able, in virtue of their thermal agitation, to carry large currents without producing a space charge, could greatly facilitate the interpretation of the discharge of electricity through gases.

The author is indebted to Professor Bergen Davis and Professor H. W. Webb for their kind permission to use the laboratory.

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GRAVITATION AND THE ELECTRON[1]

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The Problem.—The translation of Dirac's theory of the electron into general relativity is not only of formal significance, for, as we know, the Dirac equations applied to an electron in a spherically symmetric electrostatic field yield in addition to the correct energy levels those—or rather the negative of those—of an "electron" with opposite charge but the same mass. In order to do away with these superfluous terms the wave function ψ must be robbed of one of its pairs $\psi_1^+, \psi_2^+;\ \psi_1^-, \psi_2^-$ of components.[2] These two pairs occur unmixed in the action principle except for the term

$$m(\psi_1^+ \bar\psi_1^- + \psi_2^+ \bar\psi_2^- + \psi_1^- \bar\psi_2^+ + \psi_2^- \bar\psi_2^+) \qquad (1)$$

which contains the mass m of the electron as a factor. But mass is a gravitational effect: it is the flux of the gravitational field through a surface enclosing the particle in the same sense that charge is the flux of the electric field.[3] In a satisfactory theory it must therefore be as impossible to introduce a non-vanishing mass without the gravitational field as it is to introduce charge without electromagnetic field. It is therefore certain that the term (1) can at most be right in the large scale, but must really be replaced by one which includes gravitation; this may at the same time remove the defects of the present theory.

The direction in which such a modification is to be sought is clear: the field equations arising from an action principle[4]—which shall give the true laws of interaction between electrons, protons and photons only after quantization—contain at present only the Schrödinger-Dirac quantity ψ, which describes the wave field of the *electron*, in addition to the four potentials φ_p of the electromagnetic field. It is unconditionally necessary to introduce the wave field of the *proton* before quantizing. But since the ψ of the electron can only involve two components, ψ_1^+, ψ_2^+ should be ascribed to the electron and ψ_1^-, ψ_2^- to the proton. Obviously the present expression, $-e\cdot\tilde{\psi}\psi$ for charge-density,[5] being necessarily negative, runs counter to this, and something must consequently be changed in this respect. Instead of one law for the conservation of charge we must have two, expressing the conservation of the number of electrons and protons separately.

If one introduces the quantities $\frac{e\varphi_p}{ch}$ instead of φ_p (and calls them φ_p), the field equations contain only the following combinations of atomistic constants: the pure number $\alpha = e^2/ch$ and h/mc, the "wave-length" of the electron. Hence the equations certainly do not alone suffice to explain the atomistic behavior of matter with the definite values of e, m and h. But the subsequent quantization introduces the quantum of action h, and this together with the wave-length h/mc will be sufficient, since the velocity of light c is determined as an absolute measure of velocity by the theory of relativity.

The introduction of the atomic constants by the quantum theory—or at least that of the wave-length—into the field equations has removed the support from under my principle of gauge-invariance, by means of which I had hoped to unify electricity and gravitation. But as I have remarked,[6] it possesses an equivalent in the field equations of quantum theory which is its perfect counterpart in formal respects: the laws are invariant under the simultaneous substitution of $e^{i\lambda}\cdot\psi$ for ψ and $\varphi_p - \frac{\partial\lambda}{\partial x_p}$ for φ_p, where λ is an arbitrary function of position in space and time. The connection of this invariance with the conservation law of electricity remains exactly as before: the fact that the action integral is unaltered by the infinitesimal variation

$$\delta\psi = i\,\lambda\cdot\psi, \qquad \delta\varphi_p = -\frac{\partial\lambda}{\partial x_p}$$

(λ an arbitrary infinitesimal function) signifies the identical fulfilment of a dependence between the material and the electromagnetic laws which arise from the action integral by variations of the ψ and φ, respectively; it means that the conservation of electricity is a double consequence of them, that it follows from the laws of matter as well as electricity. This new principle of gauge invariance, which may go by the same name, has the character of general relativity since it contains an arbitrary function λ, and can certainly only be understood with reference to it.

It was such considerations as these, and not the desire for formal generalizations, which led me to attempt the incorporation of the Dirac theory into the scheme of general relativity. We establish the metric in a world point P by a "Cartesian" system of axes (instead of the g_{pq}) consisting of four vectors $\mathbf{e}(\alpha)$ $\{\alpha = 0, 1, 2, 3\}$ of which $\mathbf{e}(1)$, $\mathbf{e}(2)$, $\mathbf{e}(3)$ are real space-like vectors while $\mathbf{e}(0)/i$ is a real time-like vector of which we expressly demand that it be directed toward the future. A rotation of these axes is an orthogonal or Lorentz transformation which leaves these conditions of reality and sign unaltered. The laws shall remain invariant when the axes in the various points P are subjected to arbitrary and independent rotations. In addition to these we need four (real) coördinates x_p ($p = 0, 1, 2, 3$) for the purpose of analytic expression. The components of $\mathbf{e}(\alpha)$ in this coördinate system are designated by $e^p(\alpha)$. We need such local cartesian axes $\mathbf{e}(\alpha)$ in each point P in order to be able to describe the quantity ψ by means of its components ψ_1^+, ψ_2^+; ψ_1^-, ψ_2^-, for the law of transformation of the components ψ can only be given for orthogonal transformations as it corresponds to a representation of the orthogonal group which cannot be extended to the group of all linear transformations. The tensor calculus is consequently an unusable instrument for considerations involving the ψ.[7] In formal aspects our theory resembles the more recent attempts of Einstein to unify electricity and gravitation.[8] But here there is no talk of "distant parallelism"; there is no indication that Nature has availed herself of such an artificial geometry. I am convinced that if there is a physical content in Einstein's latest formal developments it must come to light in the present connection. It seems to me that it is now hopeless to seek a unification of gravitation and electricity without taking material waves into account.

Use of the Indices.—If $t(\alpha)$ be the components of an arbitrary vector at point P with respect to the axes $\mathbf{e}(\alpha)$, then

$$t^p = \sum_\alpha e^p(\alpha)\, t(\alpha) \tag{2}$$

are its contravariant components in the coördinate system x_p. Conversely, from the covariant components t_p referred to the coördinates one obtains the components $t(\alpha)$ along the axes by the equations

$$t(\alpha) = \sum_p e^p(\alpha) t_p. \tag{3}$$

Equations (2), (3) regulate the transition from one kind of indices to the other (Greek indices referring to the axes, Latin sub- or superscripts to coördinates.) In the inverse transitions the quantities $e_p(\alpha)$, which are defined by

$$e_p(\alpha)\, e^q(\alpha) = \delta_p^q$$

and which also satisfy

$$e_p(\alpha)\, e^p(\beta) = \delta\,(\alpha, \beta)$$

occur as coefficients. The Kronecker δ is 1 or 0 according to whether its indices agree or not.

Symmetry and Conservation of the Energy Density.—The invariant action

$$\int \mathbf{H}\, dx \qquad (dx = dx_0\, dx_1\, dx_2\, dx_3)$$

contains *matter* (in the extended sense) and *gravitation*, the first being represented by the ψ and possibly such additional quantities as the electromagnetic potentials φ_p, the latter by the components $e^p(\alpha)$ of the $\mathbf{e}(\alpha)$. Variation of the first kind of quantities gives rise to the equations of matter, variation of the $e^p(\alpha)$ to the gravitational equations. We disregard for the present that part of the action which depends only on the $e^p(\alpha)$, as introduced by Einstein in his classical theory of gravitation (1916), and consider only that part $\mathbf{H}$ which occurs even in the special theory of relativity. By an arbitrary infinitesimal variation of the $e^p(\alpha)$ which shall vanish outside of a finite portion of the world, an equation

$$\delta \int \mathbf{H} dx = \int \mathbf{t}_p(\alpha)\, \delta e^p(\alpha) \cdot dx \tag{4}$$

is obtained which defines the components $\mathbf{t}_p(\alpha)$ of the "energy density." In consequence of the equations of matter, which are assumed to hold, it is immaterial if or how the quantities describing matter are varied. Because of the invariance of the action (4) must vanish for variations $\delta e^p(\alpha)$ obtained by 1) subjecting the axes $\mathbf{e}(\alpha)$ to an infinitesimal rotation which may depend arbitrarily on position and 2) subjecting the coordinates x_p to an arbitrary infinitesimal transformation, the axes $\mathbf{e}(\alpha)$ being unaltered. The first process is described by

$$\delta e^p(\alpha) = o(\alpha\beta) e^p(\beta)$$

where $o(\alpha\beta)$ constitute an anti-symmetric matrix whose elements are arbitrary (infinitesimal) functions of position. This requirement yields the symmetry law:

$$\mathbf{t}_p(\beta) e^p(\alpha) = \mathbf{t}(\alpha, \beta)$$

depends symmetrically on the two indices, α and β. But it must be observed that this law is not identically fulfilled, as in the old field theory, but only in consequence of the equations of matter, for if the wave field ψ be held unchanged the components ψ_ρ must undergo a transformation which is induced by the rotation of the axes. If $\delta x_p = \xi^p$ be the change which the coördinates of point P undergo in the second process, then the components of the unaltered vector $\mathbf{e}(\alpha)$ in P will undergo the change

$$\delta' e^p(\alpha) = \frac{\partial \xi^p}{\partial x_q} \cdot e^q(\alpha) \cdot$$

This must, on the other hand, be given by

$$\delta e^p(\alpha) + \frac{\partial e^p(\alpha)}{\partial x_q}\xi^q$$

where δ means the difference at two points P which have the same values x_p of coördinates before and after the deformation. From this there arises in the usual way[9]—again assuming the validity of the equations of matter—the differential quasi-conservation law for energy (and linear momentum) whose four components are

$$\frac{\partial \mathbf{t}_p^q}{\partial x_q} + \mathbf{t}_q(\alpha)\frac{\partial e^q(\alpha)}{\partial x_p} = 0. \quad (5)$$

(Only in the special theory of relativity, where the second member is lacking, is it a true conservation law.)

It is not necessary that the integral of $\mathbf{H}$ be invariant, but only that its variation be. This is the case when $\mathbf{H}$ differs from a scalar density by a divergence; we then say that the integral is "practically invariant." Similarly it is only necessary that it be "practically real," i.e., that the difference between $\mathbf{H}$ and its complex conjugate be a divergence.

Gradient of ψ.—Let the wave field ψ be given. The invariant change $\delta\psi$ of ψ on going from the point P to a neighboring point P' is to be determined as follows. The axes $\mathbf{e}(\alpha)$ in P are taken to P' by parallel displacement: $\mathbf{e}'(\alpha)$. $\psi_\rho = \psi_\rho(P)$ being the components of ψ with respect to the axes $\mathbf{e}(\alpha)$ at P, let ψ'_ρ be the components of ψ in P' relative to this displaced system: $\delta\psi_\rho = \psi'_\rho - \psi_\rho$. These $\delta\psi_\rho$ depend only on the choice of axes in P and transform in the same way as the ψ_ρ on rotation of these axes. The axes $\mathbf{e}'(\alpha)$ are obtained from the $\mathbf{e}(\alpha)$ in P' by an infinitesimal orthogonal transformation $do(\alpha\beta)$; consequently the ψ'_ρ are obtained from the components $\psi_\rho(P')$ by the corresponding linear transformation[10] dE and we have, $d\psi_\rho$ being the differential $\psi_\rho(P') - \psi_\rho(P)$:

$$\delta\psi = d\psi + dE.\psi.$$

If $\mathbf{e}(\alpha)$ be taken as the vector $\overrightarrow{PP'}$ (multiplied by an infinitesimal factor) we write (on ignoring this factor) $o(\alpha,\beta\gamma)$, $E(\alpha)$, $\psi(\alpha)$ in place of $do(\beta\gamma)$, dE, $\delta\psi$:

$$\psi(\alpha) = \left(e^p(\alpha)\frac{\partial}{\partial x_p} + E(\alpha)\right)\psi \quad \text{or} \quad \psi_p = \left(\frac{\partial}{\partial x_p} + E_p\right)\psi.$$

The calculation of o is accomplished by means of the formula

$$e^p(\gamma)\{o(\alpha,\beta\gamma) + o(\beta,\gamma\alpha)\} = \frac{\partial e^p(\alpha)}{\partial x_q}e_q(\beta) - \frac{\partial e^p(\beta)}{\partial x_q}e_q(\alpha).$$

The right-hand side of this expression is the "commutator product" of the two vector fields $\mathbf{e}(\alpha)$ and $\mathbf{e}(\beta)$, an invariant (under transformations of the coördinates x_p), known from the Lie theory.

Introduction of Dirac's Action.—Let $S(\alpha)$ denote linear transformations which transform ψ_1^+, ψ_2^+ and ψ_1^-, ψ_2^- among themselves. They are described by the matrices[11]

$$S(0) = \begin{Vmatrix} i & 0 \\ 0 & i \end{Vmatrix}, \quad S(1) = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}, \quad S(2) = \begin{Vmatrix} 0 & -i \\ i & 0 \end{Vmatrix}, \quad S(3) = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}$$

for ψ_1^+, ψ_2^+; for ψ_1^-, ψ_2^- the expression for $S(0)$ is unchanged, $S(1)$, $S(2)$ and $S(3)$ assume the opposite sign. The essential fact is that the quantities

$$\tilde{\psi}' S(\alpha)\psi$$

transform like the four components $t(\alpha)$ of a vector on rotation of the axes $\mathbf{e}(\alpha)$, ψ' being a quantity of the same kind as ψ. (In particular, $\tilde{\psi} S(\alpha)\psi$ is the four-vector flux of probability.) Therefore

$$\tilde{\psi} S(\alpha)\psi(\alpha) = \tilde{\psi} e^p(\alpha)S(\alpha) \frac{\partial \psi}{\partial x_p} + \tilde{\psi} S(\alpha)E(\alpha)\psi$$

is a scalar, and after dividing by the absolute value ϵ of the determinant

$$| \, e^p(\alpha) \, |$$

we obtain a scalar density $i\mathbf{H}$ whose integral can be employed as action. Division by ϵ will be indicated by changing the ordinary letter into the corresponding gothic. The calculation yields

$$\frac{1}{\epsilon} S(\alpha)E(\alpha) = \frac{1}{2} \frac{\partial \mathbf{e}^p(\alpha)}{\partial x_p} S(\alpha) + \frac{1}{2} \mathbf{I}(\alpha)S'(\alpha),$$

where $S'(\alpha)$ is a transformation analogous to $S(\alpha)$: it agrees with $S(\alpha)$ for ψ_1^+, ψ_2^+, but is $-S(\alpha)$ for ψ_1^-, ψ_2^-.

$$\begin{aligned} \epsilon\mathbf{I}(\alpha) &= o(\beta,\gamma\delta) + o(\gamma,\delta\beta) + o(\delta,\beta\gamma) \\ &= \Sigma \pm \frac{\partial e^p(\beta)}{\partial x_q} e_q(\gamma)\, e^p(\delta) \end{aligned}$$

where the summation (in addition to that over p and q) is alternating and extends over the six permutations of $\beta\gamma\delta$ while $\alpha\beta\gamma\delta$ is an even permutation of the indices 0, 1, 2, 3.

We have yet to investigate whether $\mathbf{H}$ is practically real. Since

$$S^p = e^p(\alpha)S(\alpha) \quad \text{and} \quad S(\alpha)E(\alpha)$$

are Hermitean matrices

$$- i\overline{\mathbf{H}} = \frac{1}{\epsilon} \left\{ \frac{\partial \tilde{\psi}}{\partial x_p} e^p(\alpha)S(\alpha)\psi + \tilde{\psi} S(\alpha)E(\alpha)\psi \right\}.$$

The first part is, on neglecting a complete divergence,

$$-\tilde{\psi}\,\frac{\partial(\mathbf{e}^p(\alpha)S(\alpha)\psi)}{\partial x_p} = -\tilde{\psi}\,\mathbf{e}^p(\alpha)S(\alpha)\,\frac{\partial\psi}{\partial x_p} - \frac{\partial \mathbf{e}^p(\alpha)}{\partial x_p}.\tilde{\psi}S(\alpha)\psi.$$

On adding and subtracting $\mathbf{H}$ and $\overline{\mathbf{H}}$ we obtain the two action quantities $\mathbf{m}$, $\mathbf{m}'$ ($\mathbf{m}$ = matter):

$$i\mathbf{m} = \tilde{\psi}S^p\,\frac{\partial\psi}{\partial x_p} + \frac{1}{2}\,\frac{\partial \mathbf{e}^p(\alpha)}{\partial x_p}.\tilde{\psi}S(\alpha)\psi \tag{6}$$

and

$$\mathbf{m}' = \mathbf{l}(\alpha).\tilde{\psi}S'(\alpha)\psi. \tag{7}$$

Both are practically, not actually, invariant, the first is practically and the second actually real.

The first is the essential content of the Dirac theory, written in general invariantive form. The corresponding tensor density of energy

$$\mathbf{t}_p(\alpha) = \mathbf{s}_p(\alpha) - e_p(\alpha)\mathbf{s}$$

where

$$\epsilon\mathbf{s}_p(\alpha) = \frac{1}{2i}\left\{\tilde{\psi}S(\alpha)\,\frac{\partial\psi}{\partial x_p} - \frac{\partial\tilde{\psi}}{\partial x_p}\,S(\alpha)\psi\right\}$$

and $\mathbf{s}$ the contracted

$$\mathbf{s}_p(\alpha)\;e^p(\alpha).$$

It has already been given in the literature for the Dirac theory (special relativity). For the electron of hydrogen in the normal state we find that the integral

$$\int \mathbf{t}_0^0\,dx_1\,dx_2\,dx_3$$

extended over a section $x_0 = t =$ const., which should yield the mass, has the value $m/\sqrt{1-\alpha^2}$ (α the fine structure constant); this is a reasonable result, since m is to be taken as proper mass and in the Bohr theory αc is the velocity of the electron in the normal state.

It is worthy of note that there occurs in addition the action $\mathbf{m}'$, which is unknown to special relativity since it vanishes for constant $e^p(\alpha)$.

The Electromagnetic Field.—In the Dirac theory the influence of an electromagnetic field is taken into account by replacing the operator $\frac{\partial}{\partial x_p}$, affecting the ψ by $\frac{\partial}{\partial x_p} + i\varphi_p$. This yields an additional term of the form $i\varphi(\alpha)\cdot\tilde{\psi}S(\alpha)\psi$ in the action. On comparing this with (6) and (7) one might think that $\epsilon.\varphi(\alpha)$ is to be identified with $\partial\mathbf{e}^p(\alpha)/\partial x_p$ or $\mathbf{l}(\alpha)$. Disregarding the material waves ψ one would then have a theory of electricity of the same kind as the latest development of Einstein; the φ are expressed in terms of the $e^p(\alpha)$ in a way which is invariant under transformations of the coördinates, but only permit the cogredient rotations of the axes in all points P (distant parallelism). However, I believe one must proceed otherwise in order to bring in the electromagnetic field. We have previously not mentioned the fact that the linear transformation of the ψ corresponding to a rotation of axes is not uniquely determined. It is indeed possible to normalize this transformation, and we have tacitly based our calculations on such a normalization; but the normalizing is itself a double-valued process and this reveals its artificial character. The mathematical connection is this: we are to find a linear transformation of the four components of ψ such that the quantities

$$\tilde{\psi} S(\alpha)\psi \tag{8}$$

suffer the given rotation.[12] Obviously it remains unaltered when ψ_1^+, ψ_2^+ are multiplied by $e^{i\lambda^+}$, ψ_1^-, ψ_2^- by $e^{i\lambda^-}$ (transformation L) where λ^+ and λ^- are arbitrary real numbers. It is readily seen that the transformations L are the only ones which induce the identical transformation of the quantities (8); i.e., which satisfy the four equations

$$\tilde{L} S(\alpha) L = S(\alpha).$$

A transformation of the four components of ψ which multiplies ψ_1^+, ψ_2^+ by a number a^+ and ψ_1^-, ψ_2^- by a number a^- will be called *spinless quantity* $A = [a^+, a^-]$.

In consequence of this the dE employed above is only determined to within the addition of an arbitrary spinless imaginary quantity idF. Hence we obtain an additive term

$$iF(\alpha) . \tilde{\psi} \mathbf{S}(\alpha)\psi$$

in the action $\mathfrak{m}$, in which the $F(\alpha)$ are real spinless quantities $[\varphi^+(\alpha), \varphi^-(\alpha)]$ constituting the components of a vector with respect to the axes. $\partial/\partial x_p$ is to be replaced by $\partial/\partial x_p + iF_p$. We now employ the letter $\mathfrak{m}$ to denote this completed action. The introduction of F obviously brings with it invariance under the simultaneous replacement of

$$\psi \text{ by } e^{iL}\cdot\psi \text{ and } F_p \text{ by } F_p - \frac{\partial L}{\partial x_p}$$

where L is a real spinless quantity which depends arbitrarily on position. This "gauge invariance" shows why it is impossible to employ the scalar (1) as an action function.

We must naturally interpret F_p as the components of electromagnetic potential. The two fields φ_p^+, φ_p^- are independent of the choice of the axes $\mathbf{e}(\alpha)$ except that they are interchanged on transition from right- to a left-handed system of axes.

$$\frac{\partial \varphi_q^+}{\partial x_p} - \frac{\partial \varphi_p^+}{\partial x_q} = \varphi_{pq}^+ \text{ and the corresponding } \varphi_{pq}^-$$

are gauge-invariant anti-symmetric tensors.

In analogy to the Maxwellian action quantity, and in accordance with the above properties, the two functions **f**, **f′** (**f** = electromagnetic field) defined by

$$\epsilon\mathbf{f} = \varphi^{+}_{pq}\,\varphi^{pq}_{-} \text{ and } \epsilon\mathbf{f}' = \varphi^{+}_{pq}\,\varphi^{pq}_{+} + \varphi^{-}_{pq}\,\varphi^{pq}_{-} \qquad (9)$$

are to be considered in choosing the scalar density of electromagnetic action. The identification of F with the electromagnetic potential is then justified by the fact that the entity which is represented by F influences and is influenced by matter in exactly the same way as the electromagnetic potentials. *If our view is correct, then the electromagnetic field is a necessary accompaniment of the matter-wave field and not of gravitation.*

If the action, insofar as its dependence on the ψ and φ is concerned, is an additive combination of **m**, **m′**, **f**, **f′** we obtain the two conservation theorems

$$\frac{\partial \rho^{p}_{+}}{\partial x_p} = 0 \text{ and } \frac{\partial \rho^{p}_{-}}{\partial x_p} = 0 \qquad (10)$$

where

$$\rho^{p}_{+} = \tilde{\psi}^{+} S^{p} \psi^{+}$$

contains only ψ_1^+ and ψ_2^+ and similarly for ρ^p_-. They are a double consequence of the field laws, that is, of the equations of matter in the narrow sense as well as of the electromagnetic equations. These two identities obtaining thus between the field equations are an immediate consequence of the gauge invariance. In consequence of (10) the two integrals n^+ and n^- of $\rho = \rho^0$:

$$n = \int \rho \, dx_1 dx_2 dx_3$$

which are to be extended over a section x_0 = const. are invariants which are independent of the "time" x_0.[13] Since they are interchanged on transition from right- to left-handed axes their values, which are absolute constants of nature, must be equal if both kinds of axes are to be equally permissible. We normalize them, in accordance with the interpretation of ψ as probability, by

$$n^+ = 1, \quad n^- = 1. \qquad (11)$$

We have already mentioned how the quasi-conservation laws for energy, momentum and moment of momentum are related to invariance under transformation of coördinates and rotation of axes $\mathbf{e}(\alpha)$.

A stationary solution is characterized by the fact that $e^p(\alpha)$ and $F(\alpha)$ are independent of time $x_0 = t$, while the ψ^+ contain the time in an exponential factor $e^{i\nu t}$, the ψ^- in a factor $e^{i\nu' t}$; ν and ν' need not be equal.

Gravitation.—We consider as the gravitational part of the action the practically (not actually) invariant density $\boldsymbol{g}$ ($\boldsymbol{g}$ = gravitation) which underlies Einstein's "classical" theory of gravitation and which depends only on the $e^p(\alpha)$ and their first derivatives. It is most appropriate to carry through anew the entire calculation, which leads through the Riemann curvature tensor, in terms of the $e^p(\alpha)$; we find

$$\epsilon \mathbf{g} = o(\alpha, \alpha\gamma)\, o(\beta, \beta\gamma) + o(\alpha, \beta\gamma)\, o(\beta, \alpha\gamma).$$

The "cosmological term" $\mathbf{g}'$ is given by $\epsilon\mathbf{g}' = 1$.

If gravitation be represented by $\mathbf{g}$, one can, as is well known, add to the material + electromagnetic energy $\mathbf{t}_p^q$ a gravitational energy $\mathbf{v}_p^q$ in such a way that the sum satisfies a true conservation law.[14] Designating the total differential of $\mathbf{g}$ considered as a function of $e^p(\alpha)$ and $e_q^p(\alpha) = \frac{\partial e^p(\alpha)}{\partial x_q}$ by

$$\delta \mathbf{g} = \mathbf{g}_p(\alpha)\delta e^p(\alpha) + \mathbf{g}_p^q(\alpha)\delta e_q^p(\alpha),$$

then

$$-\mathbf{v}_p^q = \mathbf{g}_r^q(\alpha)\frac{\partial e^{r(\alpha)}}{\partial x_p} - \delta_p^q \mathbf{g}.$$

We obtain thus an invariant constant mass m which must be one of the characteristic universal constants of Nature.

Doubts, Prospects.—(11) is to be interpreted as the law of conservation of the number (or charge) of electrons and protons. Therefore we ascribe ψ^+ and ψ^- to the electron and to the proton, respectively. Taking $\mathbf{f}$ as the electromagnetic part of the action, which seems plausible to me, we then obtain Maxwell's equations in the sense that the proton generates the field φ^+ and the electron φ^-; whereas in accordance with the equations of matter φ^+ will effect only the electron and φ^- the proton. This is not as obstruse as it may sound; on the contrary, the previous theory leads to entirely false results if the potential due to the electron, which at large distances neutralizes that due to the nucleus, reacts on the electron itself, as Schrödinger has pointed out with emphasis.[15] It may indeed seem queer that ψ^+ and ψ^- are here equally permissible, since we know that positive and negative electricity are fundamentally different—that protons and electrons have different mass. But if we neglect the gravitational and electrical energy in comparison with the material the mass m falls into two parts m^+ and m^- which are, however, not strictly constant. It is possible that m^+ and m^- are different if our equations admit two classes of solutions which are interchanged on transition from right- to left-handed axes—as in the Dirac theory, the spherically symmetric hydrogen problem admits several solutions for the normal state which are not themselves spherically symmetric but which are transformed among themselves by rotation.

As far as we know $\mathbf{m}$, $\mathbf{g}$ and one of the two quantities $\mathbf{f}$ or $\mathbf{f}'$ are indispensable for the explanation of the phenomena. I am inclined to believe that the action is composed additively of $\mathbf{m}$, $\mathbf{f}$ and $\mathbf{g}$.

It should be noted that our field equations contain neither the theory of a single electron nor that of a single proton. One might rather consider them as the laws governing a hydrogen atom consisting of an electron and a proton; but here again, the problem of interaction between the two may first require quantization. What we have obtained is solely a field scheme which can only be applied to and compared with experience after the quantization has been accomplished. We know from the Pauli exclusion principle[16] what commutation rules are to be applied in the quantization of ψ^+; those for ψ^- must be the same in our theory. The commutation relations between ψ^+ and ψ^- are as yet entirely unknown. Those of the electromagnetic field (photons) are almost completely known. In this respect we know nothing concerning the gravitational field. The commutation rules for F are here almost completely fixed by those for ψ by the condition that these latter be unaltered when ψ is given the increment $\delta\psi = iF(\alpha)\psi$. That the rules thus obtained are in agreement with experience is indeed a support for our theory; i.e., it tells us why the "anti-symmetric," Pauli-Fermi statistics for electrons leads to the "symmetric" Bose-Einstein statistics for photons. A definite decision can, however, first be reached when the barrier which hems the progress of quantum theory is overcome: the quantization of the field equations.

[1] I am indebted to Prof. H. P. Robertson for the translation of my German manuscript.

[2] I employ the same notation as in my book *Gruppentheorie und Quantenmechanik*, Leipzig, **1928** (cited as *G Q*) except that I here write $\psi_1^+ \psi_2^+$, $\psi_1^- \psi_2^-$ in place of $\psi_1 \psi_2$, $\psi_3 \psi_4$. Cf. in particular § 25, 39, 44. $\frac{h}{2\pi}$ is Planck's constant.

[3] Cf. H. Weyl, *Space—Time—Matter*, London, **1922** (cited as *S T M*), § 33.

[4] *G Q*, pp. 199, 200.

[5] The circumflex indicates transition to the conjugate of the transposed matrix (Hermitean conjugate). The four components of ψ are considered as the elements of a matrix with four rows and one column.

[6] *G Q*, p. 88.

[7] Attempts to employ only the tensor calculus have been made by Tetrode (*Z. Physik*, **50**, 336 (1928)); J. M. Whittaker (*Proc. Camb. Phil. Soc.*, **25**, 501 (1928)), and others; I consider them misleading.

[8] *Sitzungsber. Berl. Akad.*, **1928**, pp. 217, 224; **1929.**

[9] Cf. the analogous considerations in *S T M*, pp. 233–237.

[10] Capital Latin letters (except P for point) denote linear transformations of the four components of ψ.

[11] In *G Q*, loc. cit., they are denoted by s'_α.

[12] It is to be borne in mind that under the influence of a proper Lorentz transformation the ψ^+ components — as well as the ψ^- — are transformed among themselves. Only when the improper operations of the Lorentz group, the reflection

$$\mathbf{e}(0) \rightarrow \mathbf{e}(0), \quad \mathbf{e}(\alpha) \rightarrow -\mathbf{e}(\alpha) \; [\alpha = 1, 2, 3],$$

is taken into account is it necessary to use both pairs of components together.

[13] *S T M*, p. 289.

[14] The derivation in *S T M*, pp. 269, 270, can be adapted to the new analytic formulation of the gravitational field.

[15] E. Schrödinger, *Ann. Physik*, **82**, 265–272 (1927); in particular p. 270.

[16] P. Jordan and E. Wigner, *Z. Physik*, **47**, 1928, 631; G Q, § 44.

Correction made on the proofs (March 4, (1929)).—The calculation of the action density **m** contains an error which should be corrected as follows. The spacial components $\alpha = 1,2,3$ of $l(\alpha)$ are pure imaginary and the temporal component $l(0)$ real, not the opposite as I had assumed. In the definition of **m**′ we must therefore divide the right-hand side by $2i$. But this has as consequence that the $\mathbf{H} = \mathbf{m} + \mathbf{m}'$ obtained from the calculation is practically real and not composed of a real and an imaginary part. We therefore obtain but *one* invariant action density for matter: $\mathbf{m} + \mathbf{m}'$. To the tensor density of energy arising from **m** must naturally be added the term arising from **m**′.

RAMAN SPECTRA OF SOLUTIONS OF SOME IONIZED SUBSTANCES

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A new type of secondary radiation has recently been found[1] to be present in the light scattered by liquids. If a medium is illuminated with a monochromatic radiation for which it is largely transparent, the scattered light is found to contain in addition to light of the original frequency, lines of altered frequency which do not arise from fluorescence. A characteristic of this phenomenon, the Raman Effect, is that the changes in frequency on scattering are independent of the frequency of the incident radiation. The changes in frequency are, however, specifically dependent on the nature of the scattering substance and, indeed, are often found to agree with previously known infra-red frequencies of the substance. There is little doubt that the frequency changes on scattering measure differences in the energy contents of stationary states of the molecules of the scattering substance.

The change in frequency accompanying the scattering of light has been measured for the case of the crystal calcite, $CaCO_3$, by Wood,[2] and by Landsberg and Mandelstam.[3] Landsberg and Mandelstam found strong scattered lines corresponding to the frequency change $\Delta\nu = 1095$ cm.$^{-1}$, and somewhat weaker ones corresponding to $\Delta\nu = 291$ cm.$^{-1}$; Wood reported the values 1087.0 and 280.2 for these shifts and in addition a faint shift with[4] $\Delta\nu = 730.2$ cm.$^{-1}$. In the light scattered by an aqueous solution of sodium nitrate, $NaNO_3$, Carrelli, Pringsheim and Rosen,[5] found shifted lines having $\Delta\nu = 1044$ cm.$^{-1}$.

Water has been found by various observers to give no sharp lines in the spectrum of its scattered light. The line given by sodium nitrate solution,