Supporting Derivation

General Cartesian Form for a Logarithmic Spiral Parametrized by Arc-Length. The logarithmic spiral is standardly expressed in polar coordinates as:

$$r = \alpha e^{\beta\theta}$$

However, this form is not useful for our purposes because it describes a logarithmic spiral whose pole (point of infinite curvature) is at the origin. We require a form that is general enough to describe a spiral "starting" at an arbitrary initial position (x_0, y_0), with initial tangent direction ϕ_0, initial curvature κ_0, and initial rate of change of curvature γ_0 at that point. We derive such a form in the complex Cartesian plane.

We begin with the following Cesáro equation, expressing curvature in terms of arc length:

$$\kappa(s) = \frac{1}{bs + a}$$

The values a and b are chosen to ensure that the initial curvature and rate of curvature are κ_0 and γ_0, respectively. Specifically:

$$a = \frac{1}{\kappa_0} \qquad \text{and} \qquad b = -\frac{\gamma_0}{\kappa_0^{\;2}}$$

The orientation function is then given by:

$$\begin{aligned}\phi(s) &= \phi_0 + \int_0^s \frac{dt}{bt + a} \\ &= \phi_0 + \frac{1}{b}[\log(bs + a) - \log(a)]\end{aligned}$$

and the position $\mathrm{E}(s) = x(s) + iy(s)$ is given by:

$$\begin{aligned}
\mathrm{E}(s) &= \mathrm{E}(0) + \int_0^s \exp i\left(\phi_0 + \frac{1}{b}[\log(bt+a) - \log(a)]\right)dt \\
&= \mathrm{E}(0) + \exp i\left(\phi_0 - \frac{1}{b}\log(a)\right)\left[\int_0^s \cos\left(\frac{\log(bt+a)}{b}\right)dt + i\int_0^s \sin\left(\frac{\log(bt+a)}{b}\right)dt\right] \\
&= \mathrm{E}(0) + \exp i\left(\phi_0 - \frac{1}{b}\log(a)\right)[I_1 + iI_2]
\end{aligned}$$

In order to obtain the general expression, we need to evaluate the two integrals I_1 and I_2.

Setting $u = \frac{1}{b}\log(bt+a)$, we obtain:

$$I_1 = \int_0^s \cos\left(\frac{\log(bt+a)}{b}\right)dt = \int_{u_1}^{u_2} e^{bu}\cos(u)du$$

where $u_1 = \frac{1}{b}\log(a)$ and $u_2 = \frac{1}{b}\log(bs+a)$. Using integration by parts,

$$\begin{aligned}
I_1 &= e^{bu}\sin(u)\Big|_{u_1}^{u_2} - b\int_{u_1}^{u_2} e^{bu}\sin(u)du \\
&= e^{bu}\sin(u)\Big|_{u_1}^{u_2} - b\left[-e^{bu}\cos(u)\Big|_{u_1}^{u_2} + b\int_{u_1}^{u_2} e^{bu}\cos(u)du\right] \\
&= e^{bu}\sin(u)\Big|_{u_1}^{u_2} + be^{bu}\cos(u)\Big|_{u_1}^{u_2} - b^2 I_1
\end{aligned}$$

and, therefore,

$$\begin{aligned}
I_1 &= \frac{1}{1+b^2}e^{bu}[\sin(u) + b\cos(u)]\Big|_{u_1}^{u_2} \\
&= \frac{bs+a}{1+b^2}\left[\sin\left(\frac{\log(bs+a)}{b}\right) + b\cos\left(\frac{\log(bs+a)}{b}\right)\right] - \frac{a}{1+b^2}\left[\sin\left(\frac{\log(a)}{b}\right) + b\cos\left(\frac{\log(a)}{b}\right)\right]
\end{aligned}$$

Similarly,

$$I_2 = \int_0^s \sin\left(\frac{\log(bt+a)}{b}\right)dt = \int_{u_1}^{u_2} e^{bu}\sin(u)du$$

where $u_1 = \frac{1}{b}\log(a)$ and $u_2 = \frac{1}{b}\log(bs+a)$. Using integration by parts as before,

$$\begin{aligned}
I_2 &= -e^{bu}\cos(u)\Big|_{u_1}^{u_2} + b\int_{u_1}^{u_2} e^{bu}\cos(u)du \\
&= -e^{bu}\cos(u)\Big|_{u_1}^{u_2} + b\left[e^{bu}\sin(u)\Big|_{u_1}^{u_2} - b\int_{u_1}^{u_2} e^{bu}\sin(u)du\right] \\
&= -e^{bu}\cos(u)\Big|_{u_1}^{u_2} + be^{bu}\sin(u)\Big|_{u_1}^{u_2} - b^2 I_2
\end{aligned}$$

and, therefore,

$$\begin{aligned}
I_2 &= \frac{1}{1+b^2}e^{bu}\left[b\sin(u) - \cos(u)\right]\Big|_{u_1}^{u_2} \\
&= \frac{bs+a}{1+b^2}\left[b\sin\left(\frac{\log(bs+a)}{b}\right) - \cos\left(\frac{\log(bs+a)}{b}\right)\right] - \frac{a}{1+b^2}\left[b\sin\left(\frac{\log(a)}{b}\right) - \cos\left(\frac{\log(a)}{b}\right)\right].
\end{aligned}$$