Submitted to the Annals of Applied Statistics
arXiv: arXiv:0000.0000

MCMC DETAILS AND ADDITIONAL SIMULATION RESULTS

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SUPPLEMENTARY MATERIAL

1. MCMC Details.

1.1. *Quantile Parameters.* All prior parameters are independent across basis functions and predictors. Polynomial parameters are also independent. That is, the intercept effect over time is independent of slope effect across time. Let $\theta^{\star}_{dmp}$ be the vector of length $H = 2$ corresponding to the regression coefficients for SBP and DBP for the d^{th} polynomial term, m^{th} basis function and p^{th} predictor. In our model $\theta^{\star}_{dmp} \sim N(\mu_{dmp}, \iota^2_{dmp})$, where ι^2_{dmp} is a precision. Below we suppress the subscripts for d, m, and p. We sample from the posterior distribution for $\theta^{\star}_{dmp}$ using random walk Metropolis, as the posterior distribution does not have a closed form. In the burn in period posterior variances are tuned to have a 30-40% acceptance ratio.

We assign μ a Gaussian prior with mean μ_0 and precision ι_0^2. Conditional on the other parameters the posterior distribution of μ is of the form

$$
\begin{aligned}
[\mu|rest] &\propto \exp\left\{-0.5\iota^2[\boldsymbol{\theta}^\star - \mathbf{1}\mu]'[\boldsymbol{\theta}^\star - \mathbf{1}\mu]\right\}\exp\left\{-0.5(\iota_0^2)[\mu-\mu_0]^2\right\} \\
&\propto \exp\left\{-0.5[\mu^2\left(\iota^2\mathbf{1}'\mathbf{1}+\iota_0^2\right) - 2\mu\left(\iota^2\mathbf{1}'\theta + \iota_0^2\mu_0\right)]\right\} \\
&= \exp\left\{-0.5\left[\mu'\Omega\mu - 2\mu'\omega\right]\right\} \\
&= \exp\left\{-0.5\left[\mu'\Omega\mu - 2\mu'\Omega(\Omega^{-1}\omega)\right]\right\}
\end{aligned}
$$

which is the kernel of a normal random variable with mean $\Omega^{-1}\omega$ and variance Ω^{-1} where $\Omega = H\iota^2 + \iota_0^2$ and $\omega = \iota^2\mathbf{1}'\theta + \iota_0^2\mu_0$.

*The authors are grateful for support from NSF grant DMS-1107046, NIH grant SR01ES014843, as well as NHLBI (R01-HL108427).

†The China Health and Nutrition Survey is funded by NICHD (R01-HD30880), although no direct support was received from grant for this analysis. We also are grateful to the Carolina Population Center (R24 HD050924) for general support.

We assign ι^2 a gamma(a, b) prior. Let $\mathbf{x} = \boldsymbol{\theta}^\star - \mathbf{1}\mu$. Conditional on the other parameters the posterior distribution of ι^2 is of the form

$$\begin{aligned}
[\iota^2|rest] &\propto \iota^{H/2} \exp\left\{-0.5\iota^2 \mathbf{x}'\mathbf{x}\right\} (\iota^2)^{(a-1)} \exp\left\{-\iota^2/b\right\} \\
&\propto (\iota^2)^{(H/2+a-1)} \exp\left\{-\iota^2 \left(0.5\mathbf{x}'\mathbf{x} + 1/b\right)\right\} \\
&= (\iota^2)^{(H/2+a-1)} \exp\left\{-\iota^2 \left((0.5b\mathbf{x}'\mathbf{x} + 1)/b\right)\right\}
\end{aligned}$$

which is the kernel of a gamma random variable with shape $H/2 + a$ and scale $(2b/(b\mathbf{x}'\mathbf{x} + 2))$.

1.2. *Copula Parameters.* For MCMC we model $\Phi^{-1}(\mathbf{U}_i) = \mathbf{W}_i = \mathbf{D}_i \left[\mathbf{Z}_i\boldsymbol{\gamma}_i + \boldsymbol{\eta_i} + \mathbf{E}_i\right]$, where $\boldsymbol{\eta_i} \sim \mathrm{N}(0, \boldsymbol{\Xi}(\alpha) \otimes \boldsymbol{\Lambda})$. The regression coefficients $\boldsymbol{\gamma}_i$ have posterior

$$\begin{aligned}
[\gamma_i|rest] &\propto \exp\left\{-0.5[\boldsymbol{W}_i - \boldsymbol{D}_i\left(\boldsymbol{Z}_i\gamma_i + \boldsymbol{\eta}_i\right)]'\boldsymbol{D}_i^{-2}[\boldsymbol{W}_i - \boldsymbol{D}_i\left(\boldsymbol{Z}_i\gamma_i + \boldsymbol{\eta}_i\right)]\right\} \exp\left\{-0.5\boldsymbol{\gamma}_i'\boldsymbol{\Delta}^{-1}\boldsymbol{\gamma}_i\right\} \\
&\propto \exp\left\{-0.5\boldsymbol{\gamma}_i'\left(\boldsymbol{Z}_i'\boldsymbol{D}_i\boldsymbol{D}_i^{-1}\boldsymbol{D}_i^{-1}\boldsymbol{D}_i\boldsymbol{Z}_i + \boldsymbol{\Delta}^{-1}\right)\gamma_i + \gamma_i\boldsymbol{Z}_i'\boldsymbol{D}_i\boldsymbol{D}_i^{-2}\left(\boldsymbol{W}_i - \boldsymbol{D}_i\boldsymbol{\eta}_i\right)\right\} \\
&= \exp\left\{-0.5\boldsymbol{\gamma}_i'\left(\mathbf{Z}_i'\mathbf{Z}_i + \boldsymbol{\Delta}^{-1}\right)\gamma_i + \gamma_i\boldsymbol{Z}_i'\left(\mathbf{D}_i^{-1}\mathbf{W}_i - \boldsymbol{\eta}_i\right)\right\}
\end{aligned}$$

which is the kernel of a multivariate normal random variable with mean $\Omega^{-1}\omega$ and variance Ω^{-1} where $\Omega = \boldsymbol{Z}_i'\boldsymbol{Z}_i + \boldsymbol{\Delta}^{-1}$ and $\omega = \boldsymbol{Z}_i'\left(\mathbf{D}_i^{-1}\mathbf{W}_i - \boldsymbol{\eta}_i\right)$.

The regression coefficients $\boldsymbol{\eta}_i$ have posterior

$$\begin{aligned}
[\boldsymbol{\eta}_i|rest] &\propto \exp\left\{-0.5[\boldsymbol{W}_i - \boldsymbol{D}_i\left(\boldsymbol{Z}_i\gamma_i + \boldsymbol{\eta}_i\right)]'\boldsymbol{D}_i^{-2}[\boldsymbol{W}_i - \boldsymbol{D}_i\left(\boldsymbol{Z}_i\gamma_i + \boldsymbol{\eta}_i\right)]\right\} \\
&\quad * \exp\left\{-0.5\boldsymbol{\eta}_i'\boldsymbol{\Psi}^{-1}\boldsymbol{\eta}_i\right\} \\
&\propto \exp\left\{-0.5\boldsymbol{\eta}_i'\left(\boldsymbol{D}_i\boldsymbol{D}_i^{-1}\boldsymbol{D}_i^{-1}\boldsymbol{D}_i + \boldsymbol{\Psi}^{-1}\right)\boldsymbol{\eta}_i + \boldsymbol{\eta}_i'\boldsymbol{D}_i\boldsymbol{D}_i^{-2}\left(\boldsymbol{W}_i - \boldsymbol{D}_i\boldsymbol{Z}_i\gamma_i\right)\right\} \\
&= \exp\left\{-0.5\boldsymbol{\eta}_i'\left(\mathbf{I} + \boldsymbol{\Psi}^{-1}\right)\boldsymbol{\eta}_i + \boldsymbol{\eta}_i'\left(\boldsymbol{D}_i^{-1}\boldsymbol{W}_i - \boldsymbol{Z}_i\gamma_i\right)\right\}
\end{aligned}$$

which is the kernel of a multivariate normal random variable with mean $\Omega^{-1}\omega$ and variance Ω^{-1} where $\Omega = \mathbf{I} + \boldsymbol{\Psi}^{-1}$ and $\omega = \boldsymbol{D}_i^{-1}\boldsymbol{W}_i - \boldsymbol{Z}_i\gamma_i$.

The elements of $\boldsymbol{\Delta}$ and $\boldsymbol{\Lambda}$ are singly updated using random walk Metropolis. The correlation parameter α is sampled using independent Metropolis updates.

2. Additional Simulation Results.

References.

[1] Reich, B. J., Bondell, H. D. and Wang, H. J. (2010). Flexible Bayesian quantile regression for independent and clustered data. *Biostatistics* **11** 337–352.

Supplement A: Title
(doi: 10.1214/00-AOASXXXXSUPP; .pdf).

Table 1

Coverage probability (CP) and mean squared error (MSE) for the $N = 100$ arm of the simulation study. We compare treating the data as independent within a subject ("Ind"), fitting with a copula ("Cop"), and the random effects model of [1] ("RBW"). Coverage and MSE were evaluated at and averaged over the quantile levels $\{0.1, 0.3, 0.5, 0.7, 0.9\}$. For datatype = 1, MSE values are less than depicted values by a factor of 10. Estimators whose MSEs were statistically significantly different than the copula model are indicated by *.

$\Delta = 0$, Datatype = 1												
	Coverage						MSE					
	$\alpha = 0.0$		$\alpha = 0.5$		$\alpha = 0.9$		$\alpha = 0.0$		$\alpha = 0.5$		$\alpha = 0.9$	
	X1	X2	X1	X2	X1	X2	X1	X2	X1	X2	X1	X2
Ind	0.91	0.95	0.88	0.94	0.73	0.94	0.02	0.05	0.03	0.05	0.06	0.06
Cop	0.95	0.96	0.95	0.97	0.94	0.96	0.02	0.05	0.03	0.05	0.05	0.04
RBW	0.83	0.90	0.86	0.89	0.88	0.87	0.08^*	0.13^*	0.10^*	0.12^*	0.12^*	0.11^*
$\Delta = 0$, Datatype = 2												
	Coverage						MSE					
	$\alpha = 0.0$		$\alpha = 0.5$		$\alpha = 0.9$		$\alpha = 0.0$		$\alpha = 0.5$		$\alpha = 0.9$	
	X1	X2	X1	X2	X1	X2	X1	X2	X1	X2	X1	X2
Ind	0.94	0.97	0.89	0.94	0.76	0.95	0.03	0.06	0.04	0.06	0.09	0.06
Cop	0.98	0.98	0.95	0.98	0.93	0.97	0.03	0.06	0.04	0.06	0.08	0.06
RBW	0.96	0.98	0.95	0.96	0.93	0.95	0.03	0.06	0.04	0.06	0.09	0.06
$\Delta = 3$, Datatype = 1												
	Coverage						MSE					
	$\alpha = 0.0$		$\alpha = 0.5$		$\alpha = 0.9$		$\alpha = 0.0$		$\alpha = 0.5$		$\alpha = 0.9$	
	X1	X2	X1	X2	X1	X2	X1	X2	X1	X2	X1	X2
Ind	0.61	0.76	0.58	0.78	0.56	0.72	0.08	0.09	0.08	0.09	0.09	0.10^*
Cop	0.92	0.91	0.91	0.90	0.89	0.91	0.08	0.07	0.07	0.07	0.07	0.06
RBW	0.85	0.70	0.85	0.69	0.86	0.67	0.14^*	0.15^*	0.15^*	0.15^*	0.16^*	0.15^*
$\Delta = 3$, Datatype = 2												
	Coverage						MSE					
	$\alpha = 0.0$		$\alpha = 0.5$		$\alpha = 0.9$		$\alpha = 0.0$		$\alpha = 0.5$		$\alpha = 0.9$	
	X1	X2	X1	X2	X1	X2	X1	X2	X1	X2	X1	X2
Ind	0.64	0.76	0.60	0.78	0.57	0.74	0.10^*	0.12	0.11	0.13^*	0.12^*	0.13
Cop	0.90	0.90	0.89	0.89	0.91	0.92	0.07	0.08	0.07	0.08	0.07	0.09
RBW	0.86	0.80	0.84	0.79	0.83	0.77	0.09	0.10	0.11	0.10	0.13^*	0.11