Supporting Information

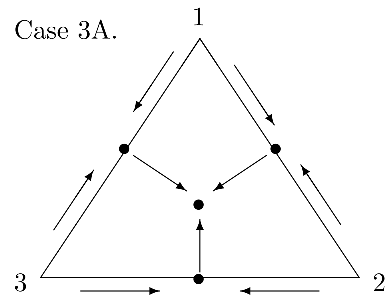
S1. Simulation algorithm

The algorithm is a variant of a technique called uniformization, see e.g., Example 4.6 in [1], which can reduce a continuous time Markov chain to discrete time. To explain this procedure, let $q(i,j)$ be the rate for jumps from i to j, $\lambda_i = \sum_{j\neq i} q(i,j)$, and $\Lambda = \max_i \lambda_i$. If the chain is in state i at the nth step of the simulation, then $X_{n+1} = j$ with probability $q(i,j)/\Lambda$ if $j \neq i$ and $X_{n+1} = i$ with probability $1 - \lambda_i/\Lambda$. Since some transitions do not result in state changes this is inefficient, but this has the advantage that the times between jumps are exponential with rate Λ, so there is no need to create the exponential random variables. If n is large then the elapsed time after n simulation steps $T_n \sim n/\Lambda$, where $a_n \sim b_n$ means $a_n/b_n \to 1$.

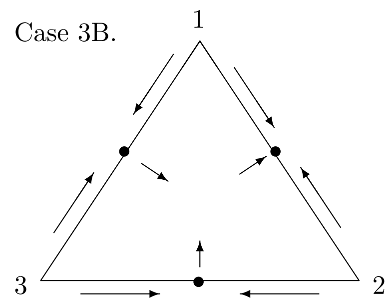
The simulation method adapts easily to interacting particle systems and to evolutionary games in particular. Let $c_{ij}(x,\xi)$ be the rate at which site x changes from i to j when the configuration is ξ, let $\lambda_i(x,\xi) = \sum_{j\neq i} c_{ij}(x,\xi)$ and let $\Lambda = \max_{i,x} \lambda_i(x,\xi)$. On each simulation step we pick a site x at random. If it is in state i it changes to j with probability $c_{ij}(x,\xi)/\Lambda$ and does not change with probability $1 - \lambda_i(x,\xi)/\Lambda$. If there are N sites then the time until the next site tries to change is a minimum of N exponential(Λ) random variables, and hence exponential($N\Lambda$) Thus if n is large the elapsed time after n simulation steps $T_n \sim n/(N\Lambda)$.

S2. Classification of 3 by 3 games

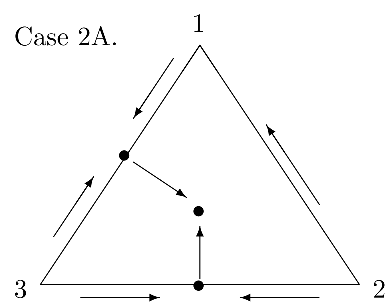
Here we describe the division of generic 3×3 games without unstable edge fixed points into 11 cases. The number in the name of each case gives the number of stable edge fixed points. Cases are further subdivided according to the number of edge fixed points that can be invaded, i.e., the freqeuncy of the third strategy will increase when rare. Whether a fixed point is invadable or not is indicated by the arrows next to the fixed points. On the other edges without fixed points, arrows give the direction of the dominance relations. Proofs of the statements we make about the behavior of the replicator equation can be found in Section 7 of [**?**].



In Case 3A, all three edge equilibria can be invaded. The replicator equation converges to the interior fixed point and it was shown in [2] that there is coexistence in the spatial game when selection is weak.

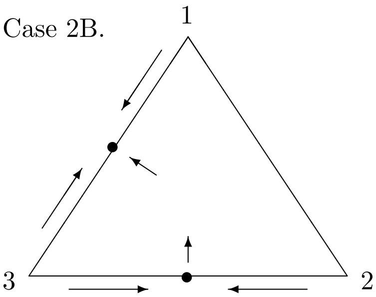


In case 3B, two of the three edge fixed points can be invaded. The replicator equation converges to the equilibrium on the 1, 2 edge, which we call $e_{1,2}$. It is impossible to have three stable edge fixed points and only 1 or 0 of them invadable.



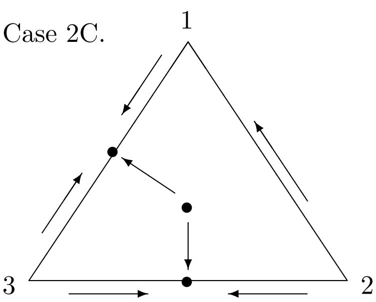
In case 2A, both edge equilibria can be invaded. The replicator equation converges to the interior fixed point and it was shown in [2] that there is coexistence in the spatial model.

Case 2B.



In case 2B, one edge fixed point can be invaded. The replicator equation converges to $e_{1,3}$. There is no arrow on the $1, 2$ edge because it is not important in which direction it points.

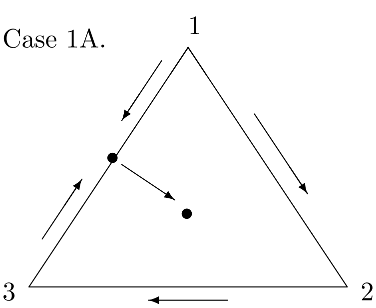
Case 2C.



In case 2C, neither edge fixed point can be invaded, so there is bistability, i.e., $e_{1,2}$ and $e_{1,3}$ are both locally stable.

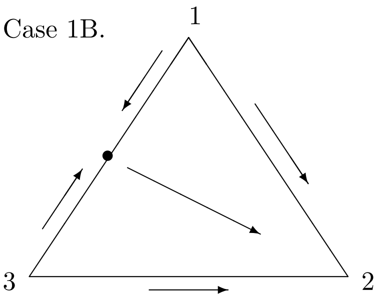
Next consider the situation in which there is one stable fixed point on the boundary. In first two cases it can be invaded.

Case 1A.



In case 1A, the pure strategy 2 can be invaded. The replicator equation converges to the interior fixed point. It was shown in [2] that there is coexistence in the spatial game when selection is weak.

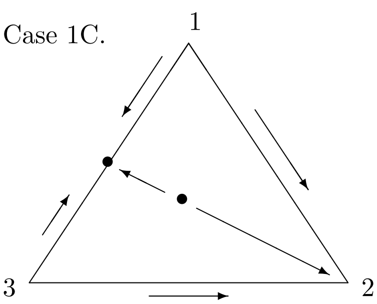
Case 1B.



In case 1B, the pure strategy 2 cannot be invaded. The replicator equation converges to the pure strategy 2.

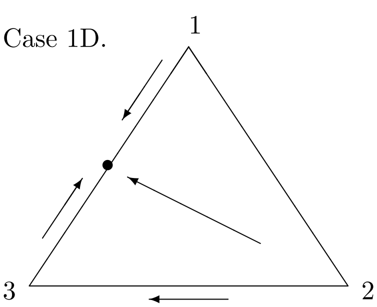
In the next two cases, there is one boundary fixed point and it cannot be invaded.

Case 1C.



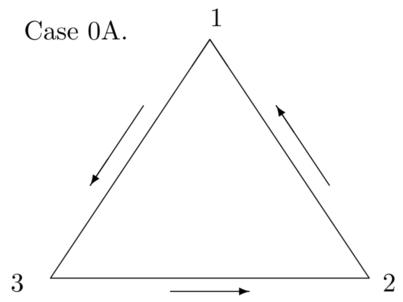
In case 1C the interior equilibrium is bistable.

Case 1D.

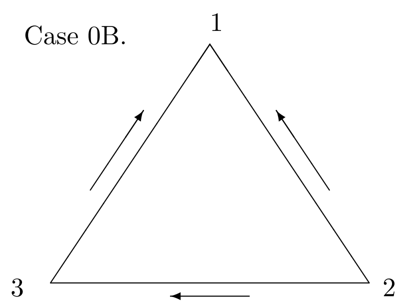


In case 1D the replicator equation converges to $e_{1,3}$ There is no arrow on the $1, 2$ edge because the result does not depend on the direction it points.

Finally we have the situation with no boundary fixed points. There are 8 possible orientations for the arrows on the edges. Two lead to rock-paper-scissor relationships between the strategies.



In the other six combinations, two arrows point toward the same pure strategy equilibrium and that is the limit in the replicator equation.



S3. Correlation length

For concreteness consider the Ising model. Let $\Lambda(L) = \{-L, -L+1, \dots L\}^2$ and for each $\xi : \Lambda_L \to \{-1, -1\}$ define

$$\mu(\omega) = \frac{1}{Z(L)} \exp\left(\beta \sum_{x \sim y} \xi_x \xi_y\right)$$

where $x \sim y$ means x and y are nearest neighbors and $Z(L)$ is a normalizing constant to make p a probability measure on $\{-1,1\}^{\Lambda(L)}$. It is a well known fact from statistical mechanics that one can let $L \to \infty$ to define probability measures on configurations $\omega : \mathbb{Z}^2 \to \{-1,1\}$. When $\beta < \beta_c$ there is only one limit that has exponentially decaying correlations. That is if we let

$$\begin{aligned} \mathrm{cov}\,(\xi(x), \xi(y)) = P(\xi(x) = 1, \xi(y) = 1) \\ - P(\xi(x) = 1) P(\xi(y) = 1) \end{aligned}$$

which is ≥ 0 by the FKG inequality then as $n \to \infty$

$$1/n \log \mathrm{cov}\,(\xi(0,0), \xi(n,0)) = -\gamma(\beta).$$

The inverse of this exponential decay rate $\xi(\beta) = 1/\gamma(\beta)$ is the correlation length. Spins that are separated by one correlation length have covariance $\approx e^{-1}$ and hence have a tendency to be aligned. However, if we look at the fraction p_L of 1 spins in a box of side L which is much larger than the correlation length then the variance of p_L will be small and this will be close to its mean 1/2. In the stochastic Ising model, boxes that are the same size as the correlation length the feequency of 1's at time t, $p_L(t)$ will show fluctuations over time due to correlations, but when the length is much larger than the correlation length $p_L(t)$ wil stay approximately constant over time. This phenomenon is best understood in the well studied Ising model but this is a general property of stochastic spatial models. For more information see Chpater 10 of [**?**].

S4. Analysis of the Multiple myeloma game.

Boundary equilibria. To study the properties of the game we begin with the two strategy games it contains.

1 vs. 2. $(A/(A+E), E/(A+E))$ is a mixed strategy equilibrium. Since $A, E > 0$ it is attracting (on the $1,2$ edge).

1 vs. 3. $(B/(B+C), C/(B+C))$ is a mixed strategy equilibrium. Since $B, C > 0$, it is attracting (on the $1,3$ edge).

2 vs. 3. 3 dominates 2.

Invadability. The next step is to determine when the third strategy will increase when rare if the other two are equilibrium.

In the $1,2$ equilibrium, fitnesses $F_1 = F_2 = AE/(A+E)$ while $F_3 = (CA+FE)/(A+E)$ so 3 can invade 1,2 (which we write as $3 \to 1,2$) if $CA + FE > AE$ or $C/E > 1 - F/A$.

In the $1,3$ equilibrium, the fitnesses $F_1 = F_3 = BC/(B+C)$, while $F_2 = (EB-DC)/(B+C)$, so 2 can invade 1,3 if $EB - DC > BC$ or $1 - DC/BE > C/E$.

Case 1. $C/E > 1 - F/A$. $3 \to 1,2$ but $2 \not\to 1,3$ so the replicator converges to the 1,3 edge fixed point.

Case 2. $1 - F/A > C/E > 1 - DC/BE$ $3 \not\to 1,2$ but $2 \not\to 1,3$ so we have bistability.

Case 3. $1 - DC/BE > C/E$. $3 \not\to 1,2$ but $2 \to 1,3$ so the replicator converges to the 1,2 edge fixed point.

S5. Convergence to boundary fixed points

Coexistence has been proved in cases 3A, 2A, 1A. Rock-paper scissors and bistable cases were considered in the main paper. Here, will give simulations for the cases in which there is convergence to a boundary fixed point: 3B, 2B, 1B, 1D, 0B. In each case we give the original game matrix G and the transformed matrix H. The invadability conditions are as previously drawn for that case.

G_1	**1**	**2**	**3**
1	0	2.5	3.25
2	3.5	0	2.5
3	1.75	1.5	0

H_1	**1**	**2**	**3**
1	0	2	4
2	4	0	3
3	1	1	0

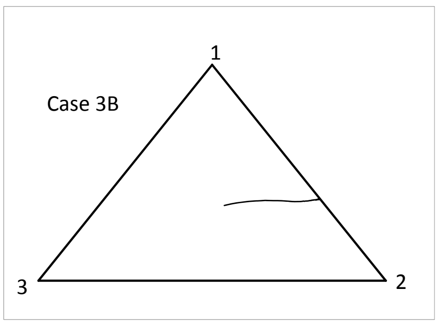


Figure 1: The replicator equation for $H_1 \to (1/3, 2/3, 0)$. In the spatial game G_1 frequencies $\to (0.378, 0.622, 0)$.

G_2	**1**	**2**	**3**
1	0	1.25	3.25
2	0.25	0	2.5
3	1.75	1.5	0

H_2	**1**	**2**	**3**
1	0	2	4
2	-1	0	3
3	1	1	0

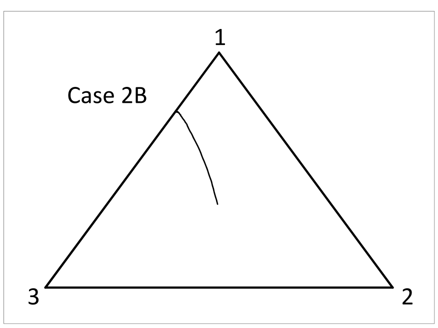


Figure 2: The replicator equation for $H_2 \to (3/4, 0, 1/4)$. In the spatial game $1 + (1/2.25)G_2$ frequencies $\to (0.7506, 0, 0.2494)$.

G_3	**1**	**2**	**3**
1	0	-1	3.25
2	1	0	0.5
3	1.75	-0.5	0

H_3	**1**	**2**	**3**
1	0	-2	4
2	2	0	1
3	1	-1	0

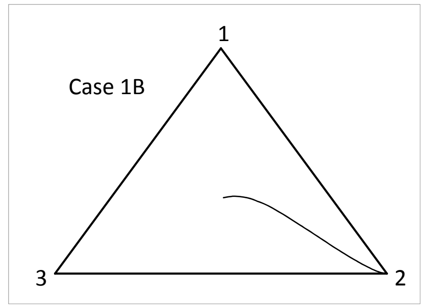


Figure 3: The replicator equation for H_3 and the frequencies in the spatial game $1 + (1/3)G_1 \to (0, 1, 0)$.

G_4	**1**	**2**	**3**
1	0	1.25	3.25
2	-0.25	0	0.125
3	2.5	-0.625	0

H_4	**1**	**2**	**3**
1	0	2	4
2	-1	0	-2
3	-2	1	0

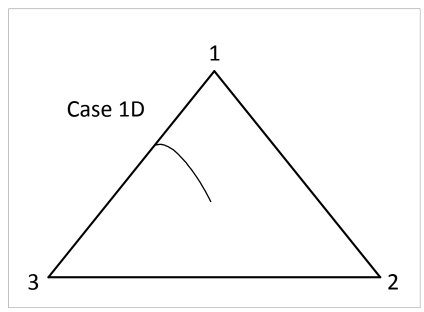


Figure 4: The replicator equation for $H_4 \to (2/3, 0, 1/3)$. In the spatial game $1 + (1/2.625)G_4$ frequencies $\to (0.636, 0, 0.364)$.

G_5	**1**	**2**	**3**
1	0	1.25	3.25
2	-0.25	0	-1.25
3	-0.5	0.25	0

H_5	**1**	**2**	**3**
1	0	2	4
2	-1	0	-2
3	2	1	0

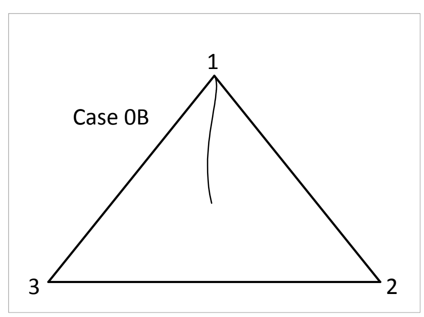


Figure 5: The replicator equation for H_5 and the frequencies in the spatial game $1 + (1/3.25)G_1 \to (1, 0, 0)$.

References

[1] Durrett, R (2012) *Essentials of Stochastic Processes.* 2nd Edition Springer, New York

[2] Durrett R (2014) Spatial evolutionary games with small selection coefficients. *Electronic J. Probability.* volume 19 (2014), paper

[3] Sethna, JP (2006) *Statistical Mechanics: Entropy, Order Parameters, and Complexity.* Oxford University Press.