Web-based Supplementary Materials for Modeling Restricted Mean Survival Time under General Censoring Mechanisms

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A.1 Notations

To begin with, we review the essential notations needed for further discussion:

i: subject index, $i \in \{1, \ldots, n\}$
D_i: treatment-free death time
T_i: dependent censoring time; e.g. treatment
C_i: independent censoring time; e.g. administrative censoring
τ: end of follow up time
L: pre-specified time point of interest, $L \leq \tau$
$X_i = D_i \wedge T_i \wedge C_i$: observation time
$Y_i = X_i \wedge L$: restricted observation time by L
$\Delta_i = I(D_i \wedge L \leq T_i \wedge C_i)$: indicator for restricted survival time $D_i \wedge L$
$\Delta_i^D = I(D_i \leq T_i \wedge C_i)$: death indicator
$\Delta_i^T = I(T_i < D_i \wedge C_i)$: dependent censoring indicator
$\Delta_i^C = I(C_i < D_i \wedge T_i)$: independent censoring indicator
$\boldsymbol{Z}_i^D(t)$: time-dependent covariates that predict death D_i
$\boldsymbol{Z}_i^T(t)$: time-dependent covariates that predict dependent censoring T_i
$\boldsymbol{Z}_i^C$: baseline covariates that predict independent censoring C_i
$\boldsymbol{Z}_i(t)$: a covariate set that stacks $\boldsymbol{Z}_i^D(t)$,$\boldsymbol{Z}_i^T(t)$, $\boldsymbol{Z}_i^C$ together and removes redundancy
$\widetilde{\boldsymbol{Z}}_i(t) = \{\boldsymbol{Z}_i(u) : 0 \leq u \leq t\}$: observation history of all the covariates up to time t
$\lambda_i^T(t)$: hazard rate for dependent censoring T_i
$\lambda_i^C(t)$: hazard rate for independent censoring C_i
$\Lambda_i^T(t) = \int_0^t \lambda_i^T(u)du$: cumulative hazard rate for dependent censoring T_i
$\Lambda_i^C(t) = \int_0^t \lambda_i^C(u)du$: cumulative hazard rate for independent censoring C_i
$N_i^D(t) = I(X_i \leq t, \Delta_i^D = 1)$: counting process for death
$N_i^T(t) = I(X_i \leq t, \Delta_i^T = 1)$: counting process for dependent censoring
$N_i^C(t) = I(X_i \leq t, \Delta_i^C = 1)$: counting process for independent censoring
$R_i(t) = I(X_i \geq t)$: at risk process
$dM_i^T(t) = dN_i^T(t) - R_i(t)d\Lambda_i^T(t)$: zero mean process for dependent censoring
$dM_i^C(t) = dN_i^C(t) - R_i(t)d\Lambda_i^C(t)$: zero mean process for independent censoring

A.2 Model Assumptions

We have made these assumptions in our paper:

(a) Assume restricted mean lifetime conditional on baseline covariates $\mu_i(L) := E\{D_i \wedge L | \boldsymbol{Z}_i^D(0)\}$ follows the model structure as below,

$$g\left[\mu_i\left(L\right)\right] \equiv g\left[E\left\{D_i \wedge L | \boldsymbol{Z}_i^D\left(0\right)\right\}\right] = \boldsymbol{\beta}_D' \boldsymbol{Z}_i^D\left(0\right),$$

where $g(*)$ is a given smooth and strictly monotone link function and $\boldsymbol{\beta}_D$ is of our primary interest.

(b) Assume Cox proportional hazards model for dependent and independent censoring time T_i and C_i:

$$\lambda_i^T\left(t\right) = \lambda_0^T\left(t\right)\exp\left\{\boldsymbol{\beta}_T' \boldsymbol{Z}_i^T\left(t\right)\right\},$$
$$\lambda_i^C\left(t\right) = \lambda_0^C\left(t\right)\exp\left(\boldsymbol{\beta}_C' \boldsymbol{Z}_i^C\right).$$

(c) Assume no unmeasured confounders for dependent censoring T_i: for any $t > 0$,

$$\lim_{h \to 0} \frac{P\left\{X_i \in [t, t+h), \Delta_i^T = 1 | X_i \geq t, \widetilde{\boldsymbol{Z}}_i\left(t\right), D_i\right\}}{h} = \lim_{h \to 0} \frac{P\left\{X_i \in [t, t+h), \Delta_i^T = 1 | X_i \geq t, \widetilde{\boldsymbol{Z}}_i\left(t\right)\right\}}{h}.$$

(d) Assume independent censoring time is independent of either death time or dependent censoring time given baseline covariates; i.e.,

$$C_i \perp T_i | \boldsymbol{Z}_i\left(0\right), C_i \perp D_i | \boldsymbol{Z}_i\left(0\right).$$

A.3 Regularity Conditions

We specify the necessary regularity conditions (i)-(vii) as below.

(i) $\{X_i, \Delta_i^D, \Delta_i^T, \Delta_i^C, \widetilde{\boldsymbol{Z}}_i(X_i)\}, i = 1, \ldots, n$ are independently and identically distributed.

(ii) $P(R_i(t) = 1) > 0$ for $t \in (0, \tau], i = 1, \ldots, n$.

(iii) $|Z_{ik}(0)| + \int_0^\tau d|Z_{ik}(t)| < M_Z < \infty$ for $i = 1, \ldots, n$, where $Z_{ik}(t)$ are the kth components of $\boldsymbol{Z}_i(t)$.

(iv) $\Lambda_i^T(\tau) < \infty, \Lambda_i^C(\tau) < \infty$ and $\Lambda_i^T(t), \Lambda_i^C(t)$ are absolutely continuous for $t \in (0, \tau]$.

(v) There exist neighborhoods $\mathcal{B}_T$ of $\boldsymbol{\beta}_T$ and $\mathcal{B}_C$ of $\boldsymbol{\beta}_C$ such that for $k = 0, 1, 2$,

$$\sup_{t\in(0,\tau],\boldsymbol{\beta}\in\mathcal{B}_T} \left\| \frac{1}{n} \sum_{i=1}^{n} \exp\left\{\boldsymbol{\beta}' \boldsymbol{Z}_i^T(t)\right\} R_i(t)\, \boldsymbol{Z}_i^T(t)^{\otimes k} - \boldsymbol{r}_T^{(k)}(t;\boldsymbol{\beta}) \right\| \xrightarrow{p} 0,$$

$$\sup_{t\in(0,\tau],\boldsymbol{\beta}\in\mathcal{B}_C} \left\| \frac{1}{n} \sum_{i=1}^{n} \exp\left(\boldsymbol{\beta}' \boldsymbol{Z}_i^C\right) R_i(t)\, \boldsymbol{Z}_i^{C\otimes k} - \boldsymbol{r}_C^{(k)}(t;\boldsymbol{\beta}) \right\| \xrightarrow{p} 0,$$

where $\boldsymbol{v}^{\otimes 0} = 1, \boldsymbol{v}^{\otimes 1} = \boldsymbol{v}, \boldsymbol{v}^{\otimes 2} = \boldsymbol{v}'\boldsymbol{v}$ and

$$\boldsymbol{r}_T^{(k)}(t;\boldsymbol{\beta}) = E\left[\exp\left\{\boldsymbol{\beta}' \boldsymbol{Z}_i^T(t)\right\} R_i(t)\, \boldsymbol{Z}_i^T(t)^{\otimes k}\right], \tag{1}$$

$$\boldsymbol{r}_C^{(k)}(t;\boldsymbol{\beta}) = E\left\{\exp\left(\boldsymbol{\beta}' \boldsymbol{Z}_i^C\right) R_i(t)\, \boldsymbol{Z}_i^{C\otimes k}\right\}. \tag{2}$$

(vi) Define $h(x) = \partial g^{-1}(x)/\partial x$, where h exists and is continuous in an open neighborhood $\mathcal{B}_D$ of $\boldsymbol{\beta}_D$.

(vii) The matrices $\boldsymbol{A}(\boldsymbol{\beta}_D), \boldsymbol{\Omega}_T(\boldsymbol{\beta}_T), \boldsymbol{\Omega}_C(\boldsymbol{\beta}_C)$ are each positive definite, where

$$\boldsymbol{A}(\boldsymbol{\beta}) = E\left[\boldsymbol{Z}_i^D(0)^{\otimes 2}\, h\left\{\boldsymbol{\beta}_D' \boldsymbol{Z}_i^D(0)\right\}\right], \tag{3}$$

$$\boldsymbol{\Omega}_T(\boldsymbol{\beta}) = E\left[\int_0^\tau \left\{\frac{\boldsymbol{r}_T^{(2)}(t;\boldsymbol{\beta})}{r_T^{(0)}(t;\boldsymbol{\beta})} - \overline{\boldsymbol{z}}_T(t;\boldsymbol{\beta})^{\otimes 2}\right\} dN_i^T(t)\right], \tag{4}$$

$$\boldsymbol{\Omega}_C(\boldsymbol{\beta}) = E\left[\int_0^\tau \left\{\frac{\boldsymbol{r}_C^{(2)}(t;\boldsymbol{\beta})}{r_C^{(0)}(t;\boldsymbol{\beta})} - \overline{\boldsymbol{z}}_C(t;\boldsymbol{\beta})^{\otimes 2}\right\} dN_i^C(t)\right], \tag{5}$$

and

$$\overline{\boldsymbol{z}}_T(t;\boldsymbol{\beta}) = \frac{\boldsymbol{r}_T^{(1)}(t;\boldsymbol{\beta})}{r_T^{(0)}(t;\boldsymbol{\beta})}, \tag{6}$$

$$\overline{\boldsymbol{z}}_C(t;\boldsymbol{\beta}) = \frac{\boldsymbol{r}_C^{(1)}(t;\boldsymbol{\beta})}{r_C^{(0)}(t;\boldsymbol{\beta})}. \tag{7}$$

A.4 Outline of Derivation

Two estimating equation mentioned in our paper are

$$\boldsymbol{\Phi}^*(\boldsymbol{\beta}) := \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{\Phi}_i^*(\boldsymbol{\beta}) := \frac{1}{n}\sum_{i=1}^{n} \Delta_i W_i(Y_i)\left[Y_i - g^{-1}\left\{\boldsymbol{\beta}' \boldsymbol{Z}_i^D(0)\right\}\right] \boldsymbol{Z}_i^D(0) = \boldsymbol{0}, \tag{8}$$

where $W_i(t) = W_i^T(t) W_i^C(t)$, $W_i^T(t) = \exp\{\Lambda_i^T(t)\}$ and $W_i^C(t) = \exp\{\Lambda_i^C(t)\}$, and

$$\boldsymbol{\Phi}(\boldsymbol{\beta}) := \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{\Phi}_i(\boldsymbol{\beta}) := \frac{1}{n}\sum_{i=1}^{n} \Delta_i \widehat{W}_i(Y_i)\left[Y_i - g^{-1}\left\{\boldsymbol{\beta}' \boldsymbol{Z}_i^D(0)\right\}\right] \boldsymbol{Z}_i^D(0) = \boldsymbol{0}, \tag{9}$$

where $\widehat{W}_i(t) = \widehat{W}_i^T(t)\widehat{W}_i^C(t)$, $\widehat{W}_i^T(t) = \exp\{\widehat{\Lambda}_i^T(t)\}$ and $\widehat{W}_i^C(t) = \exp\{\widehat{\Lambda}_i^C(t)\}$.

We will first show (8) is unbiased, and then (9) satisfies that $\sqrt{n}\boldsymbol{\Phi}(\boldsymbol{\beta}_D)$ converges to a zero-mean Normal with variance $\boldsymbol{B}(\boldsymbol{\beta}_D) = E\{\boldsymbol{B}_i(\boldsymbol{\beta}_D)^{\otimes 2}\}$, where

$$\boldsymbol{B}\left(\boldsymbol{\beta}_D\right) = E\{\boldsymbol{B}_i\left(\boldsymbol{\beta}_D\right)^{\otimes 2}\}, \tag{10}$$

$$\begin{aligned}\boldsymbol{B}_i\left(\boldsymbol{\beta}\right) = &\boldsymbol{\epsilon}_i\left(\boldsymbol{\beta}\right) + \boldsymbol{K}_T(\boldsymbol{\beta})\boldsymbol{\Omega}_T(\boldsymbol{\beta}_T)^{-1}\boldsymbol{U}_i^T\left(\boldsymbol{\beta}_T\right) + \int_0^L \boldsymbol{H}_T(u;\boldsymbol{\beta})r_T^{(0)}(u;\boldsymbol{\beta}_T)^{-1}dM_i^T\left(u\right)\\ &+ \boldsymbol{K}_C(\boldsymbol{\beta})\boldsymbol{\Omega}_C(\boldsymbol{\beta}_C)^{-1}\boldsymbol{U}_i^C\left(\boldsymbol{\beta}_C\right) + \int_0^L \boldsymbol{H}_C(u;\boldsymbol{\beta})r_C^{(0)}(u;\boldsymbol{\beta}_C)^{-1}dM_i^C\left(u\right),\end{aligned} \tag{11}$$

$$\boldsymbol{\epsilon}_i\left(\boldsymbol{\beta}\right) = \Delta_i W_i\left(Y_i\right)\left[Y_i - g^{-1}\{\boldsymbol{\beta}'\boldsymbol{Z}_i^D\left(0\right)\}\right]\boldsymbol{Z}_i^D\left(0\right), \tag{12}$$

$$\boldsymbol{U}_i^T\left(\boldsymbol{\beta}_T\right) = \int_0^t \{\boldsymbol{Z}_i^T\left(u\right) - \overline{\boldsymbol{z}}_T(u;\boldsymbol{\beta}_T)\}dM_i^T\left(u\right), \tag{13}$$

$$\boldsymbol{U}_i^C\left(\boldsymbol{\beta}_C\right) = \int_0^t \{\boldsymbol{Z}_i^C - \overline{\boldsymbol{z}}_C(u;\boldsymbol{\beta}_C)\}dM_i^C\left(u\right), \tag{14}$$

$$\boldsymbol{K}_T\left(\boldsymbol{\beta}\right) = E\{\boldsymbol{\epsilon}_j\left(\boldsymbol{\beta}\right)\boldsymbol{D}_i^T\left(Y_i\right)'\}, \tag{15}$$

$$\boldsymbol{K}_C\left(\boldsymbol{\beta}\right) = E\{\boldsymbol{\epsilon}_j\left(\boldsymbol{\beta}\right)\boldsymbol{D}_i^C\left(Y_i\right)'\}, \tag{16}$$

$$\boldsymbol{H}_T\left(t;\boldsymbol{\beta}\right) = E[\boldsymbol{\epsilon}_j\left(\boldsymbol{\beta}\right)\exp\{\boldsymbol{\beta}_T'\boldsymbol{Z}_i^T\left(t\right)\}R_i\left(t\right)], \tag{17}$$

$$\boldsymbol{H}_C\left(t;\boldsymbol{\beta}\right) = E\{\boldsymbol{\epsilon}_j\left(\boldsymbol{\beta}\right)\exp(\boldsymbol{\beta}_C'\boldsymbol{Z}_i^C)R_i\left(t\right)\}, \tag{18}$$

$$\boldsymbol{D}_i^T\left(t\right) = \int_0^t \{\boldsymbol{Z}_i^T\left(u\right) - \overline{\boldsymbol{z}}_T\left(u;\boldsymbol{\beta}_T\right)\}d\Lambda_i^T\left(u\right), \tag{19}$$

$$\boldsymbol{D}_i^C\left(t\right) = \int_0^t \{\boldsymbol{Z}_i^C - \overline{\boldsymbol{z}}_C\left(u;\boldsymbol{\beta}_C\right)\}d\Lambda_i^C\left(u\right), \tag{20}$$

for any subject $i = 1, \ldots, n$, and $\boldsymbol{\Omega}_T(\boldsymbol{\beta}), \boldsymbol{\Omega}_C(\boldsymbol{\beta})$ are already defined in (4) and (5).

Let $\widehat{\boldsymbol{\beta}}_D$ denote the solution to (9). We will show that

(a) **(Consistency)** as $n \to \infty$, $\widehat{\boldsymbol{\beta}}_D$ converges in probability to $\boldsymbol{\beta}_D$.

(b) **(Asymptotic Properties)** as $n \to \infty$, $\sqrt{n}(\widehat{\boldsymbol{\beta}}_D - \boldsymbol{\beta}_D)$ converges to a zero-mean Normal with variance $\boldsymbol{A}(\boldsymbol{\beta}_D)^{-1}\boldsymbol{B}(\boldsymbol{\beta}_D)\boldsymbol{A}(\boldsymbol{\beta}_D)^{-1}$ with $\boldsymbol{A}(\boldsymbol{\beta})$ and $\boldsymbol{B}(\boldsymbol{\beta})$ defined in (3) and (10).

A.5 Unbiased Estimating Equation

Theorem 1 *Under regularity conditions (i)-(vii), the estimating equation (8) is unbiased at the true value of $\boldsymbol{\beta}_D$; i.e. $E\{\boldsymbol{\Phi}^*(\boldsymbol{\beta}_D)\} = 0$.*

Proof 1 *As defined in our paper, the i_{th} error term in (8) are independently and identically distributed. It would be enough to show that $E\{\boldsymbol{\epsilon}_i(\boldsymbol{\beta}_D)\} = \boldsymbol{0}$. This holds because the conditional expectation on $\boldsymbol{Z}_i^D(0)$ is unbiased:*

$$\begin{aligned}&E\left\{\boldsymbol{\epsilon}_i\left(\boldsymbol{\beta}_D\right)|\boldsymbol{Z}_i^D\left(0\right)\right\}\\ &= \boldsymbol{Z}_i^D\left(0\right)E\left\{W_i\left(Y_i\right)\Delta_i Y_i|\boldsymbol{Z}_i^D\left(0\right)\right\} - \boldsymbol{Z}_i^D\left(0\right)g^{-1}\left\{\boldsymbol{\beta}_D'\boldsymbol{Z}_i^D\left(0\right)\right\}E\left\{W_i\left(Y_i\right)\Delta_i|\boldsymbol{Z}_i^D\left(0\right)\right\}\\ &= \boldsymbol{Z}_i^D\left(0\right)E\left[E\left\{W_i\left(Y_i\right)\Delta_i Y_i|D_i\right\}|\boldsymbol{Z}_i^D\left(0\right)\right] - \boldsymbol{Z}_i^D\left(0\right)g^{-1}\left\{\boldsymbol{\beta}_D'\boldsymbol{Z}_i^D\left(0\right)\right\}E\left[E\left\{W_i\left(Y_i\right)\Delta_i|D_i\right\}|\boldsymbol{Z}_i^D\left(0\right)\right]\\ &= \boldsymbol{Z}_i^D\left(0\right)E\left[E\left\{\frac{I\left(T_i \geq D_i \wedge L, C_i \geq D_i \wedge L\right)}{P\left(T_i \geq D_i \wedge L, C_i \geq D_i \wedge L\right)}\left(D_i \wedge L\right)|D_i\right\}|\boldsymbol{Z}_i^D\left(0\right)\right]\\ &- \boldsymbol{Z}_i^D\left(0\right)g^{-1}\left\{\boldsymbol{\beta}_D'\boldsymbol{Z}_i^D\left(0\right)\right\}E\left[E\left\{\frac{I\left(T_i \geq D_i \wedge L, C_i \geq D_i \wedge L\right)}{P\left(T_i \geq D_i \wedge L, C_i \geq D_i \wedge L\right)}|D_i\right\}|\boldsymbol{Z}_i^D\left(0\right)\right]\\ &= \boldsymbol{Z}_i^D\left(0\right)E\left\{D_i \wedge L|\boldsymbol{Z}_i^D\left(0\right)\right\} - \boldsymbol{Z}_i^D\left(0\right)g^{-1}\left\{\boldsymbol{\beta}_D'\boldsymbol{Z}_i^D\left(0\right)\right\}\\ &= \boldsymbol{0}.\end{aligned}$$

Then averaging over the baseline covariates, $E\{\boldsymbol{\epsilon}_i(\boldsymbol{\beta}_D)\}$ and therefore $E\{\boldsymbol{\Phi}_i^*(\boldsymbol{\beta}_D)\}$ will be $\boldsymbol{0}$.

Theorem 2 *Under regularity conditions (i)-(vii), as $n \to \infty$, $\sqrt{n}\boldsymbol{\Phi}(\boldsymbol{\beta}_D)$ converges to a zero-mean Normal with variance $\boldsymbol{B}(\boldsymbol{\beta}_D)$ defined in (10).*

Proof 2 *As shown in Zhang and Schaubel (2011), the weight involved with dependent censoring time T_i can be written as*

$$\sqrt{n}\left\{\widehat{W}_i^T(t) - W_i^T(t)\right\} = \frac{1}{\sqrt{n}} W_i^T(t) \left\{ \boldsymbol{D}_i^T(t)' \boldsymbol{\Omega}_T(\boldsymbol{\beta}_T)^{-1} \sum_{j=1}^n \boldsymbol{U}_j^T(\boldsymbol{\beta}_T) + \sum_{j=1}^n J_{ij}^T(t) \right\} + o_p(1),$$

with defined $\boldsymbol{D}_i^T(t), \boldsymbol{U}_i^T(\boldsymbol{\beta}_T), \boldsymbol{\Omega}_T(\boldsymbol{\beta}), \boldsymbol{r}_T^{(k)}(t), \overline{\boldsymbol{z}}_T(t;\boldsymbol{\beta})$ in (19), (13), (4) , (1), (6) and

$$J_{ij}^T(t) = \int_0^t \exp\left\{\boldsymbol{\beta}_T' \boldsymbol{Z}_i^T(u)\right\} R_i(u)\, r_T^{(0)}(u;\boldsymbol{\beta}_T)^{-1}\, dM_j^T(u).$$

And we can derive the similar formula for independent censoring time C_i,

$$\sqrt{n}\left\{\widehat{W}_i^C(t) - W_i^C(t)\right\} = \frac{1}{\sqrt{n}} W_i^C(t) \left\{ \boldsymbol{D}_i^C(t)' \boldsymbol{\Omega}_C(\boldsymbol{\beta}_C)^{-1} \sum_{j=1}^n \boldsymbol{U}_j^C(\boldsymbol{\beta}_C) + \sum_{j=1}^n J_{ij}^C(t) \right\} + o_p(1),$$

with defined $\boldsymbol{D}_i^C(t), \boldsymbol{U}_i^C(\boldsymbol{\beta}_C), \boldsymbol{\Omega}_C(\boldsymbol{\beta}), \boldsymbol{r}_C^{(k)}(t), \overline{\boldsymbol{z}}_C(t;\boldsymbol{\beta})$ in (20), (14), (5) , (2), (7)and

$$J_{ij}^C(t) = \int_0^t \exp\left(\boldsymbol{\beta}_C' \boldsymbol{Z}_i^C\right) R_i(u)\, r_C^{(0)}(u;\boldsymbol{\beta}_C)^{-1}\, dM_j^C(u).$$

Rewrite the target vector as

$$\begin{aligned}
\sqrt{n}\boldsymbol{\Phi}(\boldsymbol{\beta}) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \Delta_i(Y_i)[Y_i - g^{-1}\{\boldsymbol{\beta}' \boldsymbol{Z}_i^D(0)\}]\boldsymbol{Z}_i^D(0)\, \widehat{W}_i^T(Y_i)\, \widehat{W}_i^C(Y_i) \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n \Delta_i(Y_i)[Y_i - g^{-1}\{\boldsymbol{\beta}' \boldsymbol{Z}_i^D(0)\}]\boldsymbol{Z}_i^D(0) \left[W_i^T(Y_i)\, W_i^C(Y_i)\right. && (21) \\
&+ W_i^C(Y_i)\left\{\widehat{W}_i^T(Y_i) - W_i^T(Y_i)\right\} && (22) \\
&+ W_i^T(Y_i)\left\{\widehat{W}_i^C(Y_i) - W_i^C(Y_i)\right\} && (23) \\
&+ \left.\left\{\widehat{W}_i^C(Y_i) - W_i^C(Y_i)\right\}\left\{\widehat{W}_i^T(Y_i) - W_i^T(Y_i)\right\}\right] && (24)
\end{aligned}$$

- *The first part (21) is just*

  $$(21) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \boldsymbol{\epsilon}_i(\boldsymbol{\beta})$$

  where $\boldsymbol{\epsilon}_i(\boldsymbol{\beta})$ was defined in (12).

- *The second part (22) involves the difference between estimated and true IPCW weights for T:*

$$\begin{aligned}
(22) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \boldsymbol{\epsilon}_i(\boldsymbol{\beta}) \left\{\widehat{W}_i^T(Y_i) - W_i^T(Y_i)\right\} \\
&= \frac{1}{n^{1.5}} \sum_{i=1}^n \boldsymbol{\epsilon}_i(\boldsymbol{\beta}) \left\{ \boldsymbol{D}_i^T(Y_i)' \boldsymbol{\Omega}_T(\boldsymbol{\beta}_T)^{-1} \sum_{j=1}^n \boldsymbol{U}_j^T(\boldsymbol{\beta}_T) + \sum_{j=1}^n J_{ij}^T(Y_i) \right\} + \boldsymbol{o}_p(1) \\
&= \frac{1}{n^{1.5}} \sum_{i=1}^n \sum_{j=1}^n \boldsymbol{\epsilon}_i(\boldsymbol{\beta})\, \boldsymbol{D}_i^T(Y_i)' \boldsymbol{\Omega}_T(\boldsymbol{\beta}_T)^{-1} \boldsymbol{U}_j^T(\boldsymbol{\beta}_T) && (25) \\
&+ \frac{1}{n^{1.5}} \sum_{i=1}^n \sum_{j=1}^n \boldsymbol{\epsilon}_i(\boldsymbol{\beta})\, J_{ij}^T(Y_i) + \boldsymbol{o}_p(1) && (26)
\end{aligned}$$

Eq. (25) is simplified as

$$(25) = \frac{1}{n^{1.5}} \sum_{i=1}^{n} \sum_{j=1}^{n} \boldsymbol{\epsilon}_i (\boldsymbol{\beta}) \boldsymbol{D}_i^T (Y_i)' \boldsymbol{\Omega}_T (\boldsymbol{\beta}_T)^{-1} \boldsymbol{U}_j^T (\boldsymbol{\beta}_T)$$
$$= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left\{ \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{\epsilon}_i (\boldsymbol{\beta}) \boldsymbol{D}_i^T (Y_i)' \right\} \boldsymbol{\Omega}_T (\boldsymbol{\beta}_T)^{-1} \boldsymbol{U}_j^T (\boldsymbol{\beta}_T)$$

where $\boldsymbol{K}_T(\boldsymbol{\beta}) = E\{\boldsymbol{\epsilon}_i(\boldsymbol{\beta})\boldsymbol{D}_i^T(Y_i)'\}$ *was defined in (15), then*

$$(25) = \frac{1}{\sqrt{n}} \boldsymbol{K}_T(\boldsymbol{\beta}) \boldsymbol{\Omega}_T (\boldsymbol{\beta}_T)^{-1} \sum_{j=1}^{n} \boldsymbol{U}_j^T (\boldsymbol{\beta}_T)$$

Since $J_{ij}^T (Y_i)$ *can be written as*

$$J_{ij}^T (Y_i) = \int_0^{Y_i} \exp\left\{ \boldsymbol{\beta}_T' \boldsymbol{Z}_i^T (u) \right\} R_i (u) \, r_T^{(0)} (u; \boldsymbol{\beta}_T)^{-1} \, dM_j^T (u)$$
$$= \int_0^{L} \exp\left\{ \boldsymbol{\beta}_T' \boldsymbol{Z}_i^T (u) \right\} I (X_i \geq u) \, I (X_i \wedge L \geq u) \, r_T^{(0)} (u; \boldsymbol{\beta}_T)^{-1} \, dM_j^T (u)$$
$$= \int_0^{L} \exp\left\{ \boldsymbol{\beta}_T' \boldsymbol{Z}_i^T (u) \right\} I (X_i \geq u) \, r_T^{(0)} (u; \boldsymbol{\beta}_T)^{-1} \, dM_j^T (u)$$
$$\equiv \int_0^{L} \exp\left\{ \boldsymbol{\beta}_T' \boldsymbol{Z}_i^T (u) \right\} R_i (u) \, r_T^{(0)} (u; \boldsymbol{\beta}_T)^{-1} \, dM_j^T (u),$$

Eq. (26) is simplified as

$$(26) = \frac{1}{n^{1.5}} \sum_{i=1}^{n} \sum_{j=1}^{n} \boldsymbol{\epsilon}_i (\boldsymbol{\beta}) \left[\int_0^{L} \exp\left\{ \boldsymbol{\beta}_T' \boldsymbol{Z}_i^T (u) \right\} R_i (u) \, r_T^{(0)} (u; \boldsymbol{\beta}_T)^{-1} \, dM_j^T (u) \right]$$
$$= \frac{1}{\sqrt{n}} \int_0^{L} \left[\frac{1}{n} \sum_{i=1}^{n} \boldsymbol{\epsilon}_i (\boldsymbol{\beta}) \exp\left\{ \boldsymbol{\beta}_T' \boldsymbol{Z}_i^T (u) \right\} R_i (u) \right] r_T^{(0)} (u; \boldsymbol{\beta}_T)^{-1} \left\{ d \sum_{j=1}^{n} M_j^T (u) \right\}.$$

where $\boldsymbol{H}_T(u;\boldsymbol{\beta}) = E[\boldsymbol{\epsilon}_i(\boldsymbol{\beta}) \exp\{\boldsymbol{\beta}_T' \boldsymbol{Z}_i^T(u)\} R_i(u)]$ *was defined in (17), then*

$$(26) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_0^{L} \boldsymbol{H}_T (u; \boldsymbol{\beta}) \, r_T^{(0)} (u; \boldsymbol{\beta}_T)^{-1} \, dM_i^T (u).$$

To sum up, (22) can be rewritten as:

$$(22) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\boldsymbol{K}_T(\boldsymbol{\beta}) \boldsymbol{\Omega}_T(\boldsymbol{\beta}_T)^{-1} \boldsymbol{U}_i^T (\boldsymbol{\beta}_T) + \int_0^{L} \boldsymbol{H}_T(u;\boldsymbol{\beta}) r_T^{(0)}(u;\boldsymbol{\beta}_T)^{-1} dM_i^T (u) \right] + \boldsymbol{o}_p (1).$$

- *Similarly, (23) can be rewritten as:*

$$(23) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\boldsymbol{K}_C(\boldsymbol{\beta}) \boldsymbol{\Omega}_C(\boldsymbol{\beta}_C)^{-1} \boldsymbol{U}_i^C (\boldsymbol{\beta}_C) + \int_0^{L} \boldsymbol{H}_C(u;\boldsymbol{\beta}) r_C^{(0)}(u;\boldsymbol{\beta}_C)^{-1} dM_i^C (u) \right] + \boldsymbol{o}_p (1).$$

where $\boldsymbol{K}_C(\boldsymbol{\beta}) = E\{\boldsymbol{\epsilon}_i(\boldsymbol{\beta})\boldsymbol{D}_i^C(Y_i)'\}$ *and* $\boldsymbol{H}_C(u;\boldsymbol{\beta}) = E\{\boldsymbol{\epsilon}_i(\boldsymbol{\beta}) \exp(\boldsymbol{\beta}_C' \boldsymbol{Z}_i^C) R_i(u)\}$ *were defined in (18).*

- *Eq. (24) can be rewritten as:*

$$
\begin{aligned}
(24) &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Delta_i \left(Y_i\right) \left[Y_i - g^{-1}\{\boldsymbol{\beta}' \boldsymbol{Z}_i^D \left(0\right)\}\right] \boldsymbol{Z}_i^D \left(0\right) \left\{\widehat{W}_i^C \left(Y_i\right) - W_i^C \left(Y_i\right)\right\} \left\{\widehat{W}_i^T \left(Y_i\right) - W_i^T \left(Y_i\right)\right\} \\
&= \frac{1}{n^{2.5}} \sum_{i=1}^{n} \Delta_i \left(Y_i\right) \left[Y_i - g^{-1}\{\boldsymbol{\beta}' \boldsymbol{Z}_i^D \left(0\right)\}\right] \boldsymbol{Z}_i^D \left(0\right) W_i^T \left(Y_i\right) \left\{\boldsymbol{D}_i^T \left(Y_i\right)' \boldsymbol{\Omega}_T \left(\boldsymbol{\beta}_T\right)^{-1} \sum_{j=1}^{n} \boldsymbol{U}_j^T \left(\boldsymbol{\beta}_T\right) + \sum_{j=1}^{n} J_{ij}^T \left(Y_i\right) + \boldsymbol{o}_p \left(\sqrt{n}\right)\right\} \\
&* W_i^C \left(Y_i\right) \left\{\boldsymbol{D}_i^C \left(Y_i\right)' \boldsymbol{\Omega}_C \left(\boldsymbol{\beta}_C\right)^{-1} \sum_{k=1}^{n} \boldsymbol{U}_k^C \left(\boldsymbol{\beta}_C\right) + \sum_{k=1}^{n} J_{ik}^C \left(Y_i\right) + \boldsymbol{o}_p \left(\sqrt{n}\right)\right\} \\
&= \frac{1}{n^{2.5}} \sum_{i=1}^{n} \boldsymbol{\epsilon}_i \left(\boldsymbol{\beta}\right) \left\{\boldsymbol{D}_i^T \left(Y_i\right)' \boldsymbol{\Omega}_T \left(\boldsymbol{\beta}_T\right)^{-1} \sum_{j=1}^{n} \boldsymbol{U}_j^T \left(\boldsymbol{\beta}_T\right) + \sum_{j=1}^{n} J_{ij}^T \left(Y_i\right) + \boldsymbol{o}_p \left(\sqrt{n}\right)\right\} \left\{\boldsymbol{D}_i^C \left(Y_i\right)' \boldsymbol{\Omega}_C \left(\boldsymbol{\beta}_C\right)^{-1} \sum_{k=1}^{n} \boldsymbol{U}_k^C \left(\boldsymbol{\beta}_C\right) + \sum_{k=1}^{n} J_{ik}^C \left(Y_i\right) + \boldsymbol{o}_p \left(\sqrt{n}\right)\right\} \\
&= \frac{1}{n^{2.5}} \sum_{i=1}^{n} \boldsymbol{\epsilon}_i \left(\boldsymbol{\beta}\right) \left\{\boldsymbol{D}_i^T \left(Y_i\right)' \boldsymbol{\Omega}_T \left(\boldsymbol{\beta}_T\right)^{-1} \sum_{j=1}^{n} \boldsymbol{U}_j^T \left(\boldsymbol{\beta}_T\right) + \sum_{j=1}^{n} J_{ij}^T \left(Y_i\right)\right\} \left\{\boldsymbol{D}_i^C \left(Y_i\right)' \boldsymbol{\Omega}_C \left(\boldsymbol{\beta}_C\right)^{-1} \sum_{k=1}^{n} \boldsymbol{U}_k^C \left(\boldsymbol{\beta}_C\right) + \sum_{k=1}^{n} J_{ik}^C \left(Y_i\right)\right\} + \boldsymbol{o}_p \left(1\right) \\
&= \frac{1}{n^{2.5}} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \boldsymbol{\epsilon}_i \left(\boldsymbol{\beta}\right) \boldsymbol{D}_i^T \left(Y_i\right)' \boldsymbol{\Omega}_T \left(\boldsymbol{\beta}_T\right)^{-1} \boldsymbol{U}_j^T \left(\boldsymbol{\beta}_T\right) \boldsymbol{D}_i^C \left(Y_i\right)' \boldsymbol{\Omega}_C \left(\boldsymbol{\beta}_C\right)^{-1} \boldsymbol{U}_k^C \left(\boldsymbol{\beta}_C\right) && (27) \\
&+ \frac{1}{n^{2.5}} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \boldsymbol{\epsilon}_i \left(\boldsymbol{\beta}\right) J_{ij}^T \left(Y_i\right) \boldsymbol{D}_i^C \left(Y_i\right)' \boldsymbol{\Omega}_C \left(\boldsymbol{\beta}_C\right)^{-1} \boldsymbol{U}_k^C \left(\boldsymbol{\beta}_C\right) && (28) \\
&+ \frac{1}{n^{2.5}} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \boldsymbol{\epsilon}_i \left(\boldsymbol{\beta}\right) \boldsymbol{D}_i^T \left(Y_i\right)' \boldsymbol{\Omega}_T \left(\boldsymbol{\beta}_T\right)^{-1} \boldsymbol{U}_j^T \left(\boldsymbol{\beta}_T\right) J_{ik}^C \left(\boldsymbol{\beta}_C\right) && (29) \\
&+ \frac{1}{n^{2.5}} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \boldsymbol{\epsilon}_i \left(\boldsymbol{\beta}\right) J_{ij}^T \left(Y_i\right) J_{ik}^C \left(Y_i\right) + \boldsymbol{o}_p \left(1\right) && (30)
\end{aligned}
$$

  Eq. (27)-(30) can be shown to be negligible.

To sum up, we can rewrite $\sqrt{n}\boldsymbol{\Phi}(\boldsymbol{\beta})$ as

$$n\boldsymbol{\Phi}\left(\boldsymbol{\beta}\right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \boldsymbol{B}_i\left(\boldsymbol{\beta}\right) + \boldsymbol{o}_p\left(1\right),$$

where as defined in (11), Since we have defined $\boldsymbol{B}(\boldsymbol{\beta}) = E\{\boldsymbol{B}_i(\boldsymbol{\beta})^{\otimes 2}\}$ in (10), then we have proven that

$$\sqrt{n}\boldsymbol{\Phi}\left(\boldsymbol{\beta}_D\right) \xrightarrow{D} Normal\left(\boldsymbol{0}, \boldsymbol{B}\left(\boldsymbol{\beta}_D\right)\right),$$

following that the mean of each term in the summation above is $\boldsymbol{0}$ at $\boldsymbol{\beta}_D$.

A.6 Consistency

Theorem 3 *Under regularity conditions (i)-(vii), as $n \to \infty$, $\widehat{\boldsymbol{\beta}}_D \xrightarrow{p} \boldsymbol{\beta}_D$.*

Proof 3 *We use the Inverse Function Theorem (Foutz, 1977) by verifying the following conditions:*

- $\partial\boldsymbol{\Phi}(\boldsymbol{\beta})/\partial\boldsymbol{\beta}'$ *exists and is continuous in an open neighborhood $\mathcal{B}_D$ of $\boldsymbol{\beta}_D$.*
- $-n^{-1}\partial\boldsymbol{\Phi}(\boldsymbol{\beta})/\partial\boldsymbol{\beta}'|_{\boldsymbol{\beta}=\boldsymbol{\beta}_D}$ *is positive definite with probability 1 as $n \to \infty$.*
- $-n^{-1}\partial\boldsymbol{\Phi}(\boldsymbol{\beta})/\partial\boldsymbol{\beta}'$ *converges in probability to a fixed function uniformly in an open neighborhood $\mathcal{B}_D$ of $\boldsymbol{\beta}_D$.*
- *Asymptotic unbiasedness of the estimating function:$-\boldsymbol{\Phi}(\boldsymbol{\beta}_D)/n \xrightarrow{p} \boldsymbol{0}$.*

  We know that

$$\frac{\partial\boldsymbol{\Phi}\left(\boldsymbol{\beta}\right)}{\partial\boldsymbol{\beta}'} = -\sum_{i=1}^{n} \Delta_i \widehat{W}_i^T\left(Y_i\right) \widehat{W}_i^C\left(Y_i\right) h\left\{\boldsymbol{\beta}' \boldsymbol{Z}_i^D\left(0\right)\right\} \boldsymbol{Z}_i^D\left(0\right)^{\otimes 2}.$$

where $h(x) = \partial g^{-1}(x)/\partial x$. We will show that this derivative vector satisfies all the necessary conditions above.

- *The first condition here holds because of the regularity condition (vi), which states that h exists and is continuous in an open neighborhood $\mathcal{B}_D$ of $\boldsymbol{\beta}_D$.*

- *As to the second condition here, we know*

$$
\begin{aligned}
&-\frac{1}{n}\frac{\partial \mathbf{\Phi}(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}'}|_{\boldsymbol{\beta}=\boldsymbol{\beta}_D}\\
&= E\left[\Delta_i W_i^T(Y_i) W_i^C(Y_i)\, h\left\{\boldsymbol{\beta}_D' \boldsymbol{Z}_i^D(0)\right\} \boldsymbol{Z}_i^D(0)^{\otimes 2}\right] + \boldsymbol{o}_p(1)\\
&= E\left[E\left\{\frac{I(T_i \wedge C_i \geq D_i \wedge L)}{P(T_i \geq D_i \wedge L)\, P(C_i \geq D_i \wedge L)}|D_i, \boldsymbol{Z}_i^D(0)\right\} h\left\{\boldsymbol{\beta}' \boldsymbol{Z}_i^D(0)\right\} \boldsymbol{Z}_i^D(0)^{\otimes 2}\right] + \boldsymbol{o}_p(1)\\
&= E\left[E\left\{\frac{I(T_i \geq D_i \wedge L)\, I(C_i \geq D_i \wedge L)}{P(T_i \geq D_i \wedge L)\, P(C_i \geq D_i \wedge L)}|D_i, \boldsymbol{Z}_i^D(0)\right\} h\left\{\boldsymbol{\beta}' \boldsymbol{Z}_i^D(0)\right\} \boldsymbol{Z}_i^D(0)^{\otimes 2}\right] + \boldsymbol{o}_p(1)\\
&= E\left[h\left\{\boldsymbol{\beta}' \boldsymbol{Z}_i^D(0)\right\} \boldsymbol{Z}_i^D(0)^{\otimes 2}\right] + \boldsymbol{o}_p(1)\\
&\equiv \boldsymbol{A}(\boldsymbol{\beta}).
\end{aligned}
$$

  where $\boldsymbol{A}(\boldsymbol{\beta})$ is defined as (3). Since we have assumed $\boldsymbol{A}(\boldsymbol{\beta}_D)$ is positive definite, the second condition holds here too.

- *The third condition holds by the law of large numbers.*

- *Finally, since we have proven that*

$$\sqrt{n}\mathbf{\Phi}(\boldsymbol{\beta}_D) \xrightarrow{D} Normal(0, \boldsymbol{B}(\boldsymbol{\beta}_D)).$$

  The last condition holds by Chebyshev's inequality.

Having verified all the four conditions, we can argue that $\widehat{\boldsymbol{\beta}}_D \xrightarrow{p} \boldsymbol{\beta}_D$ follows from Inverse Function Theorem.

A.7 Asymptotic Distribution

Theorem 4 *Under regularity conditions (i)-(vii), as $n \to \infty$,*

$$\sqrt{n}\left(\widehat{\boldsymbol{\beta}}_D - \boldsymbol{\beta}_D\right) \xrightarrow{D} Normal\left(\mathbf{0}, \boldsymbol{A}(\boldsymbol{\beta}_D)^{-1}\boldsymbol{B}(\boldsymbol{\beta}_D)\boldsymbol{A}(\boldsymbol{\beta}_D)^{-1}\right).$$

Proof 4 *Taylor expansion of $\mathbf{\Phi}\left(\widehat{\boldsymbol{\beta}}_D\right)$ around $\boldsymbol{\beta}_D$ is:*

$$\mathbf{0} = \mathbf{\Phi}\left(\widehat{\boldsymbol{\beta}}_D\right) = \mathbf{\Phi}(\boldsymbol{\beta}_D) + \frac{\partial \mathbf{\Phi}(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}}|_{\boldsymbol{\beta}=\widetilde{\boldsymbol{\beta}}}\left(\widehat{\boldsymbol{\beta}}_D - \boldsymbol{\beta}_D\right),$$

where $\widetilde{\boldsymbol{\beta}}$ lies between $\widehat{\boldsymbol{\beta}}_D$ and $\boldsymbol{\beta}_D$. So

$$
\begin{aligned}
\sqrt{n}\left(\widehat{\boldsymbol{\beta}}_D - \boldsymbol{\beta}_D\right) &= -\left\{\frac{\partial \mathbf{\Phi}(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}}|_{\boldsymbol{\beta}=\widetilde{\boldsymbol{\beta}}}\right\}^{-1} \sqrt{n}\mathbf{\Phi}(\boldsymbol{\beta}_D)\\
&= \left[-\frac{1}{n}\sum_{i=1}^{n} \Delta_i \widehat{W}_i(Y_i)\, \boldsymbol{Z}_i^D(0)^{\otimes 2}\, h\left\{\widetilde{\boldsymbol{\beta}}' \boldsymbol{Z}_i^D(0)\right\}\right]^{-1} \sqrt{n}\mathbf{\Phi}(\boldsymbol{\beta}_D)\\
&= \boldsymbol{A}(\boldsymbol{\beta}_D)^{-1} \sqrt{n}\mathbf{\Phi}(\boldsymbol{\beta}_D) + \boldsymbol{o}_p(1).
\end{aligned}
$$

Following Theorem 2, it holds that

$$\sqrt{n}\left(\widehat{\boldsymbol{\beta}}_D - \boldsymbol{\beta}_D\right) \xrightarrow{D} Normal\left(\mathbf{0}, \boldsymbol{A}(\boldsymbol{\beta}_D)^{-1} \boldsymbol{B}(\boldsymbol{\beta}_D)\, \boldsymbol{A}(\boldsymbol{\beta}_D)^{-1}\right).$$

B Model Selection Criteria

We suggest using Concordance Statistics (IOC), Mean Absolute Deviation (MAD) and Mean Squared Deviation (MSD) to select the proper link function. To simplify the notation, we denote $D_i^L = D_i \wedge L$ and its predicted value as $\widehat{D_i^L} = g^{-1}\{\boldsymbol{\beta}_D' \boldsymbol{Z}_i^D(0)\}$. Due to the occurrence of censoring, we observe $X_i = D_i^L \wedge T_i \wedge C_i$ for subject i.

Our version of IOC is adapted from Frank Harrell's formula of concordance (Harrell, 1996; Heagerty, 2005, Uno et al., 2011):

$$\mathrm{IOC} = \frac{\sum_{i=1}^n \sum_{j=1}^n \Delta_i \widehat{W}_i\left(Y_i\right) \widehat{W}_j\left(Y_i\right) I\left(Y_i < Y_j, \widehat{D_i^L} < \widehat{D_j^L}\right)}{\sum_{i=1}^n \sum_{j=1}^n \Delta_i \widehat{W}_i\left(Y_i\right) \widehat{W}_j\left(Y_i\right) I\left(Y_i < Y_j\right)}.$$

It converges to a censoring distribution free quantity $P(\widehat{D_i^L} < \widehat{D_j^L} | D_i^L < D_j^L)$ because

(i) as to the numerator,

$$\begin{aligned}
&\frac{1}{n^2} \Delta_i \widehat{W}_i\left(Y_i\right) \widehat{W}_j\left(Y_i\right) I\left(Y_i < Y_j, \widehat{D_i^L} < \widehat{D_j^L}\right) \\
&\xrightarrow{p} E\left\{I\left(D_i^L \leq T_i \wedge C_i\right) W_i\left(Y_i\right) W_j\left(Y_i\right) I\left(Y_i < Y_j\right) I\left(\widehat{D_i^L} < \widehat{D_j^L}\right)\right\} \\
&= E\left\{I\left(D_i^L \leq T_i \wedge C_i\right) W_i\left(D_i^L\right) W_j\left(D_i^L\right) I\left(T_j \wedge C_j > D_i^L\right) I\left(D_j^L > D_i^L\right) I\left(\widehat{D_i^L} < \widehat{D_j^L}\right)\right\} \\
&= E\left[E\left\{\frac{I\left(T_i \wedge C_i \geq D_i \wedge L\right) I\left(T_j \wedge C_j > D_i \wedge L\right)}{P\left(T_i \wedge C_i > D_i \wedge L\right) P\left(T_j \wedge C_j > D_i \wedge L\right)} I\left(D_i^L < D_j^L, \widehat{D_i^L} < \widehat{D_j^L}\right) | \boldsymbol{Z}_i^D(0), D_i\right\}\right] \\
&\xrightarrow{p} P\left(D_i^L < D_j^L, \widehat{D_i^L} < \widehat{D_j^L}\right).
\end{aligned}$$

(ii) Similarly, the denominator follows that

$$\frac{1}{n^2} \Delta_i \widehat{W}_i(Y_i) \widehat{W}_j(Y_i) I(Y_i < Y_j) \xrightarrow{p} P(D_i^L < D_j^L).$$

(iii) So

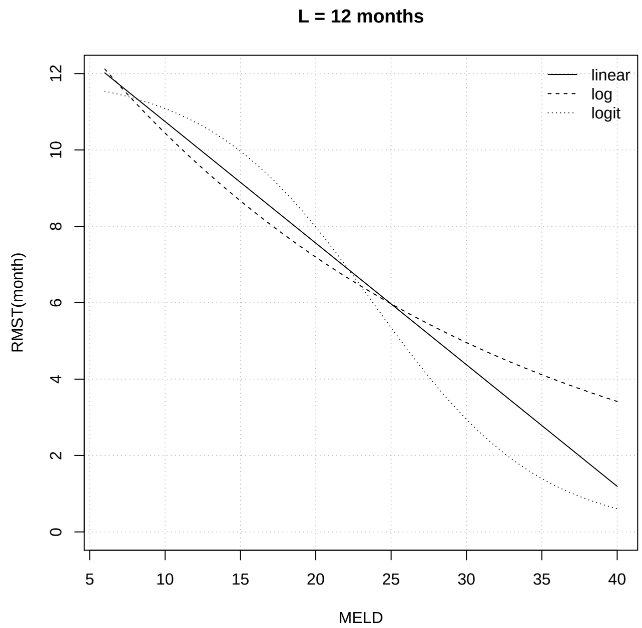
$$\mathrm{IOC} \xrightarrow{p} P(\widehat{D_i^L} < \widehat{D_j^L} | D_i^L < D_j^L).$$

We can also use the similar trick to prove that

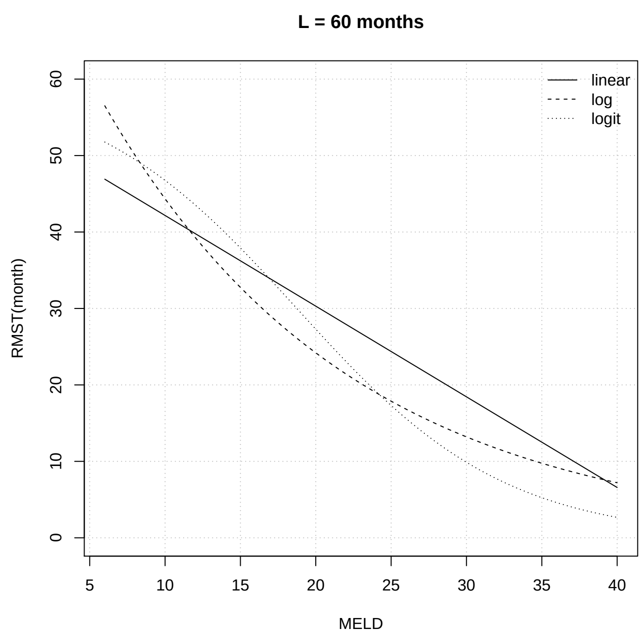
$$\begin{aligned}
\mathrm{MAD} &:= \frac{1}{n} \sum_{i=1}^n \Delta_i \widehat{W}_i\left(Y_i\right) \left|Y_i - g^{-1}\left\{\widehat{\boldsymbol{\beta}}_D' \boldsymbol{Z}_i^D\left(0\right)\right\}\right| \xrightarrow{p} E\left|D_i^L - \widehat{D_i^L}\right|, \\
\mathrm{MSD} &:= \frac{1}{n} \sum_{i=1}^n \Delta_i \widehat{W}_i\left(Y_i\right) \left[Y_i - g^{-1}\left\{\widehat{\boldsymbol{\beta}}_D' \boldsymbol{Z}_i^D\left(0\right)\right\}\right]^2 \xrightarrow{p} E\left[D_i^L - \widehat{D_i^L}\right]^2.
\end{aligned}$$

C More Results in Application Data Analysis

Below are the plots of RMST within 1 year and 5 years post wait-list for chronic ESLD patients with different MELD scores.



Web Figure 1: Fitted RMST ($L = 1$ year) by MELD score for a reference patient: white, male, age=50, Region=5, year=2005, not hospitalized, not on dialysis, blood Type=O, BMI $\in (20, 25]$, sodium=130



Web Figure 2: Fitted RMST ($L = 5$ years) by MELD score for a reference patient: white, male, age=50, Region=5, year=2005, not hospitalized, not on dialysis, blood Type=O, BMI $\in (20, 25]$, sodium=130

Below are the estimated effects of prognostic factors on pre-transplant survival time within 1 year and 5 year post wait-list for chronic ESLD patients.

Web Table 1: Estimated covariate effects on RMST in the absence of liver transplantation ($L = 12$ months)

	Linear			Log			Logistic		
$\boldsymbol{Z}_i^D(0)$	$\widehat{\boldsymbol{\beta}}_D$	ASE_1	p	$\widehat{\boldsymbol{\beta}}_D$	ASE_1	p	$\widehat{\boldsymbol{\beta}}_D$	ASE_1	p
Intercept	12.02	0.14	< 0.01	2.49	0.01	0.44	3.22	0.1	< 0.01
Year-2005	0.1	0.01	< 0.01	0.01	< 0.01	< 0.01	0.06	0.01	< 0.01
Age-50 (Years)	-0.05	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	-0.04	< 0.01	< 0.01
Sodium-130 (mmol/l)	0.13	0.01	< 0.01	0.01	< 0.01	< 0.01	0.08	< 0.01	< 0.01
MELD Score-6	-0.32	< 0.01	< 0.01	-0.04	< 0.01	< 0.01	-0.18	< 0.01	< 0.01
UNOS Region	*Reference Group: 5*								
1	-0.32	0.12	0.01	-0.02	0.01	0.02	-0.31	0.09	< 0.01
2	-0.52	0.09	< 0.01	-0.04	0.01	< 0.01	-0.41	0.06	< 0.01
3	-0.63	0.12	< 0.01	-0.03	0.01	< 0.01	-0.54	0.08	< 0.01
4	-0.08	0.08	0.29	< 0.01	0.01	0.75	-0.09	0.07	0.18
6	-0.01	0.14	0.93	0.03	0.01	0.02	-0.2	0.1	0.05
7	0.06	0.11	0.59	0.02	0.01	0.04	0.03	0.08	0.69
8	-0.15	0.1	0.15	0.01	0.01	0.49	-0.17	0.08	0.04
9	-0.45	0.09	< 0.01	-0.04	0.01	< 0.01	-0.27	0.08	< 0.01
10	-0.51	0.12	< 0.01	-0.04	0.01	< 0.01	-0.47	0.09	< 0.01
11	-0.79	0.11	< 0.01	-0.06	0.01	< 0.01	-0.59	0.08	< 0.01
Gender	*Reference Group: Male*								
Female	0.01	0.05	0.88	-0.01	< 0.01	0.03	0.03	0.04	0.47
Race	*Reference Group: White*								
Black	0.19	0.11	0.07	0.02	0.01	0.02	0.05	0.07	0.5
Hispanic	-0.02	0.07	0.76	< 0.01	0.01	0.44	-0.05	0.05	0.34
Asian	0.2	0.12	0.1	0.02	0.01	0.11	0.21	0.11	0.05
Others	-0.29	0.23	0.22	-0.02	0.02	0.28	-0.1	0.18	0.57
Blood Type	*Reference Group: O*								
A	-0.05	0.05	0.3	< 0.01	< 0.01	0.44	< 0.01	0.04	0.96
B	-0.05	0.09	0.57	-0.01	0.01	0.46	0.01	0.06	0.89
AB	-0.28	0.21	0.19	-0.03	0.01	0.04	-0.23	0.15	0.13
Diagnosis	*Reference Group: No or Yes*								
Hepatitis C	-0.09	0.09	0.32	< 0.01	0.01	0.83	-0.16	0.06	0.01
Noncholestatic	0.29	0.09	< 0.01	0.04	0.01	< 0.01	0.13	0.06	0.04
Cholestatic	-0.05	0.12	0.68	-0.01	0.01	0.43	-0.11	0.09	0.25
Acute Hepatic Necrosis	0.9	0.22	< 0.01	0.05	0.02	0.01	0.86	0.17	< 0.01
Metastatic Disease	-0.45	0.19	0.02	-0.04	0.02	0.06	-0.27	0.13	0.04
Malignant Neoplasm	-1.64	0.1	< 0.01	-0.19	0.01	< 0.01	-1.02	0.07	< 0.01
BMI	*Reference Group: $(20, 25]$*								
$(0, 20]$	-0.46	0.12	< 0.01	-0.04	0.01	< 0.01	-0.24	0.08	< 0.01
$(25, 30]$	0.08	0.07	0.21	0.01	0.01	0.02	0.03	0.05	0.48
> 30	0.08	0.07	0.23	0.01	0.01	0.06	0.04	0.05	0.47
Hospitalized	*Reference Group: Not Hospitalized*								
ICU	-1.63	0.18	< 0.01	-0.57	0.05	< 0.01	-0.98	0.1	< 0.01
not ICU	-1.43	0.13	< 0.01	-0.3	0.02	< 0.01	-0.56	0.06	< 0.01
Dialysis	*Reference Group: No or Yes*								
Yes	0.81	0.15	< 0.01	0.11	0.02	< 0.01	0.56	0.08	< 0.01

An offset of $L = 12$ months is applied for log link.

Web Table 2: Estimated covariate effects on RMST in the absence of liver transplantation ($L = 60$ months)

	Linear			Log			Logistic		
$\boldsymbol{Z}_i^D(0)$	$\widehat{\boldsymbol{\beta}}_D$	ASE_1	p	$\widehat{\boldsymbol{\beta}}_D$	ASE_1	p	$\widehat{\boldsymbol{\beta}}_D$	ASE_1	p
Intercept	46.91	0.88	< 0.01	4.03	0.03	0.06	1.84	0.09	< 0.01
Year-2005	-3.2	0.07	< 0.01	-0.13	< 0.01	< 0.01	-0.32	0.01	< 0.01
Age-50 (Years)	-0.3	0.02	< 0.01	-0.01	< 0.01	< 0.01	-0.03	< 0.01	< 0.01
Sodium-130 (mmol/l)	0.66	0.04	< 0.01	0.03	< 0.01	< 0.01	0.07	< 0.01	< 0.01
MELD Score-6	-1.19	0.02	< 0.01	-0.06	< 0.01	< 0.01	-0.14	< 0.01	< 0.01
UNOS Region				*Reference Group: 5*					
1	-2.37	0.78	< 0.01	-0.07	0.03	0.02	-0.15	0.08	0.07
2	-1.86	0.64	< 0.01	-0.06	0.02	0.01	-0.14	0.06	0.03
3	-4.36	0.76	< 0.01	-0.13	0.03	< 0.01	-0.34	0.08	< 0.01
4	0	0.63	1	0.02	0.02	0.3	0.04	0.06	0.57
6	0.66	0.94	0.48	0.03	0.03	0.28	0.1	0.1	0.34
7	-0.55	0.68	0.41	-0.02	0.03	0.34	-0.01	0.07	0.86
8	-1.59	0.83	0.05	-0.01	0.03	0.82	-0.07	0.08	0.4
9	-1.05	0.65	0.1	-0.06	0.02	< 0.01	-0.1	0.07	0.15
10	-3.94	0.83	< 0.01	-0.14	0.03	< 0.01	-0.37	0.08	< 0.01
11	-3.46	0.77	< 0.01	-0.13	0.03	< 0.01	-0.31	0.08	< 0.01
Gender				*Reference Group: Male*					
Female	0.54	0.36	0.14	< 0.01	0.01	0.81	0.03	0.04	0.38
Race				*Reference Group: White*					
Black	-0.99	0.61	0.1	-0.04	0.03	0.16	-0.06	0.07	0.37
Hispanic	0.5	0.49	0.3	0.01	0.02	0.52	0.04	0.05	0.39
Asian	1.63	0.87	0.06	0.02	0.03	0.55	0.14	0.1	0.17
Others	-0.86	1.56	0.58	-0.01	0.06	0.89	-0.02	0.16	0.89
Blood Type				*Reference Group: O*					
A	0.06	0.36	0.86	0.01	0.01	0.61	0.03	0.04	0.47
B	-0.51	0.59	0.38	-0.01	0.02	0.44	-0.04	0.06	0.54
AB	0.55	1.07	0.61	0.01	0.04	0.8	0.03	0.13	0.81
Diagnosis				*Reference Group: No or Yes*					
Hepatitis C	-1.54	0.54	< 0.01	-0.05	0.02	0.01	-0.21	0.06	< 0.01
Noncholestatic	2.34	0.55	< 0.01	0.1	0.02	< 0.01	0.23	0.06	< 0.01
Cholestatic	-0.98	0.77	0.2	-0.04	0.03	0.08	-0.15	0.08	0.07
Acute Hepatic Necrosis	2.4	1.5	0.11	0.01	0.04	0.84	0.1	0.15	0.51
Metastatic Disease	-2.58	1.41	0.07	-0.07	0.06	0.25	-0.26	0.17	0.12
Malignant Neoplasm	-8.59	0.69	< 0.01	-0.44	0.04	< 0.01	-0.99	0.08	< 0.01
BMI				*Reference Group:* $(20, 25]$					
$(0, 20]$	-2.64	0.86	< 0.01	-0.09	0.03	< 0.01	-0.22	0.09	0.02
$(25, 30]$	-0.15	0.46	0.75	< 0.01	0.02	0.8	-0.03	0.05	0.59
> 30	-0.12	0.47	0.8	< 0.01	0.02	0.88	-0.05	0.05	0.33
Hospitalized				*Reference Group: Not Hospitalized*					
ICU	-1.87	0.75	0.01	-0.63	0.11	< 0.01	-0.83	0.16	< 0.01
not ICU	-1.76	0.59	< 0.01	-0.27	0.05	< 0.01	-0.36	0.09	< 0.01
Dialysis				*Reference Group: No or Yes*					
Yes	2.01	0.77	0.01	0.13	0.05	0.01	0.47	0.11	< 0.01

An offset of $L = 60$ months is applied for log link.