)\Phi\left((C/\sigma - s_1)(1-I_0^2)^{1/2}\right) = 0.5; \tag{2}$$

that is, s_1 is the solution to Equation (2). This completes the proof. □

Web Appendix D: Complete simulation results

The simulation settings have been detailed in the main text. Web Tables 2–4 present the complete results.

Web Table 2: Simulation results for meta-analyses containing 10 studies with outliers in Scenario I. RMSE: root mean squared error; CP: coverage probability of 95% confidence interval. †Size for homogeneous studies ($\tau^2 = 0$) and power for heterogeneous studies ($\tau^2 > 0$) at the significance level $\alpha = 0.05$. ‡The sizes/powers outside the parentheses are produced by the resampling method; those inside the parentheses are obtained using Q's theoretical distribution under the null hypothesis.

Outlier pattern	Size/power†			RMSE			CP (%)		
	Q‡	Q_r	Q_m	$\hat{\tau}^2_{\text{DL}}$	$\hat{\tau}^2_r$	$\hat{\tau}^2_m$	$\hat{\tau}^2_{\text{DL}}$	$\hat{\tau}^2_r$	$\hat{\tau}^2_m$
Scenario I (contamination) with $\tau^2 = 0$ (homogeneity) and $s_i \sim U(0.5, 1)$:									
No outliers	0.04 (0.05)	0.04	0.04	0.18	0.22	0.16	100	100	100
C	0.74 (0.74)	0.53	0.47	1.04	0.80	0.56	99	98	100
(C, C)	0.97 (0.97)	0.93	0.91	1.70	1.68	1.16	92	92	98
$(C, -C)$	0.98 (0.98)	0.93	0.92	2.14	1.55	1.24	91	91	97
(C, C, C)	0.99 (0.99)	0.99	0.99	2.21	2.66	1.91	48	55	78
$(C, C, -C)$	1.00 (1.00)	0.99	1.00	3.01	2.57	2.07	48	57	79
Scenario I (contamination) with $\tau^2 = 1$ (heterogeneity) and $s_i \sim U(0.5, 1)$:									
No outliers	0.75 (0.74)	0.72	0.70	0.72	0.82	0.71	76	88	82
C	0.99 (0.98)	0.97	0.97	2.24	1.89	1.37	98	98	99
(C, C)	1.00 (1.00)	1.00	1.00	3.57	3.58	2.59	92	93	98
$(C, -C)$	1.00 (1.00)	1.00	1.00	4.44	3.49	2.77	92	92	98
(C, C, C)	1.00 (1.00)	1.00	1.00	4.47	5.29	3.94	67	71	88
$(C, C, -C)$	1.00 (1.00)	1.00	1.00	6.35	5.60	4.54	63	70	86
Scenario I (contamination) with $\tau^2 = 1$ (heterogeneity) and $s_i \sim U(1, 2)$:									
No outliers	0.26 (0.26)	0.22	0.22	1.27	1.52	1.21	76	88	81
C	0.88 (0.89)	0.78	0.75	5.34	4.28	3.06	98	98	99
(C, C)	0.99 (0.99)	0.98	0.98	8.73	8.68	6.15	92	92	98
$(C, -C)$	1.00 (1.00)	0.99	0.99	10.81	8.09	6.48	91	92	97
(C, C, C)	1.00 (1.00)	1.00	1.00	11.02	13.15	9.66	56	61	83
$(C, C, -C)$	1.00 (1.00)	1.00	1.00	15.66	13.46	10.97	56	62	82
Scenario I (contamination) with $\tau^2 = 1$ (heterogeneity) and $s_i \sim U(2, 5)$:									
No outliers	0.08 (0.08)	0.08	0.07	4.23	5.28	3.77	77	87	82
C	0.81 (0.81)	0.64	0.60	27.07	20.71	14.06	98	98	100
(C, C)	0.98 (0.98)	0.96	0.95	44.75	44.94	30.35	90	90	97
$(C, -C)$	1.00 (1.00)	0.96	0.96	55.85	39.94	31.44	90	91	97
(C, C, C)	1.00 (1.00)	1.00	1.00	56.81	69.04	49.87	44	53	75
$(C, C, -C)$	1.00 (1.00)	1.00	1.00	81.37	68.52	55.13	45	54	74

Web Table 3: Simulation results for meta-analyses containing 30 studies with outliers in Scenario I. RMSE: root mean squared error; CP: coverage probability of 95% confidence interval. †Size for homogeneous studies ($\tau^2 = 0$) and power for heterogeneous studies ($\tau^2 > 0$) at the significance level $\alpha = 0.05$. ‡The sizes/powers outside the parentheses are produced by the resampling method; those inside the parentheses are obtained using Q's theoretical distribution under the null hypothesis.

Outlier pattern	Size/power†			RMSE			CP (%)		
	$Q^‡$	Q_r	Q_m	$\hat{\tau}^2_{\text{DL}}$	$\hat{\tau}^2_r$	$\hat{\tau}^2_m$	$\hat{\tau}^2_{\text{DL}}$	$\hat{\tau}^2_r$	$\hat{\tau}^2_m$
Scenario I (contamination) with $\tau^2 = 0$ (homogeneity) and $s_i \sim U(0.5, 1)$:									
No outliers	0.05 (0.06)	0.05	0.05	0.10	0.12	0.10	98	99	99
C	0.55 (0.55)	0.27	0.25	0.37	0.24	0.20	97	97	98
(C, C)	0.89 (0.89)	0.66	0.60	0.63	0.42	0.35	88	90	94
$(C, -C)$	0.92 (0.92)	0.61	0.61	0.68	0.40	0.36	89	90	94
(C, C, C)	0.98 (0.98)	0.90	0.87	0.88	0.64	0.53	65	74	83
$(C, C, -C)$	0.99 (0.98)	0.89	0.88	0.99	0.61	0.55	64	73	83
Scenario I (contamination) with $\tau^2 = 1$ (heterogeneity) and $s_i \sim U(0.5, 1)$:									
No outliers	0.98 (0.99)	0.98	0.98	0.40	0.43	0.41	88	93	91
C	1.00 (1.00)	1.00	1.00	0.84	0.63	0.55	97	97	98
(C, C)	1.00 (1.00)	1.00	1.00	1.37	1.00	0.85	93	94	96
$(C, -C)$	1.00 (1.00)	1.00	1.00	1.45	0.97	0.85	93	94	96
(C, C, C)	1.00 (1.00)	1.00	1.00	1.86	1.44	1.22	76	83	90
$(C, C, -C)$	1.00 (1.00)	1.00	1.00	2.05	1.40	1.25	77	84	91
Scenario I (contamination) with $\tau^2 = 1$ (heterogeneity) and $s_i \sim U(1, 2)$:									
No outliers	0.48 (0.49)	0.42	0.43	0.74	0.81	0.75	89	93	91
C	0.89 (0.89)	0.78	0.77	1.97	1.36	1.17	98	97	98
(C, C)	0.99 (0.99)	0.94	0.94	3.33	2.29	1.93	91	92	96
$(C, -C)$	0.99 (0.99)	0.94	0.94	3.50	2.17	1.93	91	92	96
(C, C, C)	1.00 (1.00)	0.99	0.99	4.60	3.41	2.85	70	80	88
$(C, C, -C)$	1.00 (1.00)	0.99	0.99	5.03	3.24	2.90	71	81	88
Scenario I (contamination) with $\tau^2 = 1$ (heterogeneity) and $s_i \sim U(2, 5)$:									
No outliers	0.11 (0.11)	0.09	0.09	2.32	2.64	2.25	88	92	91
C	0.70 (0.70)	0.43	0.41	9.89	5.96	5.02	97	97	99
(C, C)	0.96 (0.96)	0.81	0.78	17.19	10.92	8.97	90	91	94
$(C, -C)$	0.96 (0.96)	0.76	0.77	18.10	10.02	8.94	90	91	95
(C, C, C)	1.00 (1.00)	0.95	0.94	23.87	16.90	13.59	65	74	82
$(C, C, -C)$	1.00 (1.00)	0.95	0.95	26.10	15.49	13.78	64	74	83

Web Table 4: Simulation results for meta-analyses containing 30 studies with outliers in Scenario II. RMSE: root mean squared error; CP: coverage probability of 95% confidence interval. †Size for homogeneous studies ($\tau^2 = 0$) and power for heterogeneous studies ($\tau^2 > 0$) at the significance level $\alpha = 0.05$. ‡The sizes/powers outside the parentheses are produced by the resampling method; those inside the parentheses are obtained using Q's theoretical distribution under the null hypothesis.

Outlier pattern	Size/power†			RMSE			CP (%)		
	Q‡	Q_r	Q_m	$\hat{\tau}^2_{\mathrm{DL}}$	$\hat{\tau}^2_r$	$\hat{\tau}^2_m$	$\hat{\tau}^2_{\mathrm{DL}}$	$\hat{\tau}^2_r$	$\hat{\tau}^2_m$
Scenario II (heavy tail) with $\tau^2 = 1$ (heterogeneity) and $s_i \sim U(0.5, 1)$:									
df = 3	0.92 (0.92)	0.89	0.88	1.45	0.59	0.56	72	79	73
df = 5	0.98 (0.98)	0.95	0.95	0.55	0.45	0.45	84	90	86
df = 10	0.98 (0.98)	0.97	0.97	0.43	0.43	0.42	88	93	90
Scenario II (heavy tail) with $\tau^2 = 1$ (heterogeneity) and $s_i \sim U(1, 2)$:									
df = 3	0.41 (0.40)	0.35	0.35	1.53	0.88	0.82	83	90	87
df = 5	0.46 (0.46)	0.40	0.40	0.82	0.82	0.77	88	93	90
df = 10	0.48 (0.49)	0.42	0.42	0.76	0.82	0.77	88	94	90
Scenario II (heavy tail) with $\tau^2 = 1$ (heterogeneity) and $s_i \sim U(2, 5)$:									
df = 3	0.10 (0.10)	0.10	0.10	2.66	2.71	2.33	88	92	91
df = 5	0.09 (0.09)	0.09	0.08	2.18	2.54	2.17	88	93	92
df = 10	0.10 (0.10)	0.08	0.09	2.18	2.48	2.12	88	93	91

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