Supplementary Material for Networks of Conforming or Non-Conforming Individuals Tend to Reach Satisfactory Decisions

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This document contains the proofs for the three theorems and associated lemmas and corollaries in the paper Agree or Agree-to-Disagree: Networks of Conforming or Non-Conforming Individuals Tend to Reach Satisfactory Decisions, in the Proceedings of the National Academy of Sciences.

I. Proof of Theorem 1

Lemma 1. *Every network of anti-coordinating agents who update asynchronously under Assumption 1, with $\tau_i = \frac{1}{2}$ for each agent $i \in \mathcal{V}$, will reach an equilibrium state in finite time.*

Proof: Let $\hat{n}_i^A$ denote the maximum number of A-neighbors of agent i that will not cause agent i to switch to B when playing A. Define the function $\Phi(k) = \sum_{i=1}^n \Phi_i(k)$, where

$$\Phi_i(k) = \begin{cases} n_i^A(k) - \hat{n}_i^A & \text{if } x_i(k) = A \\ \hat{n}_i^A + 1 - n_i^A(k) & \text{if } x_i(k) = B \end{cases}.$$

$\Phi(k)$ is clearly lower bounded by $\Phi(k) \geq -\sum_{i=1}^n \deg_i$ for all k. Consider a time step k, and let i denote the index of the active agent at that time. One of the following three cases must happen:

1) Agent i does not switch strategies at time $k+1$. This implies $\Phi(k+1) = \Phi(k)$.

2) Agent i switches from A to B at time $k+1$. This implies $n_i^A(k) \geq \hat{n}_i^A + 1$. Then, since $n_i^A(k) = n_i^A(k+1)$, we have

$$\begin{aligned} \Phi_i(k+1) - \Phi_i(k) &= \hat{n}_i^A + 1 - n_i^A(k) - n_i^A(k) + \hat{n}_i^A \\ &= 2(\hat{n}_i^A - n_i^A(k)) + 1 \leq -1. \end{aligned} \tag{3}$$

Moreover, for each $j \in \mathcal{N}_i$, if $x_j(k) = A$, it holds that

$$\begin{aligned} \Phi_j(k+1) - \Phi_j(k) &= n_j^A(k+1) - \hat{n}_j^A - n_j^A(k) + \hat{n}_j^A \\ &= -1, \end{aligned} \tag{4}$$

and if $x_j(k) = B$, it holds that

$$\Phi_j(k+1) - \Phi_j(k) = -n_j^A(k+1) + n_j^A(k) = 1. \tag{5}$$

According to (2), the fact that agent i switches from A to B at time $k+1$ implies $n_i^A(k) \geq \frac{1}{2}\deg_i$, regardless of how z_i is defined. Hence, by combining (3), (4), and (5), we have

$$\begin{aligned} &\Phi(k+1) - \Phi(k) \\ &\quad = \sum_{j \in \mathcal{N}_i \cup \{i\}} \Phi_j(k+1) - \Phi_j(k) \\ &\quad = \underbrace{\Phi_i(k+1) - \Phi_i(k)}_{\leq -1} \underbrace{- n_i^A(k) + (\deg_i - n_i^A(k))}_{\leq 0} \\ &\quad \leq -1. \end{aligned} \tag{6}$$

3) Agent i switches from B to A at time $k+1$. This implies $n_i^A(k) \leq \hat{n}_i^A$. Hence,

$$\Phi_i(k+1) - \Phi_i(k) = 2(n_i^A(k) - \hat{n}_i^A) - 1 \leq -1. \tag{7}$$

Moreover, for each $j \in \mathcal{N}_i$, if $x_j(k) = A$, it holds that

$$\Phi_j(k+1) - \Phi_j(k) = n_j^A(k+1) - n_j^A(k) = 1, \tag{8}$$

and if $x_j(k) = B$, it holds that

$$\Phi_j(k+1) - \Phi_j(k) = -n_j^A(k+1) + n_j^A(k) = -1. \tag{9}$$

According to (2), the fact that agent i switches from B to A at time $k+1$ implies $n_i^A(k) \leq \frac{1}{2}\deg_i$, regardless of how z_i is defined. Hence, by combining (7), (8), and (9), we have

$$\begin{aligned}
&\Phi(k+1) - \Phi(k) \\
&\quad = \underbrace{\Phi_i(k+1) - \Phi_i(k)}_{\leq -1} + \underbrace{n_i^A(k) - (\deg_i - n_i^A(k))}_{\leq 0} \\
&\quad \leq -1.
\end{aligned}$$

By summarizing the above three cases, we have that

$$\Phi(k+1) \leq \Phi(k) \qquad \forall k \geq 0. \tag{10}$$

Moreover, we have shown that every time an agent switches strategies, the function $\Phi(k)$ decreases by at least one. The case where all thresholds are equal to $\frac{1}{2}$ is thus a generalized ordinal potential game, by the definition given in [1]. However, as shown in [2], this does not necessarily imply convergence to an equilibrium. Hence, we complete the proof by contradiction.

Assume that the network does not reach an equilibrium in finite time. Hence, at every time step $k = 0, 1, \ldots$, there exists an agent i^k whose strategy violates its threshold. Denote by $\tilde{k}$ the first time after k at which agent i^k is active. The existence of $\tilde{k}$ is guaranteed by Assumption 1. At time $\tilde{k}$, agent i^k's threshold either remains violated, implying the agent will switch strategies at time $\tilde{k}+1$, or is no longer violated, implying that some neighbors have changed their strategies during the time interval $[k+1, \tilde{k}]$. Hence, at least one switch occurs in each interval $\mathcal{I}^k = [k+1, \tilde{k}+1]$. Now consider the sequence of intervals $\mathcal{I}^{k_1}, \mathcal{I}^{k_2}, \ldots$ where the indices $k_j, j = 1, 2, \ldots$, are such that $k_{j+1} > \tilde{k}_j + 1$. This sequence is infinite, and the intersection of each pair of intervals from the sequence is empty. Therefore, an infinite number of switches occur in the network over time. Namely, there exists an infinite time sequence $(\kappa^j)_{j=1}^{\infty}, \kappa^j \in \mathcal{I}^{k_j}$, such that an agent switches strategies at each κ^j. Hence, either Case 2 or 3 occurs at each κ^j, resulting in $\Phi(\kappa^j + 1) \leq \Phi(\kappa^j) - 1$. Hence, in view of (10),

$$\Phi(k) \leq \Phi(\kappa^j) - 1 \qquad \forall k \geq \kappa^j + 1. \tag{11}$$

Since (11) holds for all $j = 1, 2, \ldots$, we get that

$$\Phi(k) \leq \Phi(\kappa^1) - j \qquad \forall k \geq \kappa^j + 1 \quad \forall j \in \mathbb{N}.$$

The above inequality implies that Φ is not lower bounded, which is a contradiction. Hence, the proof is complete. ■

We define the *augmented network game* $\hat{\Gamma} := (\hat{\mathbb{G}}, \frac{1}{2}\mathbf{1}, -)$ based on Γ as follows. Let $\hat{\mathbb{G}} = (\hat{\mathcal{V}}, \hat{\mathcal{E}})$. Define a *$V$-block* as a triplet of nodes $\{v_1, v_2, v_3\} \subseteq \hat{\mathcal{V}}$ along with the edges $\{\{v_1, v_2\}, \{v_1, v_3\}\} \subseteq \hat{\mathcal{E}}$. For each agent $i \in \mathcal{V}$, we introduce a *dual agent* $\hat{i} \in \hat{\mathcal{V}}$ with the same initial strategy, i.e., $x_{\hat{i}}(0) = x_i(0)$, and with $z_{\hat{i}} = z_i$. Corresponding to each dual agent $\hat{i}$, there are m_i number of V-blocks in $\hat{\mathbb{G}}$ such that the v_1-node of each block is connected to $\hat{i}$, with m_i being defined as follows: if $\tau_i = \frac{1}{2}$, then $m_i = 0$; otherwise, m_i depends on which one of the following three conditions on τ_i holds:

$$m_i = \begin{cases} |(1-2\tau_i)\deg_i| & \tau_i \deg_i \in \mathbb{Z} \\ |\deg_i - r - 1| & \exists r \in 2\mathbb{Z} : \frac{r}{2} < \tau_i \deg_i < \frac{r+1}{2} \\ |\deg_i - r| & \exists r \in 2\mathbb{Z}+1 : \frac{r}{2} \leq \tau_i \deg_i < \frac{r+1}{2} \end{cases} \tag{12}$$

where $2\mathbb{Z}$ and $2\mathbb{Z}+1$ denote the set of even and odd numbers, respectively. If $\tau_i < \frac{1}{2}$, then the initial strategies of each V-block connected to the dual agent $\hat{i}$ are $x_{v_1}(0) = A$ and $x_{v_2}(0) = x_{v_3}(0) = B$; and if $\tau_i > \frac{1}{2}$, then $x_{v_1}(0) = B$ and $x_{v_2}(0) = x_{v_3}(0) = A$. In total, $\hat{\mathcal{V}}$ has $n + \sum_{i=1}^{n} m_i$ agents, the thresholds of all of which are set to $\frac{1}{2}$. For $\hat{\mathcal{E}}$, in addition to the edges for the V-blocks, there is an edge between any two dual agents $\hat{i}$ and $\hat{j}$ in $\hat{\mathcal{V}}$, if and only if there is an edge between i and j in $\mathcal{V}$.

Lemma 2. *The strategy of each V-block agent is fixed over time.*

Proof: We prove by contradiction. Assume there exists some time when the strategy of one of the V-block agents changes. Let r denote the first time this happens, and denote the V-block agent who changes her strategy by

i. If i is a v_1-node and $x_i(0) = A$, then $x_i(r-1) = A$ and $x_i(r) = B$. Agent i has two neighbors in the V-block who each play strategy B at time $r-1$. Since $\deg_i = 3$, we have

$$\left.\begin{aligned} n_i^A(r-1) &\leq 1 \\ \tau_i \deg_i &= \frac{3}{2} \end{aligned}\right\} \Rightarrow n_i^A(r-1) < \tau_i \deg_i \Rightarrow x_i(r) = A,$$

which is a contradiction. If $x_i(0) = B$, then agent i has two A-neighbors in the V-block and $x_i(r) = A$. It follows that

$$\left.\begin{aligned} n_i^A(r-1) &\geq 2 \\ \tau_i \deg_i &= \frac{3}{2} \end{aligned}\right\} \Rightarrow n_i^A(r-1) > \tau_i \deg_i \Rightarrow x_i(r) = B,$$

again a contradiction. Now if i is either a v_2 or a v_3 node, and $x_i(0) = A$, then $x_i(r-1) = A$, and i has only one neighbor, v_1, whose strategy is B at time $r-1$. Hence,

$$\left.\begin{aligned} n_i^A(r-1) &= 0 \\ \tau_i \deg_i &= \frac{1}{2} \end{aligned}\right\} \Rightarrow n_i^A(r-1) < \tau_i \deg_i \Rightarrow x_i(r) = A,$$

a contradiction. If on the other hand, $x_i(0) = B$, then

$$\left.\begin{aligned} n_i^A(r-1) &= 1 \\ \tau_i \deg_i &= \frac{1}{2} \end{aligned}\right\} \Rightarrow n_i^A(r-1) > \tau_i \deg_i \Rightarrow x_i(r) = B,$$

which is a contradiction and completes the proof. ■

The following lemma takes the first step towards establishing equivalence of the dynamics between the original and augmented network games. In particular, we need to show that the thresholds of $\frac{1}{2}$ in the augmented network will be satisfied (respectively violated) whenever the thresholds of the corresponding agents in the original network are satisfied (respectively violated).

Lemma 3. *Let n_i^A denote an instance of $n_i^A(k)$ for some agent i. If $\tau_i < \frac{1}{2}$, then*

$$\operatorname{sign}(n_i^A - \tau_i \deg_i) = \operatorname{sign}\left(n_i^A + m_i - \frac{1}{2}(\deg_i + m_i)\right), \tag{13}$$

and if $\tau_i > \frac{1}{2}$, then

$$\operatorname{sign}(n_i^A - \tau_i \deg_i) = \operatorname{sign}\left(n_i^A - \frac{1}{2}(\deg_i + m_i)\right). \tag{14}$$

Proof: First consider the situation when $\tau_i < \frac{1}{2}$. In general, one of the following cases takes place:

1) $\tau_i \deg_i \in \mathbb{Z}$: If $n_i^A = \tau_i \deg_i$, using (12) we have

$$n_i^A = \frac{1}{2}(\deg_i - m_i) \Rightarrow n_i^A + m_i = \frac{1}{2}(\deg_i + m_i).$$

The cases $n_i^A < \tau_i \deg_i$ and $n_i^A > \tau_i \deg_i$ can be shown using the same approach, which verifies (13) for this case.

2) $\exists r \in 2\mathbb{Z} : \frac{r}{2} \leq \tau_i \deg_i < \frac{r+1}{2}$: Here, $\tau_i \deg_i \notin \mathbb{Z}$ implies that $n_i^A \neq \tau_i \deg_i$, so we need only to check the inequality cases. Using (12), $n_i^A > \tau_i \deg_i$ implies

$$\left.\begin{aligned} & n_i^A > \frac{r}{2} \\ & \frac{r}{2}, n_i^A \in \mathbb{Z} \end{aligned}\right\} \Rightarrow n_i^A \geq \frac{r}{2} + 1 = \frac{1}{2}(\deg_i - m_i) + \frac{1}{2}$$
$$\Rightarrow n_i^A + m_i > \frac{1}{2}(\deg_i + m_i),$$

and $n_i^A < \tau_i \deg_i$ implies

$$n_i^A < \frac{r+1}{2} = \frac{1}{2}(\deg_i - m_i)$$
$$\Leftrightarrow n_i^A + m_i < \frac{1}{2}(\deg_i + m_i).$$

Hence, (13) is confirmed for this case.

3) $\exists r \in 2\mathbb{Z}+1 : \frac{r}{2} \leq \tau_i \deg_i < \frac{r+1}{2}$*:* Again, $\tau_i \deg_i \notin \mathbb{Z}$ implies that $n_i^A \neq \tau_i \deg_i$. Then $n_i^A > \tau_i \deg_i$ implies

$$n_i^A > \frac{r}{2} = \frac{1}{2}(\deg_i - m_i) \Leftrightarrow n_i^A + m_i > \frac{1}{2}(\deg_i + m_i),$$

and $n_i^A < \tau_i \deg_i$ implies

$$\left.\begin{array}{l} n_i^A < \dfrac{r+1}{2} \\ n_i^A \in \mathbb{Z}, r \in 2\mathbb{Z}+1 \end{array}\right\} \Rightarrow n_i^A \leq \frac{r-1}{2}$$
$$\Leftrightarrow n_i^A \leq \frac{1}{2}(\deg_i - m_i) - \frac{1}{2}$$
$$\Rightarrow n_i^A + m_i < \frac{1}{2}(\deg_i + m_i).$$

Hence, (13) holds for this case and for all $\tau_i < \frac{1}{2}$.

If $\tau_i > \frac{1}{2}$, then one of the following occurs:

1) $\tau_i \deg_i \in \mathbb{Z}$*:* If $n_i^A = \tau_i \deg_i$, then (12) implies $n_i^A = \frac{1}{2}(\deg_i + m_i)$. The cases $n_i^A < \tau_i \deg_i$ and $n_i^A > \tau_i \deg_i$ can be shown using the same approach, which verifies (14) for this case.

2) $\exists r \in 2\mathbb{Z} : \frac{r}{2} \leq \tau_i \deg_i < \frac{r+1}{2}$*:* First, we know that $n_i^A \neq \tau_i \deg_i$. Using (12), $n_i^A > \tau_i \deg_i$ implies

$$n_i^A \geq \frac{r}{2} + 1 = \frac{1}{2}(\deg_i + m_i) + \frac{1}{2} > \frac{1}{2}(\deg_i + m_i),$$

and $n_i^A < \tau_i \deg_i$ implies

$$n_i^A < \frac{r+1}{2} = \frac{1}{2}(\deg_i + m_i) \Leftrightarrow n_i^A < \frac{1}{2}(\deg_i + m_i).$$

Hence, (14) holds for this case.

3) $\exists r \in 2\mathbb{Z}+1 : \frac{r}{2} \leq \tau_i \deg_i < \frac{r+1}{2}$*:* Once again, we know that $n_i^A \neq \tau_i \deg_i$. Using (12), $n_i^A > \tau_i \deg_i$ implies

$$n_i^A > \frac{r}{2} = \frac{1}{2}(\deg_i + m_i),$$

and $n_i^A < \tau_i \deg_i$ implies

$$n_i^A \leq \frac{r-1}{2} = \frac{1}{2}(\deg_i + m_i) - \frac{1}{2} < \frac{1}{2}(\deg_i + m_i).$$

Hence, (14) holds for this case and for all $\tau_i > \frac{1}{2}$, which completes the proof. ■

Next, we show in Lemma 4 that if whenever an agent in $\mathbb{G}$ activates, its dual in $\hat{\mathbb{G}}$ also activates (while neglecting the time steps that a V-block agent is active), then the dynamics of each node in $\mathbb{G}$ are the same as the dynamics of its dual node in $\hat{\mathbb{G}}$ (again while neglecting the time steps that a V-block agent is active).

Consider the network $\mathbb{G}$ and let $i_{\mathbb{G}}^k$ denote the active agent at time k. Correspondingly, denote by $(i_{\mathbb{G}}^k)_{k=0}^{\infty}$, the sequence of active agents in $\mathbb{G}$. Similarly define $(i_{\hat{\mathbb{G}}}^k)_{k=0}^{\infty}$ as the sequence of active agents in $\hat{\mathbb{G}}$. Consider $(i_{\hat{\mathbb{G}}}^k)_{k=0}^{\infty}$ and exclude those agents $i_{\hat{\mathbb{G}}}^k$ that belong to one of the V-blocks, to get the subsequence $(i_{\hat{\mathbb{G}}}^{h_k})_{k=0}^{\infty}$. Denote the sequence of superscripts of $(i_{\hat{\mathbb{G}}}^{h_k})_{k=0}^{\infty}$ by $(h_k)_{k=0}^{\infty}$ which corresponds to the times at which the non-V-block agents in $\hat{\mathbb{G}}$ are active.

Lemma 4. *If* $(i_{\mathbb{G}}^k)_{k=0}^{\infty} = (i_{\hat{\mathbb{G}}}^{h_k})_{k=0}^{\infty}$*, then for* $k = 0, 1, \ldots$*, it holds that*

$$x_i(k) = x_{\hat{i}}(h_k) \qquad \forall i \in \mathcal{V}. \tag{15}$$

where $\hat{i} \in \hat{\mathcal{V}}$ is the dual of agent i.

Proof: The proof is done via induction on k. By the definition of $\hat{\mathbb{G}}$, (15) holds for $k = 0$. Assume that (15) holds for $k = r \in \mathbb{Z}_{\geq 0}$.

Consider agent $i^r_{\mathbb{G}}$ and its dual $\hat{i}^r_{\mathbb{G}}$ whose threshold and degree are $\frac{1}{2}$ and $\deg_i + m_i$, respectively. Agent $i^r_{\mathbb{G}}$ updates at time $k = r + 1$, and agent $\hat{i}^r_{\mathbb{G}}$ updates at $k_{\hat{\mathbb{G}}} = h_r + 1$ where $k_{\hat{\mathbb{G}}}$ denotes the time in the augmented network game $\hat{\Gamma}$. If $\tau_i = \frac{1}{2}$, then both agents have the same threshold and number of A-neighbors. Hence, they update to the same strategy at the next time step. If $\tau_i < \frac{1}{2}$, then in view of Lemma 2 and since (15) holds for $k = r$, $\hat{i}^r_{\mathbb{G}}$ has $n^A_{i^r}(r) + m_i$ A-neighbors. Therefore, according to (13) in Lemma 3, $\hat{i}^r_{\mathbb{G}}$ updates to the same strategy that $i^r_{\mathbb{G}}$ does. On the other hand, if $\tau_i > \frac{1}{2}$, then $\hat{i}^r_{\mathbb{G}}$ has $n^A_{i^r}(r)$ A-neighbors. Hence, according to (14) in Lemma 3, $\hat{i}^r_{\mathbb{G}}$ updates to the same strategy that $i^r_{\mathbb{G}}$ does. Therefore, in all cases, agent $\hat{i}^r_{\mathbb{G}}$ updates to the same strategy that agent $i^r_{\mathbb{G}}$ does. That is,

$$x_{i^r_{\mathbb{G}}}(r+1) = x_{\hat{i}^r_{\mathbb{G}}}(h_r + 1). \tag{16}$$

On the other hand, since no other agent has become active at times h_r or r,

$$x_i(r+1) = x_i(r) \qquad \forall i \in \mathcal{V} - \{i^r_{\mathbb{G}}\}, \tag{17}$$

$$x_{\hat{i}}(h_r+1) = x_{\hat{i}}(h_r) \qquad \forall i \in \mathcal{V} - \{i^r_{\mathbb{G}}\}. \tag{18}$$

Due to the induction statement for $k = r$, it holds that $x_i(r) = x_{\hat{i}}(h_r)$ for all $i \in \mathcal{V} - \{i^r_{\mathbb{G}}\}$. Hence, (17) and (18) result in

$$x_i(r+1) = x_{\hat{i}}(h_r+1) \qquad \forall i \in \mathcal{V} - \{i^r_{\mathbb{G}}\}.$$

Therefore, according to (16),

$$x_i(r+1) = x_{\hat{i}}(h_r+1) \qquad \forall i \in \mathcal{V}. \tag{19}$$

Now since at each of the time steps $h_r+1, h_r+2, \dots, h_{r+1}-1$, the active agent is a V-block agent whose strategy remains fixed by Lemma 2, (19) results in

$$x_i(r+1) = x_{\hat{i}}(h_{r+1}) \qquad \forall i \in \mathcal{V}.$$

Hence, (15) holds for $k = r + 1$, which completes the proof by induction. ■

The remaining step in proving Theorem 1 is to show that agents with arbitrary thresholds will indeed reach an equilibrium state in finite time.

Proof of Theorem 1: Towards a proof by contradiction, suppose that the original network game never converges, i.e., there exists an agent $j \in \mathcal{V}$ such that

$$\forall k^*, \big(\exists k > k^* : x_j(k) \neq x_j(k^*)\big).$$

Construct the sequence of active agents $(i^k_{\hat{\mathbb{G}}})^\infty_{k=0}$ by inserting an agent $\hat{i}$ uniformly at random from the set of augmented nodes $\hat{\mathcal{V}} - \mathcal{V}$ after every n elements of the sequence $(i^k_{\mathbb{G}})^\infty_{k=0}$. This is clearly a persistent activation sequence on the network $\hat{\mathbb{G}}$. By Lemma 1, we know that

$$\exists \tilde{k}^* : \big(\forall k > \tilde{k}^*, x_{\hat{j}}(k) = x_{\hat{j}}(\tilde{k}^*)\big). \tag{20}$$

On the other hand, by eliminating the V-block agents in $(i^k_{\hat{\mathbb{G}}})^\infty_{k=0}$, we arrive at $(i^k_{\mathbb{G}})^\infty_{k=0}$. Hence, in view of Lemma 4, (20) implies that

$$\exists k^* : \big(\forall k > k^*, x_j(k) = x_j(k^*)\big),$$

which contradicts our initial statement. Therefore, $x(k)$ will reach an equilibrium in finite time. ■

II. Proof of Theorem 2

The proof of Theorem 2 follows similar steps as the anti-coordinating case. The key difference is that the potential function becomes

$$\Phi_i(k) = \begin{cases} \check{n}^A_i - n^A_i(k) & \text{if } x_i(k) = A \\ n^A_i(k) - \check{n}^A_i + 1 & \text{if } x_i(k) = B \end{cases}, \tag{21}$$

where $\check{n}_i^A$ is defined as the minimum number of A-neighbors required for an A-playing agent to continue playing A. The maximum number of A-neighbors that a B agent can tolerate before switching to A is then given by $\check{n}_i^A - 1$. As shown in the following lemma, this function also decreases by at least 1 with every change of strategy for the network game $\Gamma := (\mathbb{G}, \frac{1}{2}\mathbf{1}, +)$.

Lemma 5. *Every network of coordinating agents who update asynchronously under Assumption 1, with $\tau_i = \frac{1}{2}$ for each agent $i \in \mathcal{V}$, will reach an equilibrium state in finite time.*

Proof: Consider the function $\Phi(k) = \sum_{i=1}^{n} \Phi_i(k)$, where Φ_i is defined in (21). Clearly $\Phi(k)$ is lower bounded by $\Phi(k) \geq -\sum_{i=1}^{n} \deg_i$ for all k. Consider a time step k, and let i denote the active agent at that time. One of the following three cases must happen:

1) Agent i does not switch strategies at time $k+1$. This implies $\Phi(k+1) = \Phi(k)$.

2) Agent i switches from A to B at time $k+1$. This implies $n_i^A(k) \leq \check{n}_i^A - 1$. Hence, since $n_i^A(k) = n_i^A(k+1)$, we have

$$\begin{aligned} \Phi_i(k+1) - \Phi_i(k) &= n_i^A(k) - \check{n}_i^A + 1 - \check{n}_i^A + n_i^A(k) \\ &= 2(n_i^A(k) - \check{n}_i^A) + 1 \leq -1. \end{aligned} \tag{22}$$

Moreover, for each $j \in \mathcal{N}_i$, if $x_j(k) = A$, it holds that

$$\begin{aligned} \Phi_j(k+1) - \Phi_j(k) &= \check{n}_i^A - n_i^A(k+1) - \check{n}_i^A + n_i^A(k) \\ &= 1, \end{aligned} \tag{23}$$

and if $x_j(k) = B$, it holds that

$$\Phi_j(k+1) - \Phi_j(k) = n_i^A(k+1) - n_i^A(k) = -1. \tag{24}$$

According to (1), the fact that agent i switches from A to B at time $k+1$ implies $n_i^A(k) \leq \frac{1}{2}\deg_i$, regardless of how z_i is defined. Hence, by combining (22), (23), and (24), we have

$$\begin{aligned} &\Phi(k+1) - \Phi(k) \\ &\quad = \underbrace{\Phi_i(k+1) - \Phi_i(k)}_{\leq -1} + \underbrace{n_i^A(k) - (\deg_i - n_i^A(k))}_{\leq 0} \\ &\quad \leq -1. \end{aligned} \tag{25}$$

3) Agent i switches from B to A at time $k+1$. This implies $n_i^A(k) \geq \check{n}_i^A$. Hence,

$$\Phi_i(k+1) - \Phi_i(k) = 2(\check{n}_i^A - n_i^A(k)) - 1 \leq -1. \tag{26}$$

Moreover, for each $j \in \mathcal{N}_i$, if $x_j(k) = A$, it holds that

$$\Phi_j(k+1) - \Phi_j(k) = -n_j^A(k+1) + n_j^A(k) = -1, \tag{27}$$

and if $x_j(k) = B$, it holds that

$$\Phi_j(k+1) - \Phi_j(k) = n_j^A(k+1) - n_j^A(k) = 1. \tag{28}$$

According to (2), the fact that agent i switches from B to A at time $k+1$ implies $n_i^A(k) \geq \frac{1}{2}\deg_i$, regardless of how z_i is defined. Hence, by combining (26), (27), and (28), we have

$$\Phi(k+1) - \Phi(k) \tag{29}$$

$$= \underbrace{\Phi_i(k+1) - \Phi_i(k)}_{\leq -1} \underbrace{-n_i^A(k) + (\deg_i - n_i^A(k))}_{\leq 0} \tag{30}$$

$$\leq -1. \tag{31}$$

By summarizing the above three cases, we have that

$$\Phi(k+1) \leq \Phi(k) \qquad \forall k \geq 0. \tag{32}$$

Moreover, we have shown that every time an agent switches strategies, the function $\Phi(k)$ decreases by at least one. The rest of the proof follows in the same way as that of Lemma 1. ■

By following the same process of constructing the network augmentation for anti-coordinating agents, we are able to extend the result of Lemma 5 to a network game with arbitrary thresholds. We define the augmented (coordinating) network game $\hat{\Gamma} := (\hat{\mathbb{G}}, \frac{1}{2}\mathbf{1}, +)$ based on the (coordinating) network game Γ, in the same way we defined the augmented network game for anti-coordinating agents, but with the following difference: If $\tau_i < \frac{1}{2}$, then the initial strategies of each V-block connected to the dual agent $\hat{i}$ are $x_{v_1}(0) = x_{v_2}(0) = x_{v_3}(0) = A$, and if $\tau_i > \frac{1}{2}$, then $x_{v_1}(0) = x_{v_2}(0) = x_{v_3}(0) = B$. Similar to Lemma 2, the following lemma guarantees the invariance of the strategies of the V-block agents.

Lemma 6. *The strategy of each (coordinating) V-block agent is fixed over time.*

Proof: The proof is done via contradiction. Assume there exists some time when the strategy of one of the V-block agents changes. Let r denote the first time this happens, and denote the V-block agent who changes her strategy by i. If i is a v_1-node and $x_i(0) = A$, then $x_i(r-1) = A$ and $x_i(r) = B$. Agent i has two neighbors in the V-block who each play strategy A at time $r-1$. Since $\deg_i = 3$, we have

$$\left.\begin{aligned} n_i^A(r-1) &\geq 2 \\ \tau_i \deg_i &= \frac{3}{2} \end{aligned}\right\} \Rightarrow n_i^A(r-1) > \tau_i \deg_i \Rightarrow x_i(r) = A,$$

which is a contradiction. If $x_i(0) = B$, then agent i has two B-neighbors in the V-block and $x_i(r) = A$. It follows that

$$\left.\begin{aligned} n_i^A(r-1) &\leq 1 \\ \tau_i \deg_i &= \frac{3}{2} \end{aligned}\right\} \Rightarrow n_i^A(r-1) < \tau_i \deg_i \Rightarrow x_i(r) = B,$$

again a contradiction. Now if i is either a v_2 or a v_3 node, and $x_i(0) = A$, then $x_i(r-1) = A$, and i has only one neighbor, v_1, whose strategy is A at time $r-1$. Hence,

$$\left.\begin{aligned} n_i^A(r-1) &= 1 \\ \tau_i \deg_i &= \frac{1}{2} \end{aligned}\right\} \Rightarrow n_i^A(r-1) > \tau_i \deg_i \Rightarrow x_i(r) = A,$$

a contradiction. If on the other hand, $x_i(0) = B$, then

$$\left.\begin{aligned} n_i^A(r-1) &= 0 \\ \tau_i \deg_i &= \frac{1}{2} \end{aligned}\right\} \Rightarrow n_i^A(r-1) < \tau_i \deg_i \Rightarrow x_i(r) = B,$$

which is a contradiction and completes the proof. ■

Next, since Lemma 3 is independent of the type of agents, i.e., coordinating or anti-coordinating, it can be used here as well. Moreover, because of Lemma 6, the result of Lemma 4 can be readily extended to a network of coordinating agents. With these lemmas in hand, and with the help of Lemma 5, the proof of Theorem 2 can be done in the same way as that of Theorem 1.

III. CONVERGENCE TIME: PROOF OF COROLLARY 3

Corollary 3. *Every network of all coordinating or all anti-coordinating agents will reach an equilibrium state after no more than $6|\mathcal{E}|$ agent switches.*

Proof: To compute the maximum number of times any agent switches strategies before such a network reaches an equilibrium, we consider the augmented network game $\hat{\Gamma}$, which will undergo the same sequence of agent switches as the original network game Γ, provided that the dual agents in $\hat{\mathcal{V}}$ activate in the same order as the corresponding agents in $\mathcal{V}$. From (6) in the proof of Lemma 1 and (25) in the proof of Lemma 5, we know that whenever an agent $i \in \hat{\mathcal{V}}$ switches strategies, $\Phi(k+1) - \Phi(k) \leq -1$. Otherwise, $\Phi(k)$ remains constant. It follows that the total number of agent switches in $\hat{\Gamma}$ is bounded from above by $\Phi(0) - \Phi(k^*)$, where k^* is the time at which the network reaches an equilibrium. To obtain such a bound, we start by decomposing the augmented network into three disjoint sets of agents such that $\hat{\mathcal{V}} = \hat{\mathcal{V}}_0 \cup \hat{\mathcal{V}}_1 \cup \hat{\mathcal{V}}_{23}$, where $\hat{\mathcal{V}}_0$ denotes the dual agents corresponding to the oringinal agents $\mathcal{V}$, $\hat{\mathcal{V}}_1$ denotes the set of v_1 agents in the V-blocks, and $\hat{\mathcal{V}}_{23}$ denotes the set of v_2 and v_3 agents in the V-blocks (we refer the reader to the proof of Theorem 1 for definitions of the augmented network). We can now expand the expression for the upper bound as follows:

$$\Phi(0) - \Phi(k^*) = \sum_{i \in \hat{\mathcal{V}}} \Phi_i(0) - \Phi_i(k^*) = \sum_{i \in \hat{\mathcal{V}}_0} \Phi_i(0) - \Phi_i(k^*) + \sum_{i \in \hat{\mathcal{V}}_1} \Phi_i(0) - \Phi_i(k^*) + \sum_{i \in \hat{\mathcal{V}}_{23}} \Phi_i(0) - \Phi_i(k^*). \tag{33}$$

Since the V-block agents never change strategies (by Lemmas 2 and 6), $\Phi_i(k)$ is constant for all agents in $\hat{\mathcal{V}}_{23}$. The final term in (33) is therefore equal to zero. The agents in $\hat{\mathcal{V}}_1$ each have one neighbor in $\hat{\mathcal{V}}_0$ who might change strategies (the other two neighbors are in $\hat{\mathcal{V}}_{23}$ and remain fixed). Since $n_i^A(k)$ can change by at most one for these agents, it follows that the maximum change in $\Phi_i(k)$ for such an agent is one. Therefore, we have

$$\sum_{i \in \hat{\mathcal{V}}_1} \Phi_i(0) - \Phi_i(k^*) \le \sum_{i \in \hat{\mathcal{V}}_1} 1 = |\hat{\mathcal{V}}_1| = \sum_{i \in \mathcal{V}} m_i < \sum_{i \in \mathcal{V}} \deg_i, \tag{34}$$

since the size of the set $\hat{\mathcal{V}}_1$ is simply the total number of V-blocks (m_i for each agent), and $m_i < \deg_i$ due to (12). Next, we consider the set $\hat{\mathcal{V}}_0$ of dual agents. For a network of anti-coordinating agents at time zero, we have for each $\hat{i} \in \hat{\mathcal{V}}_0$

$$\Phi_{\hat{i}}(0) = \begin{cases} n_{\hat{i}}^A(0) - \hat{n}_{\hat{i}}^A \le \deg_{\hat{i}} - \frac{1}{2}\deg_{\hat{i}} + 1 & \text{if } x_{\hat{i}}(0) = A \\ \hat{n}_{\hat{i}}^A + 1 - n_{\hat{i}}^A(0) \le \frac{1}{2}\deg_{\hat{i}} + 1 - 0 & \text{if } x_{\hat{i}}(0) = B \end{cases},$$

where we used the facts that $\tau_{\hat{i}} \deg_{\hat{i}} - 1 \le \hat{n}_{\hat{i}}^A \le \tau_{\hat{i}} \deg_{\hat{i}}$ and that the thresholds in the augmented network $\tau_{\hat{i}}$ are all equal to $\frac{1}{2}$. Similarly, for a network of coordinating agents, we have

$$\Phi_{\hat{i}}(0) = \begin{cases} \check{n}_{\hat{i}}^A - n_{\hat{i}}^A(0) \le \frac{1}{2}\deg_{\hat{i}} + 1 - 0 & \text{if } x_{\hat{i}}(0) = A \\ n_{\hat{i}}^A(0) - \check{n}_{\hat{i}}^A + 1 \le \deg_{\hat{i}} - \frac{1}{2}\deg_{\hat{i}} + 1 & \text{if } x_{\hat{i}}(0) = B \end{cases},$$

since it holds that $\tau_{\hat{i}} \deg_{\hat{i}} \le \check{n}_{\hat{i}}^A \le \tau_{\hat{i}} \deg_{\hat{i}} + 1$. The result is the following upper bound:

$$\Phi_{\hat{i}}(0) \le \frac{1}{2}\deg_{\hat{i}} + 1 \text{ for all } \hat{i} \in \hat{\mathcal{V}}_0. \tag{35}$$

For a network of anti-coordinating agents at equilibrium (at time k^*), we have

$$\Phi_{\hat{i}}(k^*) = \begin{cases} n_{\hat{i}}^A(k^*) - \hat{n}_{\hat{i}}^A \ge 0 - \frac{1}{2}\deg_{\hat{i}} & \text{if } x_{\hat{i}}(k^*) = A \\ \hat{n}_{\hat{i}}^A + 1 - n_{\hat{i}}^A(k^*) \ge \frac{1}{2}\deg_{\hat{i}} - \deg_{\hat{i}} & \text{if } x_{\hat{i}}(k^*) = B \end{cases}.$$

Similarly, for a network of coordinating agents, we have

$$\Phi_{\hat{i}}(k^*) = \begin{cases} \check{n}_{\hat{i}}^A - n_{\hat{i}}^A(k^*) \ge \frac{1}{2}\deg_{\hat{i}} - \deg_{\hat{i}} & \text{if } x_{\hat{i}}(k^*) = A \\ n_{\hat{i}}^A(k^*) - \check{n}_{\hat{i}}^A + 1 \ge 0 - \frac{1}{2}\deg_{\hat{i}} & \text{if } x_{\hat{i}}(k^*) = B \end{cases}.$$

This yields the following lower bound:

$$\Phi_{\hat{i}}(k^*) \ge -\frac{1}{2}\deg_{\hat{i}} \text{ for all } \hat{i} \in \hat{\mathcal{V}}_0. \tag{36}$$

Using (35) and (36), we can bound the change in potential for the dual agents as follows:

$$\sum_{\hat{i} \in \hat{\mathcal{V}}_0} \Phi_{\hat{i}}(0) - \Phi_{\hat{i}}(k^*) \le \sum_{\hat{i} \in \hat{\mathcal{V}}_0} (\deg_{\hat{i}} + 1).$$

For each dual agent $\hat{i} \in \hat{\mathcal{V}}_0$, let i denote the corresponding original agent in $\mathcal{V}$. Since $\deg_{\hat{i}} = \deg_i + m_i$ and $m_i < \deg_i$ due to (12), it holds that $\deg_{\hat{i}} \le 2\deg_i - 1$. It follows that

$$\sum_{\hat{i} \in \hat{\mathcal{V}}_0} \Phi_{\hat{i}}(0) - \Phi_{\hat{i}}(k^*) \le 2 \sum_{i \in \mathcal{V}} \deg_i. \tag{37}$$

Substituing (34) and (37) into (33) results in

$$\Phi(0) - \Phi(k^*) \leq 3 \sum_{i \in \mathcal{V}} \deg_i = 6|\mathcal{E}|.$$

Finally, Lemma 4 implies that the sequence of agent switches between an original and augmented network are equivalent, as long as the dual agents activate in the same sequence as the agents in the original network. This completes the proof. ∎

IV. Proof of Theorem 3

Proof of Theorem 3: Since the updates to $x(k+1)$ depend only on the state $x(k)$, and since agent activations do not depend on time, the network game can be modeled as a Markov chain with dimension 2^n. The state transition probabilities depend on the probabilities that each of the sets $\mathcal{A}_k$ will occur, along with the corresponding update dynamics. To prove almost sure convergence of the network game, it suffices to show that this Markov chain is *absorbing*, which requires satisfying two conditions [3, Definition 11.1, p416]. The first condition is that there exists at least one absorbing state. Absorbing states are equivalent to Nash equilibria of the network game, whose existence we have established in Corollaries 2 and 1. The second condition is that there exists a path in the Markov chain from every non-absorbing state to an absorbing state. Theorems 2 and 1 established the existence of such paths, which consist of finite sequences of asynchronous updates. It follows from Assumption 2, i.e., agent updates are independent and have support on $\mathbb{R}_{\geq 0}$, that the probabilities of each agent being the only active agent in a given time step are strictly positive (they can be computed from the probability distributions for the inter-activation times of each agent). Therefore, both conditions are met, and the Markov chain is absorbing, which implies that the corresponding network game will almost surely reach an equilibrium state in finite time [3, Theorem 11.3, p417]. ∎

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