Web-based Supplementary Materials for "A general framework for the regression analysis of pooled biomarker assessments"

Yan Liu[1], Christopher S. McMahan[1*], and Colin M. Gallagher[1]

[1]Department of Mathematical Sciences, Clemson University, Clemson, SC 29634, U.S.A.

*email: mcmaha2@clemson.edu

Appendix A: Implementation Details

Herein further details are provided regarding the sampling strategies that were used to implement the proposed methodology. In particular, the sampling strategies tailored to implement the proposed methodology, both in the presence and absence of measurement error, for the models considered in the simulation study and data application are provided below. Further, a technique for specifying $\boldsymbol{\theta}^*$ is provided. Note, following the presented development one can easily derive similar sampling strategies for other parametric models.

A.1: Non-Error Laden Observations

Consider the situation in which the pooled biomarker levels are measured without error; i.e., $\widetilde{Y}_{p_j}$ is directly observed. In this scenario and under the assumptions discussed in Section 2 of the corresponding manuscript, the probability density function of the observed biomarker measurement for the jth pool can be expressed as

$$f_j(\widetilde{y}_{p_j}|\mathbf{x}_j,\boldsymbol{\theta}) = \int\cdots\int c_j f\left(c_j\widetilde{Y}_{p_j} - \sum_{i=2}^{c_j}\widetilde{y}_{ij}\middle|\mathbf{x}_{1j},\boldsymbol{\theta}\right)\frac{h_j(\widetilde{\boldsymbol{y}}_{(-1)j}|\mathbf{x}_j,\boldsymbol{\theta})}{h_j^*(\widetilde{\boldsymbol{y}}_{(-1)j}|\mathbf{x}_j,\boldsymbol{\theta}^*)}h_j^*(\widetilde{\boldsymbol{y}}_{(-1)j}|\mathbf{x}_j,\boldsymbol{\theta}^*)d\widetilde{\boldsymbol{y}}_{(-1)j},$$

where $h_j(\widetilde{\boldsymbol{y}}_{(-1)j}|\mathbf{x}_j,\boldsymbol{\theta}) = \prod_{i=2}^{c_j} f(\widetilde{y}_{ij}|\mathbf{x}_{ij},\boldsymbol{\theta})$ and $h_j^*(\widetilde{\boldsymbol{y}}_{(-1)j}|\mathbf{x}_j,\boldsymbol{\theta}^*)$ is an importance distribution, whose support corresponds to $h_j(\widetilde{\boldsymbol{y}}_{(-1)j}|\mathbf{x}_j,\boldsymbol{\theta})$, from which the samples $\widetilde{\boldsymbol{Y}}^m_{(-1)j} = (\widetilde{Y}^m_{2j},\ldots,\widetilde{Y}^m_{c_jj})'$, for $m = 1,...,M$, can be drawn. When the individual level biomarker levels are assumed to follow a distribution whose support consists of the whole real line, then one can specify

$h_j^*(\widetilde{\boldsymbol{y}}_{(-1)j}|\mathbf{x}_j, \boldsymbol{\theta}^*) = h_j(\widetilde{\boldsymbol{y}}_{(-1)j}|\mathbf{x}_j, \boldsymbol{\theta}^*)$. Alternatively, if the individual level biomarker levels are assumed to follow a distribution with strictly positive support, then the support of the importance distribution, for the reasons of computational efficiency, should be constrained such that $\sum_{i=2}^{c_j} \widetilde{y}_{ij} \leq c_j \widetilde{Y}_{p_j}$, for $j = 1, \ldots, J$. Note, in all of the expressions above, $\boldsymbol{\theta}^*$ is a predetermined value of $\boldsymbol{\theta}$ and can be specified via the technique described below.

Under the assumption that the individual biomarker levels, $\widetilde{Y}_{ij}$, follow a distribution whose support is the entire real line (e.g., the normal distribution or t-distribution), it is suggested that one specify $h_j^*(\widetilde{\boldsymbol{y}}_{(-1)j}|\mathbf{x}_j, \boldsymbol{\theta}^*) = h_j(\widetilde{\boldsymbol{y}}_{(-1)j}|\mathbf{x}_j, \boldsymbol{\theta}^*)$, for the purposes of obtaining the MCMLE according to the methodology described in Section 2 of the corresponding manuscript. It is worth while to point out that under the assumption of normality an analytical expression for the MLE can be obtained, see the discussion provided in Web Appendix D below.

For parametric models having strictly positive support, the following technique can be employed to efficiently sample $\widetilde{\boldsymbol{Y}}_{(-1)j}^m = (\widetilde{Y}_{2j}^m, \ldots, \widetilde{Y}_{c_j j}^m)'$, for $m = 1, ..., M$. This technique was also considered in Frigyik, Kapila, and Gupta (2010) and Mitchell et al. (2014). Consider drawing Z_{ij}^m independently from $f(\widetilde{y}_{ij}|\mathbf{x}_{ij}, \boldsymbol{\theta}^*)$, for $i = 1, ..., c_j$. Then take $\widetilde{\boldsymbol{Y}}_{(-1)j}^m = (\widetilde{Y}_{2j}^m, \ldots, \widetilde{Y}_{c_j j}^m)'$, where $\widetilde{Y}_{ij}^m = Z_{ij}^m (c_j \widetilde{Y}_{p_j}) / \sum_{i=1}^{c_j} Z_{ij}^m$, for $i = 2, ..., c_j$. Proceeding in this fashion guarantees that $c_j \widetilde{Y}_{p_j} > \sum_{i=2}^{c_j} \widetilde{Y}_{ij}^m$. Note, based on the distribution of $\boldsymbol{Z}_j^m = (Z_{1j}^m, ..., Z_{c_j j}^m)'$ one can derive the resulting distribution of the $\widetilde{\boldsymbol{Y}}_{(-1)j}^m$. Typically, the distribution of $\widetilde{\boldsymbol{Y}}_{(-1)j}^m$ does not have an inviting form which is easily sampled from, but the aforementioned process can be implemented in order to draw the Monte Carlo samples. The form of the importance distribution is only needed to appropriately calculate the weights in the Monte Carlo approximation of the log-likelihood.

The importance distribution, from which $\widetilde{\boldsymbol{Y}}_{(-1)j}^m$ is being sampled, for the log-normal and gamma regression models can be expressed as

$$h_j^*(\widetilde{\boldsymbol{Y}}_{(-1)j}^m|\mathbf{x}_j, \boldsymbol{\theta}^*) = c_j \widetilde{Y}_{p_j} \sqrt{\frac{2\pi\sigma^{*2}}{c_j}} \exp\left[\frac{\left\{\sum_{i=1}^{c_j}(\log \widetilde{Y}_{ij}^m - \mathbf{x}_{ij}'\boldsymbol{\beta}^*)\right\}^2}{2c_j\sigma^{*2}}\right] \prod_{i=1}^{c_j} f(\widetilde{Y}_{ij}^m \mid \mathbf{x}_{ij}, \boldsymbol{\theta}^*),$$

and

$$h_j^*(\widetilde{\boldsymbol{Y}}_{(-1)j}^m|\mathbf{x}_j,\boldsymbol{\theta}^*) = c_j\widetilde{Y}_{p_j}\frac{\Gamma(c_ja)}{\left(\sum_{i=1}^{c_j} b_{ij}\widetilde{Y}_{ij}^m\right)^{c_ja}}\prod_{i=1}^{c_j}\frac{(b_{ij})^a}{\Gamma(a)}(\widetilde{Y}_{ij}^m)^{a-1},$$

respectively, where $\widetilde{Y}_{1j}^m = c_j\widetilde{Y}_{p_j} - \sum_{i=2}^{c_j}\widetilde{Y}_{ij}^m$, $a = 1/\sigma^{*2}$, $b_{ij} = a\exp\left(-\mathbf{x}_{ij}^{'}\boldsymbol{\beta}^*\right)$ under the log link, and $b_{ij} = a\mathbf{x}_{ij}^{'}\boldsymbol{\beta}^*$ under the inverse link.

A.2: Error Laden Observations

Consider the scenario in which the pooled biomarker assessments are subject to additive measurement error; i.e., $Y_{p_j} = \widetilde{Y}_{p_j} + \varepsilon_{p_j}$, where $\varepsilon_{p_j} \sim f_\varepsilon$. It is assumed that f_ε is known. In practice, if the distribution of the measurement error is unknown then f_ε can be replaced by an estimate, see Section 5 of the corresponding manuscript for a discussion of how this can be done. Under these assumptions, the probability density function of Y_{p_j} can be expressed as

$$g_j(Y_{p_j}|\mathbf{x}_j,\boldsymbol{\theta}) = \int\cdots\int c_jf\left(c_jY_{p_j} - \sum_{i=2}^{c_j}\widetilde{y}_{ij} - c_j\varepsilon_{p_j}\middle|\mathbf{x}_{1j},\boldsymbol{\theta}\right)h_j(\widetilde{\boldsymbol{y}}_{(-1)j},\varepsilon_{p_j}|\mathbf{x}_j,\boldsymbol{\theta})d\widetilde{\boldsymbol{y}}_{(-1)j}d\varepsilon_{p_j},$$

where $h_j(\widetilde{\boldsymbol{y}}_{(-1)j},\varepsilon_{p_j}|\mathbf{x}_j,\boldsymbol{\theta}) = f_\varepsilon(\varepsilon_{p_j})\prod_{i=2}^{c_j} f(\widetilde{y}_{ij}|\mathbf{x}_{ij},\boldsymbol{\theta})$. Again the goal is to develop an importance distribution $h_j^*(\widetilde{\boldsymbol{y}}_{(-1)j},\varepsilon_{p_j}|\mathbf{x}_j,\boldsymbol{\theta}^*)$ from which the samples $\widetilde{\boldsymbol{Y}}_{(-1)j}^m = (\widetilde{Y}_{2j}^m,\ldots,\widetilde{Y}_{c_jj}^m)'$ and $\varepsilon_{p_j}^m$, for $m = 1,...,M$, can be drawn. Further, in all of the expressions above, $\boldsymbol{\theta}^*$ is a predetermined value of $\boldsymbol{\theta}$ and can be specified via the technique described below.

Under the assumption that the individual biomarker levels, $\widetilde{Y}_{ij}$, follow a distribution whose support is the entire real line (e.g., the normal distribution or t-distribution), it is suggested that one specify $h_j^*(\widetilde{\boldsymbol{y}}_{(-1)j},\varepsilon_{p_j}|\mathbf{x}_j,\boldsymbol{\theta}^*) = h_j(\widetilde{\boldsymbol{y}}_{(-1)j},\varepsilon_{p_j}|\mathbf{x}_j,\boldsymbol{\theta}^*)$, for the purposes of obtaining the MCMLE according to the methodology described in Section 2 of the corresponding manuscript. It is worth while to point out that under the assumption of normality an analytical expression for the MLE can be obtained when a common pool size is being utilized (i.e., $c_j = c$ for all j), see the discussion provided in Web Appendix D below.

For parametric models having strictly positive support, the following technique can be employed to efficiently sample $\widetilde{\boldsymbol{Y}}_{(-1)j}^m = (\widetilde{Y}_{2j}^m,\ldots,\widetilde{Y}_{c_jj}^m)'$ and $\varepsilon_{p_j}^m$, for $m = 1,...,M$. First independently sample $\varepsilon_{p_j}^m$ from $f_\varepsilon(\varepsilon)I(Y_{p_j} > \varepsilon)$ and Z_{ij}^m from $f(\widetilde{y}_{ij}|\mathbf{x}_{ij},\boldsymbol{\theta}^*)$, for $i = 1,...,c_j$. Using these samples then construct $\widetilde{\boldsymbol{Y}}_{(-1)j}^m = (\widetilde{Y}_{2j}^m,\ldots,\widetilde{Y}_{c_jj}^m)'$, where $\widetilde{Y}_{ij}^m = Z_{ij}^mc_j\left(Y_{p_j} - \varepsilon_{p_j}^m\right)/\sum_{i=1}^{c_j} Z_{ij}^m$.

Proceeding in this fashion guarantees that $c_jY_{p_j} > \sum_{i=2}^{c_j}\widetilde{Y}_{ij}^m + c_j\varepsilon_{p_j}^m$ for all j and m. The density function of the importance distribution can subsequently be expressed as

$$h_j^*(\widetilde{\boldsymbol{Y}}_{(-1)j}^m, \varepsilon_{p_j}^m|\mathbf{x}_j, \boldsymbol{\theta}^*) = h_j^*(\widetilde{\boldsymbol{Y}}_{(-1)j}^m|\mathbf{x}_j, \boldsymbol{\theta}^*)f_\varepsilon(\varepsilon_{p_j}^m)I(Y_{p_j} > \varepsilon_{p_j}^m), \tag{1}$$

where $h_j^*(\widetilde{\boldsymbol{Y}}_{(-1)j}^m|\mathbf{x}_j, \boldsymbol{\theta}^*)$ is provided in Section A.1 for both the log-normal and gamma regression models, with the unobserved $\widetilde{Y}_{p_j}$ being replaced by $Y_{p_j} - \varepsilon_{p_j}^m$.

A.3: Choosing $\boldsymbol{\theta}^*$

In order to efficiently implement the proposed methodology one must specify an appropriate initial value $\boldsymbol{\theta}^*$ for the importance distributions developed in Sections A.1 and A.2. Herein a technique for determining an appropriate initial value of $\boldsymbol{\theta}^*$ is proposed. This approach was utilized for all of the simulation studies and the data analysis presented in the corresponding manuscript. Further, this approach can be implemented for the case in which the observed observations are subject to or free of measurement error, and as such the discussion is focused on the latter scenario (i.e., $\widetilde{Y}_{p_j}$ are observed). In the case that the observed data consists of the error laden measurements (i.e., Y_{p_j}) one should simply replace $\widetilde{Y}_{p_j}$ by Y_{p_j} in the following expressions in order to determine an appropriate initial value of $\boldsymbol{\theta}^*$. Additionally, the techniques for specifying $\boldsymbol{\theta}^*$ are dependent on the underlying distributional assumptions, and are as such presented for the parametric models considered in the corresponding manuscript. Similar techniques could be developed for other parametric models.

Normal regression model: Under this model the individual biomarker concentrations are assumed to conditionally, given the covariates, follow a normal distribution; i.e. $\widetilde{Y}_{ij} \sim N(\mathbf{x}_{ij}'\boldsymbol{\beta}, \sigma^2)$. Consequently, under the assumptions presented in the corresponding manuscript, the biomarker concentration of the jth pool also follows a normal distribution; i.e., $\widetilde{Y}_{p_j} \sim N(\overline{\mathbf{x}}_j'\boldsymbol{\beta}, \sigma^2/c_j)$, where $\sum_{i=1}^{c_j}\mathbf{x}_{ij}/c_j = \overline{\mathbf{x}}_j$. Thus, $\boldsymbol{\beta}$ and σ can be estimated using the techniques outlined in Web Appendix D. These estimators provide an initial value which can be used to implement the proposed methodology.

Log-normal regression model: Under this model the individual biomarker concentrations are assumed to conditionally, given the covariates, follow a log-normal distribution. Consequently, the method presented in Section 4 of Mitchell et al. (2014) is recommended to determine $\boldsymbol{\theta}^*$.

Shifted T regression model: Under this model the individual biomarker concentrations are assumed to conditionally, given the covariates, follow a shifted t-distribution. For this regression model, one has that

$$E(\widetilde{Y}_{ij}|\mathbf{x}_{ij}) = \mathbf{x}_{ij}'\boldsymbol{\beta} \text{ and } \mathrm{Var}(\widetilde{Y}_{ij}|\mathbf{x}_{ij}) = \frac{\nu}{\nu-2}.$$

Thus, the mean and variance of the pooled observations can be written as

$$E(\widetilde{Y}_{p_j}|\mathbf{x}_j) = \frac{1}{c_j}\sum_{i=1}^{c_j}\mathbf{x}_{ij}'\boldsymbol{\beta} = \overline{\mathbf{x}}_j'\boldsymbol{\beta} \text{ and } \mathrm{Var}\left(\widetilde{Y}_{p_j}\middle|\mathbf{x}_j\right) = \frac{\nu}{c_j(\nu-2)}$$

where $\sum_{i=1}^{c_j}\mathbf{x}_{ij}/c_j = \overline{\mathbf{x}}_j$. In order to identify a good initial value $\boldsymbol{\theta}^* = (\boldsymbol{\beta}^{*\prime}, \nu^*)'$ for the proposed methodology, the following approach was considered. Obtain $\boldsymbol{\beta}^*$ as the ordinary least squares estimator

$$\boldsymbol{\beta}^* = \arg\min_{\boldsymbol{\beta}} \sum_{j=1}^{J}\left(\widetilde{Y}_{p_j} - \overline{\mathbf{x}}_j'\boldsymbol{\beta}\right)^2.$$

and ν^* as the value which satisfies

$$\frac{\nu^*}{\nu^*-2} = \frac{\sum_{j=1}^{J} c_j\left(\widetilde{Y}_{p_j} - \overline{\mathbf{x}}_j'\boldsymbol{\beta}^*\right)^2}{J-p-1}.$$

Gamma regression model: Under this model the individual biomarker concentrations are assumed to conditionally, given the covariates, follow a gamma distribution. For this regression model, one has that

$$E(\widetilde{Y}_{ij}|\mathbf{x}_{ij}) = \eta^{-1}(\mathbf{x}_{ij}'\boldsymbol{\beta}) \quad \text{and} \quad \frac{\sqrt{\mathrm{Var}(\widetilde{Y}_{ij}|\mathbf{x}_{ij})}}{E(\widetilde{Y}_{ij}|\mathbf{x}_{ij})} = \sigma,$$

where $\eta(\cdot)$ is the link function, which is most commonly specified to be either the log or inverse link, and σ is the Coefficent of Variation. Based on these characteristics, one has that the variance of the pooled response can be expressed as

$$\text{Var}\left(\widetilde{Y}_{p_j}\middle|\mathbf{x}_j\right) = \frac{1}{c_j^2}\sum_{i=1}^{c_j} Var\left(\widetilde{Y}_{ij}\middle|\mathbf{x}_{ij}\right) = \frac{\sigma^2}{c_j^2}\sum_{i=1}^{c_j}\left\{\eta^{-1}\left(\mathbf{x}_{ij}'\boldsymbol{\beta}\right)\right\}^2.$$

After algebraic manipulating the above expression, one then has that

$$\text{Var}\left(\frac{\widetilde{Y}_{p_j}}{d_j}\middle|\mathbf{x}_j\right) = \sigma^2, \tag{2}$$

where

$$d_j = \sqrt{\frac{1}{c_j^2}\sum_{i=1}^{c_j}\left[\eta^{-1}\left(\mathbf{x}_{ij}'\boldsymbol{\beta}\right)\right]^2}.$$

In order to identify a good initial value $\boldsymbol{\theta}^* = (\boldsymbol{\beta}^{*\prime}, \sigma^*)'$ for the proposed methodology, the following approach was considered. First, consider approximating the expected value of $\widetilde{Y}_{ij}$ through a first order Taylor series expansion about a point $\boldsymbol{\beta}^0 = (\beta_0^0, ..., \beta_p^0)'$; i.e.,

$$E\left[\widetilde{Y}_{ij}\middle|\mathbf{x}_{ij}\right] \;=\; \eta^{-1}(\mathbf{x}_{ij}'\boldsymbol{\beta}) \approx \eta^{-1}\left(\mathbf{x}_{ij}'\boldsymbol{\beta}^0\right) + \sum_{l=0}^{p}\dot{\eta}^{-1}(\mathbf{x}_{ij}'\boldsymbol{\beta}^0)\left(\beta_l - \beta_l^0\right)x_{ij,l},$$

where $\dot{\eta}^{-1}(u_0) = \frac{\partial\eta^{-1}(u)}{\partial u}|_{u=u_0}$. Based on this first order approximation, one can also approximate the expected value of the $\widetilde{Y}_{p_j}$ as

$$E\left[\widetilde{Y}_{p_j}\middle|\mathbf{x}_{ij}\right] \approx \frac{1}{c_j}\sum_{i=1}^{c_j}\eta^{-1}\left(\mathbf{x}_{ij}'\boldsymbol{\beta}^0\right) + \frac{1}{c_j}\sum_{l=0}^{p}\sum_{i=1}^{c_j}\dot{\eta}^{-1}(\mathbf{x}_{ij}'\boldsymbol{\beta}^0)\left(\beta_l - \beta_l^0\right)x_{ij,l} = \alpha_j^0 + \overline{\mathbf{x}}_j^{0\prime}\boldsymbol{\beta},$$

where

$$\begin{aligned}\overline{\mathbf{x}}_j^0 &= \frac{1}{c_j}\sum_{i=1}^{c_j}\dot{\eta}^{-1}(\mathbf{x}_{ij}'\boldsymbol{\beta}^0)\mathbf{x}_{ij},\\ \alpha_j^0 &= \frac{1}{c_j}\sum_{i=1}^{c_j}\eta^{-1}\left(\mathbf{x}_{ij}'\boldsymbol{\beta}^0\right) - \overline{\mathbf{x}}_j^{0\prime}\boldsymbol{\beta}^0.\end{aligned}$$

Consequently, iteratively reweighted least squares is implemented to obtain $\boldsymbol{\beta}^*$. In particular the following steps can be used to obtain $\boldsymbol{\beta}^*$:

1) Choose an initial value of $\boldsymbol{\beta}^{(0)}$ and set $t = 0$.

2) Compute $\overline{\mathbf{x}}_j^t$, $\alpha_j^{(t)}$, and $d_j^{(t)}$, where

$$\begin{aligned}
\overline{\mathbf{x}}_j^{(t)} &= \frac{1}{c_j}\sum_{i=1}^{c_j} \dot{\eta}^{-1}(\mathbf{x}_{ij}'\boldsymbol{\beta}^{(t)})\mathbf{x}_{ij}, \\
\alpha_j^{(t)} &= \frac{1}{c_j}\sum_{i=1}^{c_j} \eta^{-1}\left(\mathbf{x}_{ij}'\boldsymbol{\beta}^{(t)}\right) - \overline{\mathbf{x}}_j^{(t)\prime}\boldsymbol{\beta}^{(t)}, \\
d_j^{(t)} &= \sqrt{\frac{1}{c_j^2}\sum_{i=1}^{c_j}\left[\eta^{-1}\left(\mathbf{x}_{ij}'\boldsymbol{\beta}^{(t)}\right)\right]^2}.
\end{aligned}$$

3) Obtain $\boldsymbol{\beta}^{(t+1)} = \left(\mathbf{X}^{(t)\prime}\mathbf{D}^{(t)}\mathbf{X}^{(t)}\right)^{-1}\mathbf{X}^{(t)\prime}\mathbf{D}^{(t)}\boldsymbol{Y}^{(t)}$, where

$$\begin{aligned}
\mathbf{X}^{(t)} &= \left(\overline{\mathbf{x}}_1^{(t)}, \cdots, \overline{\mathbf{x}}_J^{(t)}\right)', \\
\boldsymbol{Y}^{(t)} &= \left(\widetilde{Y}_{p_1} - \alpha_1^{(t)}, \cdots, \widetilde{Y}_{p_J} - \alpha_J^{(t)}\right)',
\end{aligned}$$

and $\mathbf{D}^{(t)}$ is a diagonal matrix with diagonal elements $\mathrm{D}_{jj}^{(t)} = (d_j^{(t)})^{-2}$ for $j = 1, \ldots, J$.

4) Set $t = t + 1$ an return to step 2).

Steps 2)-4) are iterated until convergence (i.e., $||\boldsymbol{\beta}^{(t+1)} - \boldsymbol{\beta}^{(t)}||$ is less than some specified tolerance) and $\boldsymbol{\beta}^*$ is taken to be $\boldsymbol{\beta}^{(t+1)}$ at the point of convergence. In the simulation section of the corresponding manuscript, the initial value was specified to be $\boldsymbol{\beta}^0 = (1, 0, \ldots, 0)'$ and a convergence tolerance of 10^{-4} was utilized.

Further, based on (2), a natural initial value of the Coefficent of Variation is given by

$$\sigma^* = \sqrt{\frac{1}{J-(p+1)}\sum_{j=1}^{J}\left(\frac{\widetilde{Y}_{p_j} - \mu_{p_j}^*}{d_j^*}\right)^2},$$

where

$$\begin{aligned}
\mu_{p_j}^* &= \frac{1}{c_j}\sum_{i=1}^{c_j}\eta^{-1}\left(\mathbf{x}_{ij}'\boldsymbol{\beta}^*\right), \\
d_j^* &= \sqrt{\frac{1}{c_j^2}\sum_{i=1}^{c_j}\left\{\eta^{-1}\left(\mathbf{x}_{ij}'\boldsymbol{\beta}^*\right)\right\}^2}.
\end{aligned}$$

Thus, for the gamma regression model the proposed methodology can be implemented after setting $\boldsymbol{\theta}^* = (\boldsymbol{\beta}^{*\prime}, \sigma^*)'$.

Appendix B: Regularity Conditions

Herein the regularity conditions under which the asymptotic properties of the proposed methodology, which are presented in Section 3 of the corresponding manuscript, can be established are provided:

i) *The parameter space* $\boldsymbol{\Theta}$ *is a compact subset of* $\mathbb{R}^k$.

ii) $f_\varepsilon(Y_{p_j}|\widetilde{Y}_{p_j})w_j(\widetilde{\boldsymbol{Y}}_j, \mathbf{x}_j, \boldsymbol{\theta}, \boldsymbol{\theta}^*)$ *is continuous in* $\boldsymbol{\theta}$ *for all* $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, *for* $j = 1, ..., J$.

iii) $E\left[\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} f_\varepsilon(Y_{p_j}|\widetilde{Y}_{p_j})w_j(\widetilde{\boldsymbol{Y}}_j, \mathbf{x}_j, \boldsymbol{\theta}, \boldsymbol{\theta}^*)\middle| Y_{p_j}, \boldsymbol{\theta}^*\right] < \infty$, *for* $j = 1, ..., J$.

iv) *The maximum likelihood estimate* $\widehat{\boldsymbol{\theta}}$ *is unique, in the interior of* $\boldsymbol{\Theta}$.

v) $\widehat{\boldsymbol{\theta}}_M \xrightarrow{P} \widehat{\boldsymbol{\theta}}$ *as* $M \to \infty$.

vi) $g_j(Y_{p_j}|\mathbf{x}_j, \boldsymbol{\theta})$ *can be differentiated twice with respect to* $\boldsymbol{\theta}$ *under the integral sign.*

vii) $\mathbf{B} = -\nabla^2 l(\widehat{\boldsymbol{\theta}})$ *is positive definite.*

viii) $\nabla^3 l_M\left(\boldsymbol{\theta}|\boldsymbol{Y}_p, \mathbf{x}, \boldsymbol{\theta}^*\right)$ *is bounded in probability uniformly in a neighborhood of* $\widehat{\boldsymbol{\theta}}$.

All of the aforementioned conditions are easy to verify with the exception of *iii)*, and as such Lemma 1, established in Web Appendix C, provides sufficient conditions for verifying *iii)* based on the assumed model for the individual level data.

Appendix C: Asymptotic Properties

C.1: A proof of Lemma 1

Lemma 1: *To establish regularity condition iii), it is sufficient to show that* $f_\varepsilon(y|\widetilde{y})$ *is uniformly bounded* $\forall\ y$ *and* $\widetilde{y}$, *and that*

$$\int \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} f\left(y|\mathbf{x},\boldsymbol{\theta}\right)dy < \infty,\ \forall\ \mathbf{x}.$$

Proof: Recall $w_j(\widetilde{\boldsymbol{y}}_j,\mathbf{x}_j,\boldsymbol{\theta},\boldsymbol{\theta}^*) = h_j(\widetilde{\boldsymbol{y}}_j|\mathbf{x}_j,\boldsymbol{\theta})/h_j^*(\widetilde{\boldsymbol{y}}_j|\mathbf{x}_j,\boldsymbol{\theta}^*)$, thus

$$\begin{aligned} E\left[\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} f_\varepsilon(Y_{p_j}|\widetilde{y}_{p_j})w_j(\widetilde{\boldsymbol{y}}_j,\mathbf{x}_j,\boldsymbol{\theta},\boldsymbol{\theta}^*)\middle|Y_{p_j},\boldsymbol{\theta}^*\right] &= \int \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} f_\varepsilon(Y_{p_j}|\widetilde{y}_{p_j})h_j(\widetilde{\boldsymbol{y}}_j|\mathbf{x}_j,\boldsymbol{\theta})d\widetilde{\boldsymbol{y}}_j \\ &= \int \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} f_\varepsilon(Y_{p_j}|\widetilde{y}_{p_j})\prod_{i=1}^{c_j} f(\widetilde{y}_{ij}|\mathbf{x}_{ij},\boldsymbol{\theta})d\widetilde{\boldsymbol{y}}_j. \end{aligned}$$

Under the assumption that $f_\varepsilon(y|\widetilde{y})$ is uniformly bounded $\forall\ y$ and $\widetilde{y}$, let D_3 denote the upper bound, then

$$E\left[\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} f_\varepsilon(Y_{p_j}|\widetilde{Y}_{p_j})w_j(\widetilde{\boldsymbol{Y}}_j,\mathbf{x}_j,\boldsymbol{\theta},\boldsymbol{\theta}^*)\middle|Y_{p_j},\boldsymbol{\theta}^*\right] \leq D_3\prod_{i=1}^{c_j}\left\{\int \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} f(\widetilde{y}_{ij}|\mathbf{x}_{ij},\boldsymbol{\theta})d\widetilde{y}_{ij}\right\} < \infty$$

C.2: A sketch of the proof of Theorem 2

Theorem 2: *Under regularity conditions iv)-viii), as* $M\to\infty$, *then*

$$\sqrt{M}\nabla l_M(\widehat{\boldsymbol{\theta}}|\boldsymbol{Y}_p,\mathbf{x},\boldsymbol{\theta}^*) \overset{d}{\to} N(\mathbf{0},\mathbf{A}), \tag{3}$$

$$-\nabla^2 l_M(\widehat{\boldsymbol{\theta}}_M|\boldsymbol{Y}_p,\mathbf{x},\boldsymbol{\theta}^*) \overset{p}{\to} \mathbf{B},$$

$$\sqrt{M}(\widehat{\boldsymbol{\theta}}_M-\widehat{\boldsymbol{\theta}}|\boldsymbol{\theta}^*) \overset{d}{\to} N(\mathbf{0},\boldsymbol{\Sigma}), \tag{4}$$

where $\boldsymbol{\Sigma}=\mathbf{B}^{-1}\mathbf{A}\mathbf{B}^{-1}$ *and* $\mathbf{B}=-\nabla^2 l(\widehat{\boldsymbol{\theta}}|\boldsymbol{Y}_p,\mathbf{x})$.

Proof: The Taylor series expansion of $\nabla l_M(\boldsymbol{\theta}|\boldsymbol{Y}_p,\mathbf{x},\boldsymbol{\theta}^*)$, about $\widehat{\boldsymbol{\theta}}_M$, provides

$$\nabla l_M(\widehat{\boldsymbol{\theta}}|\boldsymbol{Y}_p,\mathbf{x},\boldsymbol{\theta}^*) = \nabla l_M(\widehat{\boldsymbol{\theta}}_M|\boldsymbol{Y}_p,\mathbf{x},\boldsymbol{\theta}^*) + \nabla^2 l_M(\widehat{\boldsymbol{\theta}}_M|\boldsymbol{Y}_p,\mathbf{x},\boldsymbol{\theta}^*)(\widehat{\boldsymbol{\theta}}-\widehat{\boldsymbol{\theta}}_M) + \mathbf{R}_M,$$

where $\nabla l_M(\widehat{\boldsymbol{\theta}}_M|\boldsymbol{Y}_p,\mathbf{x},\boldsymbol{\theta}^*)=\mathbf{0}$. Under the regularity conditions, provided above, it is easy to establish that $\sqrt{M}\mathbf{R}_M \overset{p}{\to} \mathbf{0}$ and that $\widehat{\mathbf{B}}(\widehat{\boldsymbol{\theta}}_M|\boldsymbol{\theta}^*) = -\nabla^2 l_M(\widehat{\boldsymbol{\theta}}_M|\boldsymbol{Y}_p,\mathbf{x},\boldsymbol{\theta}^*) \overset{p}{\to} \mathbf{B}$; i.e., $\widehat{\mathbf{B}}(\widehat{\boldsymbol{\theta}}_M|\boldsymbol{\theta}^*)$

is a consistent estimator of $\mathbf{B}$. Consequently, for M sufficiently large it is expected that $\widehat{\mathbf{B}}(\widehat{\boldsymbol{\theta}}_M|\boldsymbol{\theta}^*)$ is nonsingular so that upon rearrangement one has

$$\sqrt{M}(\widehat{\boldsymbol{\theta}}_M - \widehat{\boldsymbol{\theta}}) = \left[\widehat{\mathbf{B}}(\widehat{\boldsymbol{\theta}}_M|\boldsymbol{\theta}^*)\right]^{-1} \sqrt{M}\nabla l_M(\widehat{\boldsymbol{\theta}}|\mathbf{Y}_p, \mathbf{x}, \boldsymbol{\theta}^*) - \sqrt{M}\mathbf{R}_M^*, \tag{5}$$

where $\sqrt{M}\mathbf{R}_M^* \xrightarrow{p} \mathbf{0}$. Therefore, establishing (4) is tantamount to establishing (3). Note,

$$\nabla l_M(\boldsymbol{\theta}|\mathbf{Y}_p, \mathbf{x}, \boldsymbol{\theta}^*) = \left(\frac{\partial l_M(\boldsymbol{\theta}|\mathbf{Y}_p, \mathbf{x}, \boldsymbol{\theta}^*)}{\partial \theta_1}, \ldots, \frac{\partial l_M(\boldsymbol{\theta}|\mathbf{Y}_p, \mathbf{x}, \boldsymbol{\theta}^*)}{\partial \theta_k}\right)',$$

where, for $r = 1, \ldots, k$,

$$\frac{\partial l_M(\boldsymbol{\theta}|\mathbf{Y}_p, \mathbf{x}, \boldsymbol{\theta}^*)}{\partial \theta_r} = \sum_{j=1}^{J} \left\{ \frac{\frac{\partial}{\partial \theta_r} g_j^M(Y_{p_j}|\mathbf{x}_j, \boldsymbol{\theta}, \boldsymbol{\theta}^*)}{g_j^M(Y_{p_j}|\mathbf{x}_j, \boldsymbol{\theta}, \boldsymbol{\theta}^*)} \right\},$$

where $g_j^M(Y_{p_j}|\mathbf{x}_j, \boldsymbol{\theta}, \boldsymbol{\theta}^*) = M^{-1} \sum_{m=1}^{M} f_\varepsilon(Y_{p_j}|\widetilde{\mathbf{Y}}_j^m) w_j(\widetilde{\mathbf{Y}}_j^m, \mathbf{x}_j, \boldsymbol{\theta}, \boldsymbol{\theta}^*)$. Thus, by the Weak Law of Large Numbers, as $M \to \infty$, then

$$\begin{aligned}
\frac{\partial}{\partial \theta_r} g_j^M(Y_{p_j}|\mathbf{x}_j, \boldsymbol{\theta}, \boldsymbol{\theta}^*) &\xrightarrow{p} E\left[\frac{\partial}{\partial \theta_r} f_\varepsilon(Y_{p_j}|\widetilde{\mathbf{Y}}_j^m) w_j(\widetilde{\mathbf{Y}}_j^m, \mathbf{x}_j, \boldsymbol{\theta}, \boldsymbol{\theta}^*)\right] = \frac{\partial}{\partial \theta_r} g_j(Y_{p_j}|\mathbf{x}_j, \boldsymbol{\theta}),\\
g_j^M(Y_{p_j}|\mathbf{x}_j, \boldsymbol{\theta}, \boldsymbol{\theta}^*) &\xrightarrow{p} E\left[f_\varepsilon(Y_{p_j}|\widetilde{\mathbf{Y}}_j^m) w_j(\widetilde{\mathbf{Y}}_j^m, \mathbf{x}_j, \boldsymbol{\theta}, \boldsymbol{\theta}^*)\right] = g_j(Y_{p_j}|\mathbf{x}_j, \boldsymbol{\theta}),
\end{aligned}$$

for any fixed value of $\boldsymbol{\theta}$. Further, by the Central Limit Theorem, one has that

$$\sqrt{M}\{\overline{\mathbf{D}}^M(\boldsymbol{\theta}) - \mathbf{D}(\boldsymbol{\theta})\} \xrightarrow{d} N\{\mathbf{0}, \mathbf{W}(\boldsymbol{\theta}|\boldsymbol{\theta}^*)\},$$

where $\overline{\mathbf{D}}^M(\boldsymbol{\theta}) = M^{-1}\sum_{m=1}^{M} \mathbf{D}^m(\boldsymbol{\theta})$, $\mathbf{D}^m(\boldsymbol{\theta}) = \{\mathbf{D}_1^m(\boldsymbol{\theta}), ..., \mathbf{D}_J^m(\boldsymbol{\theta})\}'$, $\mathbf{D}(\boldsymbol{\theta}) = \{\mathbf{D}_1(\boldsymbol{\theta}), ..., \mathbf{D}_J(\boldsymbol{\theta})\}'$,

$$\mathbf{D}_j^m(\boldsymbol{\theta})' = \begin{pmatrix} \frac{\partial}{\partial \theta_1} f_\varepsilon(Y_{p_j}|\widetilde{\mathbf{Y}}_j^m) w_j(\widetilde{\mathbf{Y}}_j^m, \mathbf{x}_j, \boldsymbol{\theta}, \boldsymbol{\theta}^*) \\ \vdots \\ \frac{\partial}{\partial \theta_k} f_\varepsilon(Y_{p_j}|\widetilde{\mathbf{Y}}_j^m) w_j(\widetilde{\mathbf{Y}}_j^m, \mathbf{x}_j, \boldsymbol{\theta}, \boldsymbol{\theta}^*) \\ f_\varepsilon(Y_{p_j}|\widetilde{\mathbf{Y}}_j^m) w_j(\widetilde{\mathbf{Y}}_j^m, \mathbf{x}_j, \boldsymbol{\theta}, \boldsymbol{\theta}^*) \end{pmatrix}, \quad \mathbf{D}_j(\boldsymbol{\theta})' = \begin{pmatrix} \frac{\partial}{\partial \theta_1} g_j(Y_{p_j}|\mathbf{x}_j, \boldsymbol{\theta}) \\ \vdots \\ \frac{\partial}{\partial \theta_k} g_j(Y_{p_j}|\mathbf{x}_j, \boldsymbol{\theta}) \\ g_j(Y_{p_j}|\mathbf{x}_j, \boldsymbol{\theta}) \end{pmatrix},$$

and $\mathbf{W}(\boldsymbol{\theta}|\boldsymbol{\theta}^*) = \mathrm{Cov}\{\mathbf{D}^1(\boldsymbol{\theta})\}$. Thus, by an application of the delta method one can obtain that

$$\sqrt{M}\{\nabla l_M(\boldsymbol{\theta}|\mathbf{Y}_p, \mathbf{x}, \boldsymbol{\theta}^*) - \nabla l(\boldsymbol{\theta}|\mathbf{Y}_p, \mathbf{x})\} \xrightarrow{d} N\{\mathbf{0}, \mathbf{U}(\boldsymbol{\theta})\mathbf{W}(\boldsymbol{\theta}|\boldsymbol{\theta}^*)\mathbf{U}(\boldsymbol{\theta})'\},$$

for any fixed value of $\boldsymbol{\theta}$, where $\mathbf{U}(\boldsymbol{\theta})$ is a $k \times J(k+1)$ matrix, whose elements are given by

$$\mathbf{U}(\boldsymbol{\theta})_{r,s} = \begin{cases} \{g_j(Y_{p_j}|\mathbf{x}_j, \boldsymbol{\theta})\}^{-1} & \text{if} \quad s = (j-1)(k+1) + r, \\ -\{\frac{\partial}{\partial \theta_r} g_j(Y_{p_j}|\mathbf{x}_j, \boldsymbol{\theta})\}\{g_j(Y_{p_j}|\mathbf{x}_j, \boldsymbol{\theta})\}^{-2} & \text{if} \quad s = j(k+1), \\ 0 & \text{o.w.} \end{cases} \tag{6}$$

Consequently, taking $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}$, one obtains the desired result; i.e.,

$$\sqrt{M}\nabla l_M(\widehat{\boldsymbol{\theta}}|\mathbf{Y}_p, \mathbf{x}, \boldsymbol{\theta}^*) \xrightarrow{d} N(\mathbf{0}, \mathbf{A}),$$

where $\mathbf{A} = \mathbf{U}(\widehat{\boldsymbol{\theta}})\mathbf{W}(\widehat{\boldsymbol{\theta}}|\boldsymbol{\theta}^*)\mathbf{U}(\widehat{\boldsymbol{\theta}})'\}$. Note $\nabla l(\widehat{\boldsymbol{\theta}}|\mathbf{Y}_p, \mathbf{x}) = \mathbf{0}$ since $\widehat{\boldsymbol{\theta}}$ is the maximum likelihood estimate. Combining this result with (5) and by an application of Slutsky's theorem yields the desired result; i.e., as $M \to \infty$,

$$\sqrt{M}(\widehat{\boldsymbol{\theta}}_M - \widehat{\boldsymbol{\theta}}|\boldsymbol{\theta}^*) \xrightarrow{d} N(\mathbf{0}, \boldsymbol{\Sigma}),$$

where $\boldsymbol{\Sigma} = \mathbf{B}^{-1}\mathbf{A}\mathbf{B}^{-1}$.

C.3: Estimating B and A

From the results discussed in the above section, one has that $\widehat{\mathbf{B}}(\widehat{\boldsymbol{\theta}}_M|\boldsymbol{\theta}^*)$ is a consistent estimator of $\mathbf{B}$ and is easily computed based on the Monte Carlo sample $\widetilde{\boldsymbol{Y}}_j^1, \ldots, \widetilde{\boldsymbol{Y}}_j^M$ from the importance distribution $h_j^*(\widetilde{\boldsymbol{Y}}_j|\mathbf{x}_j, \boldsymbol{\theta}^*)$, for $j = 1, ..., J$. Similarly, consider $\mathbf{U}^M(\boldsymbol{\theta})$, a $k \times J(k+1)$ matrix, whose elements are given by

$$\mathbf{U}^M(\boldsymbol{\theta})_{r,s} = \begin{cases} \{g_j^M(Y_{p_j}|\mathbf{x}_j, \boldsymbol{\theta}, \boldsymbol{\theta}^*)\}^{-1} & \text{if} \quad s = (j-1)(k+1) + r, \\ -\{\frac{\partial}{\partial \theta_r} g_j^M(Y_{p_j}|\mathbf{x}_j, \boldsymbol{\theta}, \boldsymbol{\theta}^*)\}\{g_j^M(Y_{p_j}|\mathbf{x}_j, \boldsymbol{\theta}, \boldsymbol{\theta}^*)\}^{-2} & \text{if} \quad s = j(k+1), \\ 0 & \text{o.w.} \end{cases}$$

Note, that for a fixed value of $\boldsymbol{\theta}$, $\mathbf{U}^M(\boldsymbol{\theta}) \xrightarrow{p} \mathbf{U}(\boldsymbol{\theta})$. Consequently, a natural estimator of $\mathbf{U}(\widehat{\boldsymbol{\theta}})$ is given by $\mathbf{U}^M(\widehat{\boldsymbol{\theta}}_M)$, since $\widehat{\boldsymbol{\theta}}_M \xrightarrow{p} \widehat{\boldsymbol{\theta}}$. Again, for a fixed value of $\boldsymbol{\theta}$, the proposed estimator of $\mathbf{W}(\boldsymbol{\theta}|\boldsymbol{\theta}^*)$ is given by

$$\mathbf{W}^M(\boldsymbol{\theta}|\boldsymbol{\theta}^*) = (M-1)^{-1}\{\mathbf{D}(\theta) - \overline{\mathbf{D}}^M(\theta)\mathbf{1}_M'\}'\{\mathbf{D}(\theta) - \overline{\mathbf{D}}^M(\theta)\mathbf{1}_M'\}$$

where $\mathbf{D}(\theta) = [\mathbf{D}^1(\theta), ..., \mathbf{D}^M(\theta)]$ is a $J(k+1) \times M$ matrix and $\mathbf{1}_M$ is an $M \times 1$ vector of all ones, and note $\mathbf{W}^M(\boldsymbol{\theta}|\boldsymbol{\theta}^*) \xrightarrow{p} \mathbf{W}(\boldsymbol{\theta}|\boldsymbol{\theta}^*)$. Thus, a natural estimator of $\mathbf{W}(\widehat{\boldsymbol{\theta}}|\boldsymbol{\theta}^*)$ is $\mathbf{W}^M(\widehat{\boldsymbol{\theta}}_M|\boldsymbol{\theta}^*)$. Consequently, an estimator of $\mathbf{A}$ is given by $\mathbf{A}(\widehat{\boldsymbol{\theta}}_M|\boldsymbol{\theta}^*) = \mathbf{U}^M(\widehat{\boldsymbol{\theta}}_M)\mathbf{W}(\widehat{\boldsymbol{\theta}}_M|\boldsymbol{\theta}^*)\mathbf{U}(\widehat{\boldsymbol{\theta}}_M)'$.

Appendix D: Analytic Expressions For The MLEs Under Normality

Under the assumption that the biomarker levels of the individuals obey the normal errors regression model (i.e., $\widetilde{Y}_{ij}|\mathbf{x}_{ij} \sim N(\mu_{ij}, \sigma^2)$, where $\mu_{ij} = \mathbf{x}_{ij}'\boldsymbol{\beta}$, $\mathbf{x}_{ij}$ is a vector of covariates for the ith individual in the jth pool, and $\boldsymbol{\beta}$ is the corresponding vector of regression parameters) the MLE of $\boldsymbol{\theta}_0 = (\boldsymbol{\beta}_0', \sigma_0^2)'$ can be expressed in closed form based on $\widetilde{Y}_{p_j}$ or Y_{p_j} when a common pool size is utilized (i.e., $c_j = c$ for all j), where it is assumed that Y_{p_j} is subject to normal additive measurement error; i.e., $Y_{p_j} = \widetilde{Y}_{p_j} + \varepsilon_{p_j}$, where $\varepsilon_{p_j} \overset{iid}{\sim} N(0, \tau^2)$. For the scenario in which the observed observations are free of measurement error the MLE of $\boldsymbol{\theta}_0$ is given by

$$\widehat{\boldsymbol{\beta}} = (\overline{\mathbf{X}}'\overline{\mathbf{X}})^{-1}\overline{\mathbf{X}}'\widetilde{\boldsymbol{Y}}_p \quad \text{and} \quad \widehat{\sigma}^2 = cJ^{-1}(\widetilde{\boldsymbol{Y}}_p - \overline{\mathbf{X}}\widehat{\boldsymbol{\beta}})'(\widetilde{\boldsymbol{Y}}_p - \overline{\mathbf{X}}\widehat{\boldsymbol{\beta}}),$$

where $\overline{\mathbf{X}} = (\overline{\mathbf{x}}_1, \ldots, \overline{\mathbf{x}}_J)'$, $\overline{\mathbf{x}}_j = c_j^{-1}\sum_{i=1}^{c_j}\mathbf{x}_{ij}$, and $\widetilde{\boldsymbol{Y}}_p = (\widetilde{Y}_{p_1}, \ldots, \widetilde{Y}_{p_J})'$. Similarly, the MLE of $\boldsymbol{\theta}_0$ in the presence of additive measurement error is given by

$$\widehat{\boldsymbol{\beta}} = (\overline{\mathbf{X}}'\overline{\mathbf{X}})^{-1}\overline{\mathbf{X}}'\mathbf{Y}_p \quad \text{and} \quad \widehat{\sigma}^2 = cJ^{-1}(\mathbf{Y}_p - \overline{\mathbf{X}}\widehat{\boldsymbol{\beta}})'(\mathbf{Y}_p - \overline{\mathbf{X}}\widehat{\boldsymbol{\beta}}) - c\tau^2,$$

where $\mathbf{Y}_p = (Y_{p_1}, ..., Y_{p_J})'$. Note, it is assumed that the distribution of the measurement error is known (i.e., τ^2 is known a priori). In practice, this quantity would have to be estimated, see Section 5 of the corresponding manuscript for a discussion of how this can be accomplished. Further, the above closed form expressions for the MLE can easily be extended to allow for the effect of multiple pool sizes (i.e., $c_j \neq c_{j'}$ for $j \neq j'$) when the observed data are free of measurement error and are given by

$$\widehat{\boldsymbol{\beta}} = (\overline{\mathbf{X}}'\mathbf{C}\overline{\mathbf{X}})^{-1}\overline{\mathbf{X}}'\mathbf{C}\widetilde{\boldsymbol{Y}}_p \quad \text{and} \quad \widehat{\sigma}^2 = J^{-1}(\widetilde{\boldsymbol{Y}}_p - \overline{\mathbf{X}}\widehat{\boldsymbol{\beta}})'\mathbf{C}(\widetilde{\boldsymbol{Y}}_p - \overline{\mathbf{X}}\widehat{\boldsymbol{\beta}}), \tag{7}$$

where $\mathbf{C}$ is a $J \times J$ diagonal matrix whose jth diagonal element is c_j. Analogous expressions for error laden observations do not appear to exist in closed form.

References

Mitchell, E., Lyles, R., Manatunga, A., Danaher, M., Perkins, N., and Schisterman, E. (2014). Regression for skewed biomarker outcomes subject to pooling. *Biometrics* **70,** 202–211.

Frigyik, B., Kapila, A., and Gupta, M. (2010). Introduction to the dirichlet distribution and related processes. UWEE Technical Report UWEETR-2010-0006. Department of Electrical Engineering, University of Washington, Seattle, WA.

Carroll, R., Ruppert, D., Stefanski, L., and Crainiceanu, C. (2006). *Measurement Error in Nonlinear Models: A Modern Perspective*, 2nd edition. London: Chapmen & Hall.

Web Table 1: Simulation study: Presented results include the empirical bias (Bias) of the 500 estimated regression coefficients and there sample standard deviation (SD) obtained from analyzing data generated according to models M1-M5, for all considered pool sizes when $J = 50$ in the absence of measurement error. Also included are the average estimated standard errors (SE) and the empirical coverage probabilities (Cov) associated with 95% Wald confidence intervals. Three model fitting procedures were implemented, the proposed methodology (MCMLE), the analytical approach described in Web Appendix A.3 (MLE), and the MCEM algorithm, with the latter two techniques only being applicable for models M1 and M2, respectively.

Model		Measure	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$
M1(MCMLE)	β_0	Bias(SD)	0.00(0.10)	0.00(0.07)	0.00(0.06)	0.00(0.05)	0.00(0.05)
		Cov(SE)	0.96(0.10)	0.92(0.07)	0.93(0.06)	0.92(0.05)	0.92(0.04)
	β_1	Bias(SD)	0.00(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.03)
		Cov(SE)	0.95(0.07)	0.94(0.05)	0.94(0.04)	0.93(0.03)	0.94(0.03)
	β_2	Bias(SD)	0.00(0.14)	0.00(0.10)	0.00(0.08)	0.00(0.07)	0.00(0.06)
		Cov(SE)	0.97(0.14)	0.92(0.10)	0.94(0.08)	0.94(0.07)	0.93(0.06)
M1(MLE)	β_0	Bias(SD)	0.00(0.10)	0.00(0.07)	0.00(0.06)	0.00(0.05)	0.00(0.05)
		Cov(SE)	0.96(0.10)	0.94(0.07)	0.94(0.06)	0.93(0.05)	0.93(0.04)
	β_1	Bias(SD)	0.00(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.03)
		Cov(SE)	0.95(0.07)	0.96(0.05)	0.94(0.04)	0.94(0.04)	0.95(0.03)
	β_2	Bias(SD)	0.00(0.14)	0.00(0.10)	0.00(0.08)	0.00(0.07)	0.00(0.06)
		Cov(SE)	0.97(0.14)	0.92(0.10)	0.96(0.08)	0.94(0.07)	0.94(0.06)
M2(MCMLE)	β_0	Bias(SD)	0.00(0.10)	0.00(0.08)	0.01(0.06)	0.01(0.05)	0.01(0.05)
		Cov(SE)	0.96(0.10)	0.91(0.07)	0.93(0.06)	0.93(0.05)	0.96(0.05)
	β_1	Bias(SD)	0.00(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.04)
		Cov(SE)	0.95(0.07)	0.93(0.05)	0.94(0.04)	0.96(0.04)	0.93(0.03)
	β_2	Bias(SD)	0.00(0.14)	-0.01(0.11)	0.00(0.09)	-0.01(0.08)	0.00(0.07)
		Cov(SE)	0.95(0.14)	0.93(0.10)	0.94(0.08)	0.93(0.07)	0.93(0.06)
M2(MCEM)	β_0	Bias(SD)	0.00(0.10)	0.00(0.08)	0.01(0.06)	0.01(0.05)	0.01(0.05)
		Cov(SE)	0.96(0.10)	0.91(0.07)	0.93(0.06)	0.93(0.05)	0.96(0.05)
	β_1	Bias(SD)	0.00(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.04)
		Cov(SE)	0.95(0.07)	0.93(0.05)	0.94(0.04)	0.96(0.04)	0.93(0.03)
	β_2	Bias(SD)	0.00(0.14)	-0.01(0.11)	0.00(0.09)	-0.01(0.08)	0.00(0.07)
		Cov(SE)	0.95(0.14)	0.93(0.10)	0.94(0.08)	0.93(0.07)	0.93(0.06)
M3(MCMLE)	β_0	Bias(SD)	-0.01(0.25)	0.02(0.19)	0.00(0.15)	-0.01(0.14)	0.01(0.12)
		Cov(SE)	0.94(0.23)	0.93(0.18)	0.94(0.15)	0.93(0.13)	0.95(0.12)
	β_1	Bias(SD)	-0.02(0.18)	0.00(0.13)	0.00(0.11)	-0.01(0.10)	-0.01(0.09)
		Cov(SE)	0.92(0.16)	0.95(0.13)	0.94(0.11)	0.95(0.09)	0.93(0.08)
	β_2	Bias(SD)	0.00(0.34)	-0.02(0.27)	0.01(0.22)	0.00(0.19)	0.00(0.17)
		Cov(SE)	0.93(0.32)	0.93(0.25)	0.94(0.21)	0.94(0.18)	0.94(0.17)
M4(MCMLE)	β_0	Bias(SD)	-0.01(0.10)	0.00(0.07)	-0.01(0.06)	0.00(0.05)	0.00(0.05)
		Cov(SE)	0.94(0.10)	0.94(0.07)	0.93(0.06)	0.93(0.05)	0.94(0.04)
	β_1	Bias(SD)	0.01(0.08)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.03)
		Cov(SE)	0.93(0.07)	0.95(0.05)	0.92(0.04)	0.95(0.04)	0.93(0.03)
	β_2	Bias(SD)	0.00(0.15)	0.00(0.10)	0.01(0.08)	-0.01(0.08)	0.00(0.07)
		Cov(SE)	0.94(0.14)	0.94(0.10)	0.93(0.08)	0.92(0.07)	0.93(0.06)
M5(MCMLE)	β_0	Bias(SD)	0.00(0.05)	0.00(0.04)	0.00(0.03)	0.00(0.03)	0.00(0.02)
		Cov(SE)	0.95(0.05)	0.93(0.03)	0.96(0.03)	0.95(0.02)	0.93(0.02)
	β_1	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.93(0.04)	0.93(0.03)	0.94(0.02)	0.94(0.02)	0.94(0.02)
	β_2	Bias(SD)	0.00(0.09)	0.00(0.06)	0.00(0.05)	0.00(0.04)	0.00(0.04)
		Cov(SE)	0.92(0.08)	0.94(0.06)	0.95(0.05)	0.94(0.04)	0.96(0.04)

Web Table 2: Simulation study: Presented results include the empirical bias (Bias) of the 500 estimated regression coefficients and there sample standard deviation (SD) obtained from analyzing data generated according to models M1-M5, for all considered pool sizes when $J = 200$ in the absence of measurement error. Also included are the average estimated standard errors (SE) and the empirical coverage probabilities (Cov) associated with 95% Wald confidence intervals. Three model fitting procedures were implemented, the proposed methodology (MCMLE), the analytical approach described in Web Appendix A.3 (MLE), and the MCEM algorithm, with the latter two techniques only being applicable for models M1 and M2, respectively.

Model		Measure	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$
M1(MCMLE)	β_0	Bias(SD)	0.00(0.05)	0.00(0.04)	0.00(0.03)	0.00(0.03)	0.00(0.02)
		Cov(SE)	0.94(0.05)	0.93(0.04)	0.93(0.03)	0.94(0.02)	0.96(0.02)
	β_1	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.95(0.04)	0.95(0.02)	0.96(0.02)	0.94(0.02)	0.94(0.02)
	β_2	Bias(SD)	0.00(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.03)
		Cov(SE)	0.95(0.07)	0.95(0.05)	0.94(0.04)	0.95(0.04)	0.93(0.03)
M1(MLE)	β_0	Bias(SD)	0.00(0.05)	0.00(0.04)	0.00(0.03)	0.00(0.03)	0.00(0.02)
		Cov(SE)	0.94(0.05)	0.93(0.04)	0.93(0.03)	0.95(0.03)	0.96(0.02)
	β_1	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.95(0.04)	0.95(0.03)	0.96(0.02)	0.95(0.02)	0.94(0.02)
	β_2	Bias(SD)	0.00(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.03)
		Cov(SE)	0.95(0.07)	0.95(0.05)	0.94(0.04)	0.96(0.04)	0.93(0.03)
M2(MCMLE)	β_0	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.03)	0.00(0.03)	0.00(0.03)
		Cov(SE)	0.95(0.04)	0.95(0.04)	0.94(0.03)	0.95(0.03)	0.94(0.02)
	β_1	Bias(SD)	0.00(0.04)	0.00(0.02)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.95(0.04)	0.96(0.03)	0.95(0.02)	0.94(0.02)	0.93(0.02)
	β_2	Bias(SD)	0.00(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.03)
		Cov(SE)	0.95(0.07)	0.96(0.05)	0.94(0.04)	0.93(0.04)	0.96(0.03)
M2(MCEM)	β_0	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.03)	0.00(0.03)	0.00(0.03)
		Cov(SE)	0.95(0.04)	0.95(0.04)	0.94(0.03)	0.96(0.03)	0.93(0.02)
	β_1	Bias(SD)	0.00(0.04)	0.00(0.02)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.00(0.04)	0.96(0.03)	0.95(0.02)	0.94(0.02)	0.94(0.02)
	β_2	Bias(SD)	0.00(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.03)
		Cov(SE)	0.95(0.07)	0.96(0.05)	0.94(0.04)	0.94(0.04)	0.96(0.03)
M3(MCMLE)	β_0	Bias(SD)	-0.01(0.12)	0.00(0.09)	0.00(0.08)	0.00(0.06)	-0.01(0.06)
		Cov(SE)	0.95(0.12)	0.94(0.09)	0.94(0.08)	0.97(0.07)	0.95(0.06)
	β_1	Bias(SD)	-0.01(0.08)	0.00(0.06)	-0.01(0.05)	0.00(0.05)	0.00(0.04)
		Cov(SE)	0.97(0.08)	0.95(0.06)	0.92(0.05)	0.94(0.05)	0.94(0.04)
	β_2	Bias(SD)	0.02(0.16)	0.00(0.13)	-0.01(0.11)	-0.01(0.10)	0.01(0.08)
		Cov(SE)	0.95(0.17)	0.95(0.13)	0.94(0.11)	0.95(0.09)	0.96(0.09)
M4(MCMLE)	β_0	Bias(SD)	0.00(0.05)	0.00(0.03)	0.00(0.03)	0.00(0.03)	0.00(0.02)
		Cov(SE)	0.94(0.05)	0.96(0.04)	0.96(0.03)	0.93(0.02)	0.96(0.02)
	β_1	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.94(0.04)	0.95(0.03)	0.93(0.02)	0.96(0.02)	0.94(0.02)
	β_2	Bias(SD)	0.00(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.03)
		Cov(SE)	0.95(0.07)	0.95(0.05)	0.94(0.04)	0.94(0.04)	0.96(0.03)
M5(MCMLE)	β_0	Bias(SD)	0.01(0.02)	0.00(0.02)	0.00(0.01)	0.00(0.01)	0.00(0.01)
		Cov(SE)	0.95(0.03)	0.95(0.02)	0.94(0.01)	0.95(0.01)	0.94(0.01)
	β_1	Bias(SD)	0.00(0.02)	0.00(0.01)	0.00(0.01)	0.00(0.01)	0.00(0.01)
		Cov(SE)	0.96(0.02)	0.94(0.01)	0.95(0.01)	0.95(0.01)	0.93(0.01)
	β_2	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.95(0.04)	0.97(0.03)	0.94(0.02)	0.95(0.02)	0.96(0.02)

Web Table 3: Simulation study: Presented results include the empirical bias (Bias) of the 500 estimated regression coefficients and there sample standard deviation (SD) obtained from analyzing data generated according to models M1-M5, for all considered pool sizes when $J = 50$ in the presence of measurement error ($\tau = 0.05$). Also included are the average estimated standard errors (SE) and the empirical coverage probabilities (Cov) associated with 95% Wald confidence intervals. Two model fitting procedures were implemented, the proposed methodology (MCMLE) and the analytical approach described in Web Appendix A.3 (MLE), with the latter technique only being applicable for model M1.

Model		Measure	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$
M1(MCMLE)	β_0	Bias(SD)	0.01(0.11)	0.00(0.07)	0.00(0.06)	0.00(0.05)	0.00(0.05)
		Cov(SE)	0.93(0.10)	0.94(0.07)	0.93(0.06)	0.94(0.05)	0.93(0.04)
	β_1	Bias(SD)	0.00(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.03)
		Cov(SE)	0.94(0.07)	0.94(0.05)	0.94(0.04)	0.91(0.04)	0.93(0.03)
	β_2	Bias(SD)	-0.01(0.15)	0.01(0.10)	0.00(0.09)	0.00(0.07)	0.00(0.06)
		Cov(SE)	0.93(0.14)	0.94(0.10)	0.93(0.08)	0.95(0.07)	0.93(0.06)
M1(MLE)	β_0	Bias(SD)	0.01(0.11)	0.00(0.07)	0.00(0.06)	0.00(0.05)	0.00(0.05)
		Cov(SE)	0.94(0.10)	0.94(0.07)	0.94(0.06)	0.95(0.05)	0.94(0.05)
	β_1	Bias(SD)	0.00(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.03)
		Cov(SE)	0.95(0.07)	0.95(0.05)	0.94(0.04)	0.92(0.04)	0.95(0.03)
	β_2	Bias(SD)	-0.01(0.15)	0.00(0.10)	0.00(0.09)	0.00(0.07)	0.00(0.06)
		Cov(SE)	0.94(0.14)	0.95(0.10)	0.94(0.08)	0.95(0.07)	0.94(0.07)
M2(MCMLE)	β_0	Bias(SD)	-0.01(0.12)	0.00(0.09)	0.00(0.07)	0.00(0.07)	0.01(0.06)
		Cov(SE)	0.95(0.11)	0.93(0.08)	0.94(0.07)	0.93(0.07)	0.93(0.06)
	β_1	Bias(SD)	-0.01(0.08)	0.00(0.06)	0.00(0.05)	0.00(0.05)	0.00(0.04)
		Cov(SE)	0.93(0.08)	0.94(0.06)	0.92(0.05)	0.92(0.04)	0.91(0.04)
	β_2	Bias(SD)	0.00(0.15)	0.00(0.12)	0.00(0.10)	0.00(0.09)	0.00(0.08)
		Cov(SE)	0.94(0.15)	0.92(0.11)	0.92(0.09)	0.93(0.08)	0.94(0.07)
M3(MCMLE)	β_0	Bias(SD)	-0.01(0.25)	0.00(0.19)	-0.01(0.16)	0.00(0.14)	0.01(0.12)
		Cov(SE)	0.94(0.24)	0.93(0.18)	0.95(0.15)	0.92(0.13)	0.95(0.12)
	β_1	Bias(SD)	0.01(0.19)	0.00(0.13)	0.00(0.11)	0.01(0.10)	0.00(0.09)
		Cov(SE)	0.95(0.17)	0.94(0.13)	0.95(0.11)	0.93(0.09)	0.94(0.08)
	β_2	Bias(SD)	0.00(0.35)	0.01(0.25)	0.00(0.22)	0.00(0.20)	-0.01(0.18)
		Cov(SE)	0.94(0.34)	0.94(0.25)	0.95(0.21)	0.93(0.19)	0.93(0.17)
M4(MCMLE)	β_0	Bias(SD)	-0.01(0.12)	0.00(0.08)	0.00(0.07)	0.00(0.06)	-0.01(0.06)
		Cov(SE)	0.94(0.11)	0.95(0.08)	0.95(0.07)	0.94(0.06)	0.93(0.06)
	β_1	Bias(SD)	0.00(0.08)	0.00(0.06)	0.00(0.05)	0.00(0.04)	0.00(0.04)
		Cov(SE)	0.96(0.08)	0.95(0.06)	0.92(0.05)	0.92(0.04)	0.95(0.04)
	β_2	Bias(SD)	0.00(0.16)	0.00(0.11)	-0.01(0.09)	0.00(0.08)	0.00(0.07)
		Cov(SE)	0.92(0.15)	0.93(0.11)	0.94(0.09)	0.94(0.08)	0.94(0.07)
M5(MCMLE)	β_0	Bias(SD)	0.00(0.05)	0.00(0.03)	0.00(0.03)	0.00(0.03)	0.00(0.02)
		Cov(SE)	0.93(0.05)	0.96(0.03)	0.93(0.03)	0.93(0.02)	0.94(0.02)
	β_1	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.95(0.04)	0.96(0.03)	0.92(0.02)	0.96(0.02)	0.93(0.02)
	β_2	Bias(SD)	0.00(0.08)	0.00(0.06)	0.00(0.05)	0.00(0.05)	0.00(0.04)
		Cov(SE)	0.96(0.08)	0.94(0.06)	0.94(0.05)	0.92(0.04)	0.95(0.04)

Web Table 4: Simulation study: Presented results include the empirical bias (Bias) of the 500 estimated regression coefficients and there sample standard deviation (SD) obtained from analyzing data generated according to models M1-M5, for all considered pool sizes when $J = 200$ in the presence of measurement error ($\tau = 0.05$). Also included are the average estimated standard errors (SE) and the empirical coverage probabilities (Cov) associated with 95% Wald confidence intervals. Two model fitting procedures were implemented, the proposed methodology (MCMLE) and the analytical approach described in Web Appendix A.3 (MLE), with the latter technique only being applicable for model M1.

Model		Measure	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$
M1(MCMLE)	β_0	Bias(SD)	0.00(0.05)	0.00(0.04)	0.00(0.03)	0.00(0.03)	0.00(0.02)
		Cov(SE)	0.93(0.05)	0.94(0.04)	0.97(0.03)	0.96(0.03)	0.93(0.02)
	β_1	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.94(0.04)	0.94(0.03)	0.94(0.02)	0.95(0.02)	0.94(0.02)
	β_2	Bias(SD)	0.00(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.03)	0.00(0.03)
		Cov(SE)	0.95(0.07)	0.95(0.05)	0.96(0.04)	0.96(0.04)	0.94(0.03)
M1(MLE)	β_0	Bias(SD)	0.00(0.05)	0.00(0.04)	0.00(0.03)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.93(0.05)	0.94(0.04)	0.97(0.03)	0.96(0.03)	0.93(0.02)
	β_1	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.94(0.04)	0.94(0.03)	0.94(0.02)	0.95(0.02)	0.95(0.02)
	β_2	Bias(SD)	0.00(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.03)	0.00(0.03)
		Cov(SE)	0.95(0.07)	0.95(0.05)	0.96(0.04)	0.96(0.04)	0.94(0.03)
M2(MCMLE)	β_0	Bias(SD)	0.01(0.08)	0.01(0.04)	0.00(0.04)	0.00(0.03)	0.01(0.03)
		Cov(SE)	0.93(0.08)	0.95(0.04)	0.93(0.04)	0.94(0.03)	0.95(0.03)
	β_1	Bias(SD)	-0.01(0.06)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.92(0.06)	0.96(0.03)	0.96(0.02)	0.95(0.02)	0.93(0.02)
	β_2	Bias(SD)	-0.01(0.10)	0.00(0.05)	0.00(0.05)	0.00(0.04)	0.00(0.04)
		Cov(SE)	0.95(0.1)	0.95(0.06)	0.93(0.05)	0.94(0.04)	0.96(0.04)
M3(MCMLE)	β_0	Bias(SD)	0.00(0.12)	0.00(0.09)	0.00(0.08)	0.00(0.06)	-0.01(0.06)
		Cov(SE)	0.95(0.12)	0.94(0.09)	0.94(0.08)	0.97(0.07)	0.95(0.06)
	β_1	Bias(SD)	0.00(0.09)	0.00(0.06)	-0.01(0.05)	0.00(0.05)	0.00(0.04)
		Cov(SE)	0.94(0.08)	0.95(0.06)	0.92(0.05)	0.94(0.05)	0.94(0.04)
	β_2	Bias(SD)	0.00(0.17)	0.00(0.13)	-0.01(0.11)	-0.01(0.10)	0.01(0.08)
		Cov(SE)	0.95(0.17)	0.95(0.13)	0.94(0.11)	0.95(0.09)	0.96(0.09)
M4(MCMLE)	β_0	Bias(SD)	0.01(0.05)	0.00(0.04)	0.00(0.03)	0.00(0.03)	0.00(0.03)
		Cov(SE)	0.95(0.05)	0.94(0.04)	0.95(0.03)	0.94(0.03)	0.95(0.03)
	β_1	Bias(SD)	-0.01(0.04)	-0.01(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.95(0.04)	0.95(0.03)	0.94(0.02)	0.93(0.02)	0.95(0.02)
	β_2	Bias(SD)	-0.02(0.07)	0.00(0.06)	0.00(0.04)	0.00(0.04)	0.00(0.04)
		Cov(SE)	0.94(0.07)	0.93(0.05)	0.96(0.05)	0.95(0.04)	0.95(0.04)
M5(MCMLE)	β_0	Bias(SD)	0.00(0.03)	0.00(0.02)	0.00(0.01)	0.00(0.01)	0.00(0.01)
		Cov(SE)	0.94(0.02)	0.95(0.02)	0.95(0.01)	0.96(0.01)	0.96(0.01)
	β_1	Bias(SD)	0.00(0.02)	0.00(0.01)	0.00(0.01)	0.00(0.01)	0.00(0.01)
		Cov(SE)	0.93(0.02)	0.94(0.01)	0.93(0.01)	0.96(0.01)	0.93(0.01)
	β_2	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.96(0.04)	0.95(0.03)	0.94(0.02)	0.96(0.02)	0.94(0.02)

Web Table 5: Simulation study: Presented results include the empirical bias (Bias) of the 500 estimated regression coefficients and there sample standard deviation (SD) obtained from analyzing data generated according to models M1-M5, for all considered pool sizes when $J = 50$ in the presence of measurement error ($\tau = 0.10$). Also included are the average estimated standard errors (SE) and the empirical coverage probabilities (Cov) associated with 95% Wald confidence intervals. Two model fitting procedures were implemented, the proposed methodology (MCMLE) and the analytical approach described in Web Appendix A.3 (MLE), with the latter technique only being applicable for model M1.

Model		Measure	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$
M1(MCMLE)	β_0	Bias(SD)	0.00(0.11)	0.00(0.07)	0.00(0.06)	0.00(0.05)	0.00(0.05)
		Cov(SE)	0.93(0.10)	0.94(0.07)	0.94(0.06)	0.95(0.05)	0.94(0.05)
	β_1	Bias(SD)	0.00(0.08)	0.00(0.05)	0.00(0.05)	0.00(0.04)	0.00(0.03)
		Cov(SE)	0.92(0.07)	0.93(0.05)	0.93(0.04)	0.93(0.04)	0.94(0.03)
	β_2	Bias(SD)	0.00(0.15)	0.01(0.10)	0.00(0.08)	0.00(0.07)	0.00(0.06)
		Cov(SE)	0.95(0.14)	0.94(0.10)	0.94(0.08)	0.95(0.07)	0.96(0.07)
M1(MLE)	β_0	Bias(SD)	0.00(0.11)	0.00(0.07)	0.00(0.06)	0.00(0.05)	0.00(0.05)
		Cov(SE)	0.93(0.10)	0.94(0.07)	0.94(0.06)	0.95(0.05)	0.95(0.05)
	β_1	Bias(SD)	0.00(0.08)	0.00(0.05)	0.00(0.05)	0.00(0.04)	0.00(0.03)
		Cov(SE)	0.93(0.07)	0.93(0.05)	0.93(0.04)	0.93(0.04)	0.95(0.04)
	β_2	Bias(SD)	0.00(0.15)	0.01(0.10)	0.00(0.08)	0.00(0.07)	0.00(0.06)
		Cov(SE)	0.95(0.15)	0.94(0.11)	0.95(0.09)	0.95(0.08)	0.97(0.07)
M2(MCMLE)	β_0	Bias(SD)	-0.01(0.12)	0.01(0.10)	0.01(0.09)	0.00(0.08)	0.00(0.08)
		Cov(SE)	0.95(0.13)	0.94(0.10)	0.94(0.09)	0.94(0.08)	0.95(0.08)
	β_1	Bias(SD)	0.01(0.09)	0.00(0.06)	0.00(0.06)	0.00(0.05)	0.01(0.05)
		Cov(SE)	0.94(0.09)	0.95(0.06)	0.94(0.05)	0.92(0.05)	0.94(0.04)
	β_2	Bias(SD)	0.02(0.16)	0.00(0.13)	0.00(0.10)	0.01(0.09)	0.00(0.09)
		Cov(SE)	0.94(0.16)	0.93(0.12)	0.94(0.10)	0.93(0.09)	0.94(0.08)
M3(MCMLE)	β_0	Bias(SD)	0.02(0.24)	0.00(0.20)	-0.01(0.16)	-0.02(0.14)	0.00(0.13)
		Cov(SE)	0.95(0.24)	0.93(0.18)	0.95(0.15)	0.91(0.13)	0.93(0.12)
	β_1	Bias(SD)	0.01(0.19)	0.00(0.13)	0.01(0.11)	0.00(0.10)	0.00(0.09)
		Cov(SE)	0.92(0.17)	0.95(0.13)	0.94(0.11)	0.94(0.09)	0.93(0.08)
	β_2	Bias(SD)	0.01(0.33)	-0.01(0.27)	0.01(0.22)	0.02(0.18)	0.00(0.18)
		Cov(SE)	0.96(0.34)	0.93(0.25)	0.95(0.21)	0.95(0.19)	0.93(0.17)
M4(MCMLE)	β_0	Bias(SD)	-0.01(0.13)	0.00(0.10)	0.00(0.08)	0.00(0.08)	0.00(0.07)
		Cov(SE)	0.93(0.12)	0.93(0.10)	0.94(0.08)	0.94(0.08)	0.95(0.07)
	β_1	Bias(SD)	-0.01(0.09)	0.00(0.06)	0.00(0.06)	0.00(0.05)	0.00(0.05)
		Cov(SE)	0.95(0.09)	0.95(0.06)	0.94(0.05)	0.94(0.05)	0.91(0.04)
	β_2	Bias(SD)	0.00(0.17)	-0.01(0.13)	0.00(0.10)	0.00(0.09)	0.00(0.08)
		Cov(SE)	0.93(0.16)	0.93(0.12)	0.94(0.10)	0.95(0.09)	0.95(0.08)
M5(MCMLE)	β_0	Bias(SD)	0.01(0.05)	0.01(0.04)	0.00(0.03)	0.00(0.03)	0.00(0.02)
		Cov(SE)	0.94(0.05)	0.92(0.04)	0.92(0.03)	0.95(0.02)	0.94(0.02)
	β_1	Bias(SD)	-0.01(0.04)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.97(0.04)	0.94(0.03)	0.94(0.02)	0.94(0.02)	0.94(0.02)
	β_2	Bias(SD)	0.00(0.08)	0.00(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.04)
		Cov(SE)	0.95(0.08)	0.92(0.06)	0.94(0.05)	0.95(0.04)	0.93(0.04)

Web Table 6: Simulation study: Presented results include the empirical bias (Bias) of the 500 estimated regression coefficients and there sample standard deviation (SD) obtained from analyzing data generated according to models M1-M5, for all considered pool sizes when $J = 100$ in the presence of measurement error ($\tau = 0.10$). Also included are the average estimated standard errors (SE) and the empirical coverage probabilities (Cov) associated with 95% Wald confidence intervals. Two model fitting procedures were implemented, the proposed methodology (MCMLE) and the analytical approach described in Web Appendix A.3 (MLE), with the latter technique only being applicable for model M1.

Model		Measure	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$
M1(MCMLE)	β_0	Bias(SD)	-0.01(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.03)
		Cov(SE)	0.94(0.07)	0.93(0.05)	0.93(0.04)	0.95(0.04)	0.95(0.03)
	β_1	Bias(SD)	0.00(0.05)	0.00(0.04)	0.00(0.03)	0.00(0.03)	0.00(0.02)
		Cov(SE)	0.94(0.05)	0.96(0.04)	0.94(0.03)	0.95(0.03)	0.94(0.02)
	β_2	Bias(SD)	0.01(0.10)	0.00(0.07)	0.00(0.06)	0.00(0.05)	0.00(0.05)
		Cov(SE)	0.94(0.10)	0.95(0.07)	0.93(0.06)	0.95(0.05)	0.96(0.05)
M1(MLE)	β_0	Bias(SD)	-0.01(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.03)
		Cov(SE)	0.94(0.07)	0.94(0.05)	0.94(0.04)	0.96(0.04)	0.96(0.03)
	β_1	Bias(SD)	0.00(0.05)	0.00(0.04)	0.00(0.03)	0.00(0.03)	0.00(0.02)
		Cov(SE)	0.95(0.05)	0.97(0.04)	0.95(0.03)	0.95(0.03)	0.94(0.02)
	β_2	Bias(SD)	0.01(0.10)	0.00(0.07)	0.00(0.06)	0.00(0.05)	0.00(0.05)
		Cov(SE)	0.94(0.10)	0.95(0.07)	0.94(0.06)	0.96(0.05)	0.96(0.05)
M2(MCMLE)	β_0	Bias(SD)	0.00(0.09)	0.00(0.07)	0.00(0.06)	0.00(0.06)	0.00(0.05)
		Cov(SE)	0.95(0.09)	0.94(0.07)	0.94(0.06)	0.95(0.06)	0.94(0.05)
	β_1	Bias(SD)	-0.01(0.06)	0.00(0.05)	0.00(0.04)	0.00(0.03)	0.00(0.03)
		Cov(SE)	0.97(0.06)	0.94(0.04)	0.95(0.04)	0.94(0.03)	0.94(0.03)
	β_2	Bias(SD)	0.00(0.11)	0.00(0.09)	0.00(0.07)	0.00(0.07)	0.00(0.06)
		Cov(SE)	0.95(0.11)	0.94(0.08)	0.95(0.07)	0.95(0.07)	0.94(0.06)
M3(MCMLE)	β_0	Bias(SD)	-0.01(0.17)	-0.01(0.13)	0.00(0.11)	0.00(0.10)	0.00(0.08)
		Cov(SE)	0.95(0.17)	0.94(0.13)	0.95(0.11)	0.92(0.09)	0.96(0.09)
	β_1	Bias(SD)	0.00(0.12)	0.00(0.09)	0.00(0.08)	0.00(0.07)	0.00(0.06)
		Cov(SE)	0.94(0.12)	0.95(0.09)	0.94(0.08)	0.94(0.07)	0.95(0.06)
	β_2	Bias(SD)	0.01(0.24)	0.02(0.18)	0.01(0.15)	0.00(0.14)	0.00(0.12)
		Cov(SE)	0.93(0.24)	0.96(0.18)	0.95(0.15)	0.93(0.13)	0.95(0.12)
M4(MCMLE)	β_0	Bias(SD)	-0.01(0.09)	0.00(0.07)	0.00(0.06)	0.00(0.06)	0.00(0.05)
		Cov(SE)	0.95(0.09)	0.96(0.07)	0.93(0.06)	0.96(0.06)	0.94(0.05)
	β_1	Bias(SD)	0.00(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.03)	0.00(0.03)
		Cov(SE)	0.93(0.06)	0.94(0.04)	0.94(0.04)	0.94(0.03)	0.94(0.03)
	β_2	Bias(SD)	0.01(0.12)	0.00(0.09)	0.00(0.07)	0.00(0.07)	0.00(0.06)
		Cov(SE)	0.94(0.11)	0.95(0.08)	0.96(0.07)	0.95(0.07)	0.96(0.06)
M5(MCMLE)	β_0	Bias(SD)	0.00(0.03)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.95(0.04)	0.95(0.02)	0.93(0.02)	0.95(0.02)	0.95(0.02)
	β_1	Bias(SD)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.01)	0.00(0.01)
		Cov(SE)	0.94(0.03)	0.95(0.02)	0.94(0.02)	0.95(0.01)	0.95(0.01)
	β_2	Bias(SD)	0.00(0.06)	0.00(0.04)	0.00(0.03)	0.00(0.03)	0.00(0.03)
		Cov(SE)	0.96(0.06)	0.94(0.04)	0.96(0.03)	0.96(0.03)	0.96(0.03)

Web Table 7: Simulation study: Presented results include the empirical bias (Bias) of the 500 estimated regression coefficients and there sample standard deviation (SD) obtained from analyzing data generated according to models M1-M5, for all considered pool sizes when $J = 200$ in the presence of measurement error ($\tau = 0.10$). Also included are the average estimated standard errors (SE) and the empirical coverage probabilities (Cov) associated with 95% Wald confidence intervals. Two model fitting procedures were implemented, the proposed methodology (MCMLE) and the analytical approach described in Web Appendix A.3 (MLE), with the latter technique only being applicable for model M1.

Model		Measure	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$
M1(MCMLE)	β_0	Bias(SD)	0.00(0.05)	0.00(0.04)	0.00(0.03)	0.00(0.03)	0.00(0.02)
		Cov(SE)	0.96(0.05)	0.95(0.04)	0.95(0.03)	0.93(0.03)	0.95(0.02)
	β_1	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.96(0.04)	0.95(0.03)	0.95(0.02)	0.94(0.02)	0.95(0.02)
	β_2	Bias(SD)	0.00(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.03)
		Cov(SE)	0.97(0.07)	0.95(0.05)	0.94(0.04)	0.95(0.04)	0.96(0.03)
M1(MLE)	β_0	Bias(SD)	0.00(0.05)	0.00(0.04)	0.00(0.03)	0.00(0.03)	0.00(0.02)
		Cov(SE)	0.96(0.05)	0.95(0.04)	0.96(0.03)	0.93(0.03)	0.95(0.02)
	β_1	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.96(0.04)	0.95(0.03)	0.95(0.02)	0.94(0.02)	0.95(0.02)
	β_2	Bias(SD)	0.00(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.03)
		Cov(SE)	0.97(0.07)	0.95(0.05)	0.95(0.04)	0.95(0.04)	0.96(0.03)
M2(MCMLE)	β_0	Bias(SD)	0.00(0.07)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.04)
		Cov(SE)	0.94(0.06)	0.93(0.05)	0.95(0.04)	0.95(0.04)	0.94(0.04)
	β_1	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.03)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.95(0.04)	0.94(0.03)	0.93(0.03)	0.95(0.02)	0.94(0.02)
	β_2	Bias(SD)	0.00(0.08)	0.00(0.06)	0.01(0.05)	0.00(0.05)	0.00(0.04)
		Cov(SE)	0.94(0.08)	0.94(0.06)	0.95(0.05)	0.95(0.05)	0.93(0.04)
M3(MCMLE)	β_0	Bias(SD)	0.00(0.12)	0.00(0.09)	0.00(0.08)	0.00(0.07)	0.00(0.06)
		Cov(SE)	0.96(0.12)	0.95(0.09)	0.93(0.08)	0.93(0.07)	0.94(0.06)
	β_1	Bias(SD)	0.00(0.08)	0.01(0.06)	0.00(0.05)	0.00(0.05)	0.00(0.04)
		Cov(SE)	0.96(0.08)	0.94(0.06)	0.95(0.05)	0.96(0.05)	0.94(0.04)
	β_2	Bias(SD)	0.00(0.16)	0.00(0.13)	0.00(0.11)	-0.01(0.09)	0.00(0.09)
		Cov(SE)	0.97(0.17)	0.95(0.13)	0.92(0.11)	0.95(0.10)	0.94(0.09)
M4(MCMLE)	β_0	Bias(SD)	0.00(0.07)	0.00(0.05)	0.00(0.05)	0.00(0.04)	0.00(0.04)
		Cov(SE)	0.95(0.06)	0.94(0.05)	0.93(0.04)	0.94(0.04)	0.94(0.04)
	β_1	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.03)	0.00(0.03)	0.00(0.02)
		Cov(SE)	0.92(0.04)	0.95(0.03)	0.94(0.03)	0.94(0.02)	0.95(0.02)
	β_2	Bias(SD)	0.00(0.08)	0.00(0.06)	0.00(0.05)	0.00(0.05)	0.00(0.04)
		Cov(SE)	0.95(0.08)	0.94(0.06)	0.96(0.05)	0.96(0.05)	0.94(0.04)
M5(MCMLE)	β_0	Bias(SD)	0.00(0.03)	0.00(0.02)	0.00(0.01)	0.00(0.01)	0.00(0.01)
		Cov(SE)	0.95(0.02)	0.94(0.02)	0.94(0.01)	0.95(0.01)	0.96(0.01)
	β_1	Bias(SD)	0.00(0.02)	0.00(0.01)	0.00(0.01)	0.00(0.01)	0.00(0.01)
		Cov(SE)	0.96(0.02)	0.96(0.01)	0.95(0.01)	0.95(0.01)	0.95(0.01)
	β_2	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.95(0.04)	0.94(0.03)	0.94(0.02)	0.96(0.02)	0.95(0.02)

Web Table 8: Simulation study: Presented results include the empirical bias (Bias) of the 500 estimated regression coefficients and there sample standard deviation (SD) obtained from analyzing data generated according to models M1-M5, for all considered pool sizes when $J = 50$ in the presence of measurement error ($\tau = 0.20$). Also included are the average estimated standard errors (SE) and the empirical coverage probabilities (Cov) associated with 95% Wald confidence intervals. Two model fitting procedures were implemented, the proposed methodology (MCMLE) and the analytical approach described in Web Appendix A.3 (MLE), with the latter technique only being applicable for model M1.

Model		Measure	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$
M1(MCMLE)	β_0	Bias(SD)	0.00(0.11)	0.00(0.09)	0.00(0.07)	0.00(0.07)	0.00(0.06)
		Cov(SE)	0.94(0.11)	0.92(0.08)	0.94(0.07)	0.94(0.06)	0.92(0.06)
	β_1	Bias(SD)	0.00(0.08)	0.00(0.06)	0.00(0.05)	0.00(0.05)	0.00(0.04)
		Cov(SE)	0.95(0.08)	0.94(0.06)	0.95(0.05)	0.94(0.04)	0.92(0.04)
	β_2	Bias(SD)	-0.01(0.14)	0.00(0.12)	0.00(0.09)	-0.01(0.09)	0.00(0.08)
		Cov(SE)	0.96(0.15)	0.94(0.11)	0.96(0.10)	0.94(0.09)	0.96(0.08)
M1(MLE)	β_0	Bias(SD)	0.00(0.11)	0.00(0.09)	0.00(0.07)	0.00(0.07)	0.00(0.06)
		Cov(SE)	0.95(0.11)	0.93(0.08)	0.95(0.07)	0.95(0.06)	0.94(0.06)
	β_1	Bias(SD)	0.00(0.08)	0.00(0.06)	0.00(0.05)	0.00(0.05)	0.00(0.04)
		Cov(SE)	0.95(0.08)	0.94(0.06)	0.96(0.05)	0.95(0.05)	0.94(0.04)
	β_2	Bias(SD)	-0.01(0.14)	0.00(0.12)	0.00(0.10)	-0.01(0.09)	0.00(0.08)
		Cov(SE)	0.96(0.16)	0.95(0.12)	0.96(0.10)	0.95(0.09)	0.96(0.09)
M2(MCMLE)	β_0	Bias(SD)	-0.01(0.16)	0.00(0.13)	0.00(0.12)	0.01(0.11)	0.00(0.11)
		Cov(SE)	0.95(0.15)	0.94(0.13)	0.94(0.12)	0.93(0.11)	0.94(0.10)
	β_1	Bias(SD)	0.01(0.10)	0.00(0.08)	0.00(0.07)	0.00(0.06)	0.00(0.05)
		Cov(SE)	0.92(0.09)	0.93(0.07)	0.92(0.06)	0.93(0.06)	0.93(0.05)
	β_2	Bias(SD)	0.01(0.19)	0.01(0.14)	0.01(0.13)	0.00(0.12)	0.00(0.11)
		Cov(SE)	0.95(0.18)	0.95(0.14)	0.92(0.12)	0.93(0.11)	0.94(0.11)
M3(MCMLE)	β_0	Bias(SD)	-0.01(0.25)	0.01(0.19)	-0.01(0.15)	0.01(0.14)	0.00(0.13)
		Cov(SE)	0.93(0.24)	0.95(0.18)	0.95(0.16)	0.94(0.14)	0.93(0.12)
	β_1	Bias(SD)	0.00(0.19)	-0.01(0.14)	-0.01(0.11)	0.00(0.10)	0.00(0.09)
		Cov(SE)	0.93(0.17)	0.92(0.13)	0.93(0.11)	0.95(0.10)	0.94(0.09)
	β_2	Bias(SD)	0.01(0.37)	0.00(0.27)	0.01(0.22)	-0.01(0.20)	0.00(0.19)
		Cov(SE)	0.94(0.34)	0.95(0.26)	0.96(0.22)	0.93(0.19)	0.94(0.18)
M4(MCMLE)	β_0	Bias(SD)	0.00(0.16)	0.00(0.13)	0.00(0.12)	0.00(0.10)	0.00(0.11)
		Cov(SE)	0.94(0.16)	0.95(0.13)	0.93(0.11)	0.95(0.10)	0.92(0.10)
	β_1	Bias(SD)	0.00(0.10)	0.00(0.07)	0.00(0.06)	0.00(0.06)	0.00(0.05)
		Cov(SE)	0.93(0.10)	0.95(0.07)	0.92(0.06)	0.94(0.05)	0.93(0.05)
	β_2	Bias(SD)	-0.02(0.19)	-0.01(0.15)	0.00(0.13)	-0.01(0.11)	0.00(0.11)
		Cov(SE)	0.94(0.18)	0.92(0.14)	0.93(0.12)	0.93(0.11)	0.92(0.11)
M5(MCMLE)	β_0	Bias(SD)	0.01(0.06)	0.00(0.04)	0.00(0.03)	0.00(0.03)	0.00(0.02)
		Cov(SE)	0.93(0.05)	0.94(0.04)	0.93(0.03)	0.94(0.03)	0.95(0.02)
	β_1	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.96(0.04)	0.95(0.03)	0.94(0.02)	0.93(0.02)	0.93(0.02)
	β_2	Bias(SD)	0.01(0.09)	0.00(0.06)	0.00(0.05)	0.00(0.04)	0.00(0.04)
		Cov(SE)	0.94(0.09)	0.95(0.06)	0.93(0.05)	0.94(0.05)	0.93(0.04)

Web Table 9: Simulation study: Presented results include the empirical bias (Bias) of the 500 estimated regression coefficients and there sample standard deviation (SD) obtained from analyzing data generated according to models M1-M5, for all considered pool sizes when $J = 100$ in the presence of measurement error ($\tau = 0.20$). Also included are the average estimated standard errors (SE) and the empirical coverage probabilities (Cov) associated with 95% Wald confidence intervals. Two model fitting procedures were implemented, the proposed methodology (MCMLE) and the analytical approach described in Web Appendix A.3 (MLE), with the latter technique only being applicable for model M1.

Model		Measure	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$
M1(MCMLE)	β_0	Bias(SD)	0.00(0.08)	0.00(0.06)	0.00(0.05)	0.00(0.05)	0.00(0.04)
		Cov(SE)	0.94(0.08)	0.93(0.06)	0.94(0.05)	0.94(0.04)	0.95(0.04)
	β_1	Bias(SD)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.03)	0.00(0.03)
		Cov(SE)	0.94(0.05)	0.94(0.04)	0.93(0.03)	0.96(0.03)	0.94(0.03)
	β_2	Bias(SD)	0.00(0.10)	0.00(0.08)	0.00(0.07)	0.00(0.07)	0.00(0.06)
		Cov(SE)	0.94(0.11)	0.94(0.08)	0.95(0.07)	0.94(0.06)	0.94(0.06)
M1(MLE)	β_0	Bias(SD)	0.00(0.08)	0.00(0.06)	0.00(0.05)	0.00(0.05)	0.00(0.04)
		Cov(SE)	0.94(0.08)	0.94(0.06)	0.94(0.05)	0.94(0.05)	0.96(0.04)
	β_1	Bias(SD)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.03)	0.00(0.03)
		Cov(SE)	0.94(0.05)	0.95(0.04)	0.94(0.04)	0.97(0.03)	0.94(0.03)
	β_2	Bias(SD)	0.00(0.10)	0.00(0.08)	0.00(0.07)	0.00(0.07)	0.00(0.06)
		Cov(SE)	0.95(0.11)	0.95(0.08)	0.95(0.07)	0.94(0.06)	0.95(0.06)
M2(MCMLE)	β_0	Bias(SD)	0.00(0.11)	0.00(0.10)	0.00(0.08)	0.00(0.08)	0.00(0.07)
		Cov(SE)	0.94(0.11)	0.94(0.09)	0.95(0.08)	0.94(0.08)	0.96(0.07)
	β_1	Bias(SD)	0.00(0.07)	0.00(0.06)	0.00(0.05)	0.00(0.04)	0.00(0.04)
		Cov(SE)	0.94(0.07)	0.93(0.05)	0.95(0.04)	0.94(0.04)	0.92(0.04)
	β_2	Bias(SD)	0.00(0.13)	0.00(0.10)	0.00(0.08)	0.01(0.08)	0.00(0.07)
		Cov(SE)	0.95(0.13)	0.93(0.11)	0.96(0.09)	0.94(0.08)	0.96(0.07)
M3(MCMLE)	β_0	Bias(SD)	-0.01(0.17)	0.00(0.13)	0.00(0.11)	0.01(0.10)	0.00(0.09)
		Cov(SE)	0.95(0.17)	0.95(0.13)	0.95(0.11)	0.94(0.10)	0.95(0.09)
	β_1	Bias(SD)	0.00(0.12)	0.00(0.09)	0.00(0.08)	0.00(0.07)	0.01(0.06)
		Cov(SE)	0.96(0.12)	0.95(0.09)	0.94(0.08)	0.94(0.07)	0.94(0.06)
	β_2	Bias(SD)	0.01(0.24)	0.01(0.19)	0.01(0.17)	0.00(0.15)	0.00(0.13)
		Cov(SE)	0.95(0.24)	0.95(0.18)	0.93(0.16)	0.93(0.14)	0.94(0.13)
M4(MCMLE)	β_0	Bias(SD)	0.00(0.11)	0.00(0.09)	-0.01(0.08)	0.00(0.08)	0.00(0.07)
		Cov(SE)	0.94(0.11)	0.96(0.09)	0.94(0.08)	0.93(0.08)	0.95(0.07)
	β_1	Bias(SD)	0.00(0.07)	0.00(0.05)	0.00(0.05)	0.00(0.04)	0.00(0.04)
		Cov(SE)	0.95(0.07)	0.95(0.05)	0.92(0.04)	0.93(0.04)	0.94(0.04)
	β_2	Bias(SD)	-0.01(0.13)	0.00(0.10)	0.00(0.09)	0.00(0.08)	0.00(0.08)
		Cov(SE)	0.95(0.13)	0.95(0.10)	0.95(0.09)	0.95(0.08)	0.94(0.08)
M5(MCMLE)	β_0	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.94(0.04)	0.93(0.03)	0.96(0.02)	0.95(0.02)	0.94(0.02)
	β_1	Bias(SD)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.01)	0.00(0.01)
		Cov(SE)	0.94(0.03)	0.94(0.02)	0.95(0.02)	0.96(0.01)	0.96(0.01)
	β_2	Bias(SD)	0.00(0.06)	0.00(0.05)	0.00(0.04)	0.00(0.03)	0.00(0.03)
		Cov(SE)	0.93(0.06)	0.94(0.04)	0.95(0.04)	0.96(0.03)	0.94(0.03)

Web Table 10: Simulation study: Presented results include the empirical bias (Bias) of the 500 estimated regression coefficients and there sample standard deviation (SD) obtained from analyzing data generated according to models M1-M5, for all considered pool sizes when $J = 200$ in the presence of measurement error ($\tau = 0.20$). Also included are the average estimated standard errors (SE) and the empirical coverage probabilities (Cov) associated with 95% Wald confidence intervals. Two model fitting procedures were implemented, the proposed methodology (MCMLE) and the analytical approach described in Web Appendix A.3 (MLE), with the latter technique only being applicable for model M1.

Model		Measure	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$
M1(MCMLE)	β_0	Bias(SD)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.03)	0.00(0.03)
		Cov(SE)	0.94(0.05)	0.94(0.04)	0.95(0.03)	0.96(0.03)	0.95(0.03)
	β_1	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.95(0.04)	0.94(0.03)	0.96(0.02)	0.94(0.02)	0.94(0.02)
	β_2	Bias(SD)	0.00(0.08)	0.00(0.06)	-0.01(0.05)	0.00(0.04)	0.00(0.04)
		Cov(SE)	0.94(0.08)	0.95(0.06)	0.94(0.05)	0.95(0.05)	0.94(0.04)
M1(MLE)	β_0	Bias(SD)	0.00(0.05)	0.00(0.04)	0.00(0.04)	0.00(0.03)	0.00(0.03)
		Cov(SE)	0.94(0.05)	0.95(0.04)	0.95(0.04)	0.96(0.03)	0.95(0.03)
	β_1	Bias(SD)	0.00(0.04)	0.00(0.03)	0.00(0.02)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.95(0.04)	0.94(0.03)	0.96(0.02)	0.94(0.02)	0.94(0.02)
	β_2	Bias(SD)	0.00(0.08)	0.00(0.06)	-0.01(0.05)	0.00(0.04)	0.00(0.04)
		Cov(SE)	0.94(0.08)	0.95(0.06)	0.95(0.05)	0.95(0.05)	0.94(0.04)
M2(MCMLE)	β_0	Bias(SD)	0.00(0.08)	0.00(0.07)	0.00(0.06)	0.00(0.05)	0.00(0.05)
		Cov(SE)	0.94(0.08)	0.93(0.06)	0.93(0.06)	0.92(0.06)	0.94(0.05)
	β_1	Bias(SD)	0.00(0.05)	0.00(0.04)	0.00(0.03)	0.00(0.03)	0.00(0.02)
		Cov(SE)	0.93(0.05)	0.94(0.04)	0.94(0.03)	0.95(0.03)	0.96(0.03)
	β_2	Bias(SD)	-0.01(0.09)	0.01(0.07)	0.00(0.07)	0.00(0.06)	0.00(0.05)
		Cov(SE)	0.96(0.09)	0.95(0.07)	0.93(0.06)	0.93(0.06)	0.94(0.05)
M3(MCMLE)	β_0	Bias(SD)	0.00(0.13)	0.00(0.09)	0.00(0.08)	0.01(0.07)	0.00(0.06)
		Cov(SE)	0.94(0.12)	0.95(0.09)	0.95(0.08)	0.97(0.07)	0.95(0.06)
	β_1	Bias(SD)	0.01(0.09)	0.00(0.07)	0.00(0.06)	0.00(0.05)	0.00(0.05)
		Cov(SE)	0.97(0.09)	0.95(0.07)	0.95(0.06)	0.94(0.05)	0.95(0.05)
	β_2	Bias(SD)	0.02(0.17)	0.01(0.13)	0.00(0.11)	0.00(0.09)	0.00(0.10)
		Cov(SE)	0.95(0.17)	0.95(0.13)	0.96(0.11)	0.96(0.10)	0.93(0.09)
M4(MCMLE)	β_0	Bias(SD)	0.00(0.08)	0.00(0.06)	0.00(0.06)	0.00(0.06)	0.01(0.05)
		Cov(SE)	0.96(0.08)	0.96(0.06)	0.93(0.06)	0.93(0.05)	0.94(0.05)
	β_1	Bias(SD)	0.00(0.05)	0.00(0.04)	0.00(0.03)	0.00(0.03)	0.00(0.03)
		Cov(SE)	0.95(0.05)	0.95(0.04)	0.94(0.03)	0.93(0.03)	0.95(0.03)
	β_2	Bias(SD)	0.01(0.09)	0.00(0.07)	0.00(0.07)	0.00(0.06)	-0.01(0.06)
		Cov(SE)	0.95(0.09)	0.95(0.07)	0.94(0.06)	0.94(0.06)	0.94(0.05)
M5(MCMLE)	β_0	Bias(SD)	0.00(0.02)	0.00(0.02)	0.00(0.01)	0.00(0.01)	0.00(0.01)
		Cov(SE)	0.95(0.03)	0.95(0.02)	0.95(0.02)	0.96(0.01)	0.94(0.01)
	β_1	Bias(SD)	0.00(0.02)	0.00(0.01)	0.00(0.01)	0.00(0.01)	0.00(0.01)
		Cov(SE)	0.96(0.02)	0.94(0.01)	0.95(0.01)	0.96(0.01)	0.95(0.01)
	β_2	Bias(SD)	0.00(0.05)	0.00(0.03)	0.00(0.03)	0.00(0.02)	0.00(0.02)
		Cov(SE)	0.93(0.04)	0.94(0.03)	0.95(0.03)	0.96(0.02)	0.95(0.02)