Appendix

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Appendix A: Bounds on the Laplace spectrum

Li and Yau in [1] proved that the Laplace spectrum has the following universal lower bound

$$\lambda_n \geq \frac{d}{d+2} 4\pi^2 \left(\frac{n}{B_d V}\right)^{2/d} \quad \forall n > 0. \tag{1}$$

We notice that this lower bound does not depend on the shape.

1) Upper bounds: Several authors have investigated the upper bounds and the relative growth rate of the eigenvalues of the Dirichlet spectrum [2]. In [3], Yang provides an upper bound for the growth rate of the components for $n \geq 1$ as

$$\lambda_{n+1} < \left[1 + \frac{4}{d}\right] \frac{1}{n} \sum_{m=1}^{n} \lambda_m. \tag{2}$$

This equation can be transformed into a sequence of upper bounds by only knowing the first eigenvalue λ_1. Although sharp for the first few eigenvalues, the upper bound is too relaxed for the remaining modes. Cheng and Yang in [4] provides a much sharper upper bound for larger values of n and is valid for $n \geq 2$.

$$\lambda_{n+1} \leq C_0(d,n) n^{\frac{2}{d}} \lambda_1, \tag{3}$$

where

$$C_0(d,n) = 1 + \frac{a(\min(d, n-1))}{d}$$
$$a(1) \leq 2.64 \;\; a(2) \leq 2.27$$
$$\text{and } a(p) = 2.2 - 4\log(1 + \frac{p-3}{50}) \text{ for } p \geq 3,$$

where the bound only depends on the first eigenvalue and furthermore it is consistent with Weyl's asymptotic growth law.

Appendix B: Proofs of Lemmas and Corollaries

Before presenting the proofs for the corollaries let us provide a lemma that will be useful throughout this section.

Lemma 2. *For any $a, b \in \mathbb{R}$ such that $a > b > 0$ the function $f(a,b) = \frac{a-b}{ab}$ increases monotonously with increasing a and decreases monotonously with increasing b.*

Proof: Since f is differentiable it suffices to look at its partial derivatives $\frac{\partial f}{\partial a} = \frac{1}{a^2}$ and $\frac{\partial f}{\partial b} = -\frac{1}{b^2}$. ■

Corollary 1. *Let $\Omega_\lambda \subset \mathbb{R}^d$ and $\Omega_\xi \subset \mathbb{R}^d$ be any two closed domains with piecewise smooth boundaries and $\{\lambda\}_{n=1}^{\infty}$ and $\{\xi\}_{n=1}^{\infty}$ be their Laplace spectrum. Then the weighted spectral distance*

$$\rho(\Omega_\lambda, \Omega_\xi) = \left[\sum_{n=1}^{\infty} \left(\frac{|\lambda_n - \xi_n|}{\lambda_n \xi_n}\right)^p\right]^{1/p}$$

converges for $p > \frac{d}{2}$. Furthermore,

$$\rho(\Omega_\lambda, \Omega_\xi) < \left\{C + K \cdot \left[\zeta\left(\frac{2p}{d}\right) - 1 - \left(\frac{1}{2}\right)^{\frac{2p}{d}}\right]\right\}^{\frac{1}{p}}, \tag{4}$$

where $\zeta(\cdot)$ is the Riemann zeta function and the coefficients C and K are given as

$$C \triangleq \sum_{i=1,2} \left[\frac{d+2}{d \cdot 4\pi^2} \cdot \left(\frac{B_d \hat{V}}{i}\right)^{\frac{2}{d}} - \frac{1}{\mu} \cdot \left(\frac{d}{d+4}\right)^{i-1}\right]^p$$
$$K \triangleq \left[\frac{d+2}{d \cdot 4\pi^2} \cdot \left(B_d \hat{V}\right)^{\frac{2}{d}} - \frac{1}{\mu} \cdot \frac{d}{d+2.64}\right]^p$$
$$\hat{V} \triangleq \max(V(\Omega_\lambda), V(\Omega_\xi)), \quad \mu \triangleq \max(\lambda_1, \xi_1),$$

where $V(\cdot)$ denotes the volume (or area in 2D) of an object.

Proof: The following inequality results from combining the bounds specified in Section with Lemma -1

$$\frac{|\lambda_n - \xi_n|}{\lambda_n \xi_n} < \frac{d+2}{d \cdot 4\pi^2} \cdot \left(\frac{B_d \hat{V}}{n}\right)^{\frac{2}{d}} - \frac{1}{\mu} \cdot \left(\frac{d}{d+4}\right)^{n-1}$$

for $n = 1, 2$ and for $n \geq 3$

$$\frac{|\lambda_n - \xi_n|}{\lambda_n \xi_n} < \frac{d+2}{d \cdot 4\pi^2} \cdot \left(\frac{B_d \hat{V}}{n}\right)^{\frac{2}{d}} - \frac{1}{\mu} \cdot \frac{1}{C_0(d,n) n^{\frac{2}{d}}}$$
$$\leq \frac{d+2}{d \cdot 4\pi^2} \cdot \left(\frac{B_d \hat{V}}{n}\right)^{\frac{2}{d}} - \frac{1}{\mu} \cdot \frac{d}{(d+2.64) n^{\frac{2}{d}}},$$

Based on this component-wise bound we can write the infinite sum without the first two terms as

$$\sum_{n=3}^{\infty} \left(\frac{|\lambda_n - \xi_n|}{\lambda_n \xi_n}\right)^p < K \sum_{n=3}^{\infty} \left(\frac{1}{n}\right)^{\frac{2p}{d}},$$

which for $p > \frac{d}{2}$ converges to

$$K \sum_{n=3}^{\infty} \left(\frac{1}{n}\right)^{\frac{2p}{d}} = \zeta\left(\frac{2p}{d}\right) - 1 - \left(\frac{1}{2}\right)^{\frac{2p}{d}}$$

and diverges for $p \leq \frac{2}{d}$. Consequently, $\rho(\Omega_\lambda, \Omega_\xi)$ converges for $p > \frac{d}{2}$. Furthermore, extending the sum with the upper bounds for $n = 1, 2$ the following upper bound for the distance between Ω_λ and Ω_ξ holds

$$\rho(\Omega_\lambda, \Omega_\xi) < \left\{C + K \cdot \left[\zeta\left(\frac{2p}{d}\right) - 1 - \left(\frac{1}{2}\right)^{\frac{2p}{d}}\right]\right\}^{\frac{1}{p}}.$$

■

Corollary 2. *$\rho(\Omega_\lambda, \Omega_\xi)$ is a pseudometric for $d \geq 2$.*

To ease notation we define

$$\varrho_n(\Omega_\lambda, \Omega_\xi) \triangleq \frac{|\lambda_n - \xi_n|}{\lambda_n \xi_n}.$$

This leads to

$$\rho(\Omega_\lambda, \Omega_\xi) = \left[\sum_{n=1}^{\infty} \varrho_n^p(\Omega_\lambda, \Omega_\xi)\right]^{\frac{1}{p}}.$$

Proof: $\forall \Omega_\lambda \subset \mathbb{R}^d,\ \Omega_\xi \subset \mathbb{R}^d$ The first three points for this proof are trivial:

- $\rho(\Omega_\lambda, \Omega_\xi) > 0$ since $\varrho_n(\Omega_\lambda, \Omega_\xi) > 0\ \forall n$.
- $\rho(\Omega_\lambda, \Omega_\lambda) = 0$ since $|\lambda_n - \lambda_n| = 0\ \forall n$.
- $\rho(\Omega_\lambda, \Omega_\xi) = \rho(\Omega_\xi, \Omega_\lambda)$ since $|\lambda_n - \xi_n| = |\xi_n - \lambda_n|\ \forall n$

In order to prove the triangle inequality let us proceed with the case $\lambda_n \geq \xi_n$. The inverse case follows exactly the same way. Now, let $\Omega_\eta \subset \mathbb{R}^d$ be an arbitrary closed domain with piecewise smooth boundaries whose spectrum is given as $\{\eta_n\}_{n=1}^{\infty}$. Investigating

$$\frac{|\lambda_n - \eta_n|}{\lambda_n \eta_n} + \frac{|\eta_n - \xi_n|}{\eta_n \xi_n}$$

we notice that for each n there are only three possible cases:

1) $\lambda_n \geq \eta_n \geq \xi_n$, for which

$$\varrho_n(\Omega_\lambda, \Omega_\eta) + \varrho_n(\Omega_\eta, \Omega_\xi) = \varrho_n(\Omega_\lambda, \Omega_\xi),$$

2) $\lambda_n \geq \xi_n \geq \eta_n$, for which $\varrho_n(\Omega_\lambda, \Omega_\eta) \geq \varrho_n(\Omega_\lambda, \Omega_\xi)$ as a result of the Lemma -1. Due $\varrho_n(\Omega_\xi, \Omega_\eta) \geq 0$:

$$\varrho_n(\Omega_\lambda, \Omega_\eta) + \varrho_n(\Omega_\eta, \Omega_\xi) \geq \varrho_n(\Omega_\lambda, \Omega_\xi)$$

3) $\eta_n \geq \lambda_n \geq \xi_n$, for which $\varrho_n(\Omega_\eta, \Omega_\xi) \geq \varrho_n(\Omega_\lambda, \Omega_\xi)$ as a result of the Lemma -1 once again. And as in the previous case, due to $\varrho_n(\Omega_\eta, \Omega_\lambda) \geq 0$ we have

$$\varrho_n(\Omega_\lambda, \Omega_\eta) + \varrho_n(\Omega_\eta, \Omega_\xi) \geq \varrho_n(\Omega_\lambda, \Omega_\xi).$$

Thus $\forall n\ \varrho_n(\Omega_\lambda, \Omega_\eta) + \varrho_n(\Omega_\eta, \Omega_\xi) \geq \varrho_n(\Omega_\lambda, \Omega_\xi)$. Since $p > 1$ as $p > \frac{d}{2}$ for $d \geq 2$ the Minkowski Inequality states

$$\begin{aligned}\rho(\Omega_\lambda, \Omega_\eta) &+ \rho(\Omega_\eta, \Omega_\xi) \\ &= \left[\sum_{n=1}^{\infty} \varrho_n^p(\Omega_\lambda, \Omega_\eta)\right]^{\frac{1}{p}} + \left[\sum_{n=1}^{\infty} \varrho_n^p(\Omega_\eta, \Omega_\xi)\right]^{\frac{1}{p}} \\ &\geq \left[\sum_{n=1}^{\infty} \left(\varrho_n(\Omega_\lambda, \Omega_\eta) + \varrho_n(\Omega_\eta, \Omega_\xi)\right)^p\right]^{1/p}.\end{aligned}$$

When combined with the previous results, the outcome is the triangle inequality:

$$\rho(\Omega_\lambda, \Omega_\eta) + \rho(\Omega_\eta, \Omega_\xi) \geq \rho(\Omega_\lambda, \Omega_\xi)$$

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Lemma 1. *Let $\Omega_\lambda \mathbb{R}^d$ represent an object with piecewise smooth boundary and $\mathcal{D}(l,t) \triangleq \frac{e^{-\lambda_l t}}{Z(t)}$ be the corresponding influence ratio of mode l at t. Then for any two spectral indices $m > n > 0$*

$$\mathcal{D}(n,t) > \mathcal{D}(m,t), \quad \forall t > 0$$

and particularly for two t values such that $t_1 > t_2$

$$\frac{\mathcal{D}(m,t_1)}{\mathcal{D}(n,t_1)} < \frac{\mathcal{D}(m,t_2)}{\mathcal{D}(n,t_2)}.$$

Proof: The proof follows the properties of the exponential function and the properties of the spectrum of the Laplace operator. For $n < m$ we know that $\lambda_n < \lambda_m$ which leads to $e^{-\lambda_n t} > e^{-\lambda_m t}\ \forall t > 0$. Since the denominators are the same for both $\mathcal{D}(n,t)$ and $\mathcal{D}(m,t)$ then

$$\mathcal{D}(n,t) > \mathcal{D}(m,t)\ \forall t > 0.$$

For the second part of the lemma, we first compute the ratio

$$\frac{\mathcal{D}(m,t)}{\mathcal{D}(n,t)} = e^{-(\lambda_m - \lambda_n)t}.$$

Now based on $\lambda_m > \lambda_n$ and $e^{-(\lambda_m - \lambda_n)t}$ is monotonously decreasing with increasing t, it follows for $t_1 > t_2$ that

$$\frac{\mathcal{D}(m,t_1)}{\mathcal{D}(n,t_1)} = e^{-(\lambda_m - \lambda_n)t_1} < e^{-(\lambda_m - \lambda_n)t_2} = \frac{\mathcal{D}(m,t_2)}{\mathcal{D}(n,t_2)}.$$

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Corollary 3. *Let Ω_λ and Ω_ξ be two objects with piecewise smooth boundaries. Then for any two scalars with $p > d/2$, $q > d/2$, $p \geq q$ and for all n with $|\lambda_n - \xi_n| > 0$ there exists a $M > n$ so that $\forall m \geq M$*

$$\frac{\left(\frac{|\lambda_m - \xi_m|}{\lambda_m \xi_m}\right)^p}{\left(\frac{|\lambda_n - \xi_n|}{\lambda_n \xi_n}\right)^p} \leq \frac{\left(\frac{|\lambda_m - \xi_m|}{\lambda_m \xi_m}\right)^q}{\left(\frac{|\lambda_n - \xi_n|}{\lambda_n \xi_n}\right)^q}$$

Proof: From Corollary 1 we know that the series

$$\sum_{m=1}^{\infty} \left(\frac{|\lambda_m - \xi_m|}{\lambda_m \xi_m}\right)^q$$

converges. Then based on Cauchy's convergence criterion for series

$$\lim_{n \to \infty} \left(\frac{|\lambda_m - \xi_m|}{\lambda_m \xi_m}\right)^q = 0.$$

In other words, $\forall \epsilon > 0$ there exists a M such that

$$\left(\frac{|\lambda_m - \xi_m|}{\lambda_m \xi_m}\right)^q < \epsilon,\ \forall m > M.$$

Let n be an arbitrary index such that $|\lambda_n - \xi_n| > 0$. Consequently, also for $|\lambda_n - \xi_n|$, there exists a M such that $\forall m > M$

$$\frac{\left(\frac{|\lambda_m - \xi_m|}{\lambda_m \xi_m}\right)^q}{\left(\frac{|\lambda_n - \xi_n|}{\lambda_n \xi_n}\right)^q} < 1.$$

Since $p \geq q$ we can find a $k \geq 1$ such that $p = kq$. Then based on the above inequality $\forall m > M$

$$\begin{aligned}\frac{\left(\frac{|\lambda_m - \xi_m|}{\lambda_m \xi_m}\right)^p}{\left(\frac{|\lambda_n - \xi_n|}{\lambda_n \xi_n}\right)^p} &= \left[\frac{\left(\frac{|\lambda_m - \xi_m|}{\lambda_m \xi_m}\right)^q}{\left(\frac{|\lambda_n - \xi_n|}{\lambda_n \xi_n}\right)^q}\right]^k \\ &\leq \frac{\left(\frac{|\lambda_m - \xi_m|}{\lambda_m \xi_m}\right)^q}{\left(\frac{|\lambda_n - \xi_n|}{\lambda_n \xi_n}\right)^q}\end{aligned}$$

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Corollary 4. *Let $\rho^N(\Omega_\lambda, \Omega_\xi)$ be the truncated approximation of $\rho(\Omega_\lambda, \Omega_\xi)$ based on the first N modes and $\overline{\rho}^N(\Omega_\lambda, \Omega_\xi)$ of $\overline{\rho}(\Omega_\lambda, \Omega_\xi)$. Then $\forall p > d/2$*

$$\lim_{N\to\infty} |\rho - \rho^N| = 0$$

and

$$\lim_{N\to\infty} |\overline{\rho} - \overline{\rho}^N| = 0.$$

Furthermore, for a given $N \geq 3$ the truncation errors $|\rho - \rho^N|$ and $|\overline{\rho} - \overline{\rho}^N|$ can be bounded by

$$\begin{aligned} |\rho - \rho^N| &< \left\{C + K \cdot \left[\zeta\left(\frac{2p}{d}\right) - 1 - \left(\frac{1}{2}\right)^{\frac{2p}{d}}\right]\right\}^{\frac{1}{p}} \qquad (5)\\ &\quad -\left\{C + K \cdot \left[\sum_{n=3}^{N}\left(\frac{1}{n}\right)^{\frac{2p}{d}}\right]\right\}^{\frac{1}{p}} \\ |\overline{\rho} - \overline{\rho}_N| &< 1 - \left\{\frac{C + K \cdot \left[\sum_{n=3}^{N}\left(\frac{1}{n}\right)^{\frac{2p}{d}}\right]}{C + K \cdot \left[\zeta\left(\frac{2p}{d}\right) - 1 - \left(\frac{1}{2}\right)^{\frac{2p}{d}}\right]}\right\}^{\frac{1}{p}} \qquad (6)\end{aligned}$$

Proof: As before, to ease notation, let us again define

$$\varrho_n \triangleq \frac{|\lambda_n - \xi_n|}{\lambda_n \xi_n}. \qquad (7)$$

Then based on Corollary 1 we know that the sum $\sum_{n=1}^{\infty} \varrho_n^p$ exists and thus also the partial sums converge

$$\lim_{N\to\infty}\left|\sum_{n=1}^{\infty}\varrho_n^p - \sum_{n=1}^{N}\varrho_n^p\right| = \lim_{N\to\infty}\sum_{n=1}^{\infty}\varrho_n^p - \sum_{n=1}^{N}\varrho_n^p = 0.$$

Based on

$\forall a, b, d \in \mathbb{R}$ with $a, b, d \geq 0$ and $a^d - b^d \to 0 \Rightarrow a - b \to 0$

and

$\varrho_n \succeq 0$

we reach

$$\lim_{N\to\infty} |\rho - \rho_N| = \lim_{N\to\infty}\left[\sum_{n=1}^{\infty}\varrho_n^p\right]^{\frac{1}{p}} - \left[\sum_{n=1}^{N}\varrho_n^p\right]^{\frac{1}{p}} = 0$$

As the denominators for both $\overline{\rho}_N$ and $\overline{\rho}$ are the same, the above limit also yields $\lim_{N\to\infty} |\overline{\rho} - \overline{\rho}_N| = 0$.

The upper bounds for the truncation errors now is a direct result of Corollary 1 as

$$\begin{aligned} |\rho - \rho^N| &= \left[\sum_{n=1}^{\infty}\varrho_n^p\right]^{\frac{1}{p}} - \left[\sum_{n=1}^{N}\varrho_n^p\right]^{\frac{1}{p}} \\ &< \left\{C + K \cdot \left[\zeta\left(\frac{2p}{d}\right) - 1 - \left(\frac{1}{2}\right)^{\frac{2p}{d}}\right]\right\}^{\frac{1}{p}} \\ &\quad -\left\{C + K \cdot \left[\sum_{n=3}^{N}\left(\frac{1}{n}\right)^{\frac{2p}{d}}\right]\right\}^{\frac{1}{p}} \\ |\overline{\rho} - \overline{\rho}_N| &= \frac{\left[\sum_{n=1}^{\infty}\varrho^p\right]^{1/p} - \left[\sum_{n=1}^{N}\varrho^p\right]^{1/p}}{\left\{C + K\left(\zeta(2p/d) - 1 - 1/2^{2p/d}\right)\right\}^{1/p}} \\ &< 1 - \left\{\frac{C + K \cdot \left[\sum_{n=3}^{N}\left(\frac{1}{n}\right)^{\frac{2p}{d}}\right]}{C + K \cdot \left[\zeta\left(\frac{2p}{d}\right) - 1 - \left(\frac{1}{2}\right)^{\frac{2p}{d}}\right]}\right\}^{\frac{1}{p}}\end{aligned}$$

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Proposition 1. *Let $x, y \in \mathbb{R}$ be positive real values such that $y > x$. Then $\forall A, B \in \mathbb{R},\ A, B > 0$*

$$\frac{A + Bx}{A + By} > \frac{x}{y}.$$

Proof: Since $A, x, y > 0$, we can find two positive real values $k_1 > 0$ and $k_2 > 0$ such that $A = k_1 x$ and $A = k_2 y$. Furthermore, $y > x$ simply implies $k_1 > k_2$. Using k_1 and k_2 now we can write

$$\frac{A + Bx}{A + By} = \frac{k_1 x + Bx}{k_2 y + By} > \frac{k_2 x + Bx}{k_2 y + By} = \frac{x}{y}.$$

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