## APPENDIX A
## DYNAMIC MODELING OF PLANAR ROBOT MANIPULATOR

For the dynamic modeling of the six-link planar robot manipulator presented in [1], it is known from [2], [3] that the dynamic equation of the robot manipulator can be given as

$$\tau = H\ddot{\theta} + c_\tau(\dot{\theta}, \theta) + g_\tau(\theta), \tag{1}$$

where $\theta$, $\dot{\theta}$, $\ddot{\theta}$ and $\tau$ denote the $n$-dimensional joint-angle, joint-velocity, joint-acceleration and joint-torque vectors, respectively; $H$ denotes the $n \times n$ dimensional inertia matrix; $c_\tau$ denotes the $n$-dimensional Coriolis/centrifugal force vector and $g_\tau$ denotes the $n$-dimensional gravitational force vector. By defining

$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} & H_{15} & H_{16} \\ H_{21} & H_{22} & H_{23} & H_{24} & H_{25} & H_{26} \\ H_{31} & H_{32} & H_{33} & H_{34} & H_{35} & H_{36} \\ H_{41} & H_{42} & H_{43} & H_{44} & H_{45} & H_{46} \\ H_{51} & H_{52} & H_{53} & H_{54} & H_{55} & H_{56} \\ H_{61} & H_{62} & H_{63} & H_{64} & H_{65} & H_{66} \end{bmatrix},$$

$$c_\tau = [c_{\tau 1} \quad c_{\tau 2} \quad c_{\tau 3} \quad c_{\tau 4} \quad c_{\tau 5} \quad c_{\tau 6}]^{\mathrm{T}},$$

$$g_\tau = [g_{\tau 1} \quad g_{\tau 2} \quad g_{\tau 3} \quad g_{\tau 4} \quad g_{\tau 5} \quad g_{\tau 6}]^{\mathrm{T}},$$

and by using the recursive Newton-Euler algorithm [2], [3], the detailed expressions of elements of matrix $H$ as well as vectors $c_\tau$ and $g_\tau$ are derived and presented as below.

$H_{11} = H_{21} + (\sum_{i=1}^{6} m_i) l_1(l_1) + (\sum_{i=2}^{6} m_i) l_1 l_2 c_2 + (\sum_{i=3}^{6} m_i) l_1 l_3 c_{23} + (\sum_{i=4}^{6} m_i) l_1 l_4 c_{234} + (m_5 + m_6) l_1 l_5 c_{2345} + m_6 l_1 l_6 c_{23456}$,

$H_{12} = H_{22} + (\sum_{i=2}^{6} m_i) l_1 l_2 c_2 + (\sum_{i=3}^{6} m_i) l_1 l_3 c_{23} + (\sum_{i=4}^{6} m_i) l_1 l_4 c_{234} + (m_5 + m_6) l_1 l_5 c_{2345} + m_6 l_1 l_6 c_{23456}$,

$H_{13} = H_{23} + (\sum_{i=3}^{6} m_i) l_1 l_3 c_{23} + (\sum_{i=4}^{6} m_i) l_1 l_4 c_{234} + (m_5 + m_6) l_1 l_5 c_{2345} + m_6 l_1 l_6 c_{23456}$,

$H_{14} = H_{24} + (\sum_{i=4}^{6} m_i) l_1 l_4 c_{234} + (m_5 + m_6) l_1 l_5 c_{2345} + m_6 l_1 l_6 c_{23456}$,

$H_{15} = H_{25} + (m_5 + m_6) l_1 l_5 c_{2345} + m_6 l_1 l_6 c_{23456}$,

$H_{16} = H_{26} + m_6 l_1 l_6 c_{23456}$,

$c_{\tau 1} = c_{\tau 2} - (\sum_{i=2}^{6} m_i) l_1 l_2 a_2 s_2 - (\sum_{i=3}^{6} m_i) l_1 l_3 a_3 s_{23} - (\sum_{i=4}^{6} m_i) l_1 l_4 a_4 s_{234} - (m_5 + m_6) l_1 l_5 a_5 s_{2345} + m_6 l_1 l_6 a_6 s_{23456}$,

$g_{\tau 1} = g_{\tau 2} + (\sum_{i=1}^{6} m_i) l_1 g c_1$;

$H_{21} = H_{31} + (\sum_{i=2}^{6} m_i) l_2(l_1 c_2 + l_2) + (\sum_{i=3}^{6} m_i) l_2 l_3 c_3 + (\sum_{i=4}^{6} m_i) l_2 l_4 c_{34} + (m_5 + m_6) l_2 l_5 c_{345} + m_6 l_2 l_6 c_{3456}$,

$H_{22} = H_{32} + (\sum_{i=2}^{6} m_i) l_2(l_2) + (\sum_{i=3}^{6} m_i) l_2 l_3 c_3 + (\sum_{i=4}^{6} m_i) l_2 l_4 c_{34} + (m_5 + m_6) l_2 l_5 c_{345} + m_6 l_2 l_6 c_{3456}$,

$H_{23} = H_{33} + (\sum_{i=3}^{6} m_i) l_2 l_3 c_3 + (\sum_{i=4}^{6} m_i) l_2 l_4 c_{34} + (m_5 + m_6) l_2 l_5 c_{345} + m_6 l_2 l_6 c_{3456}$,

$H_{24} = H_{34} + (\sum_{i=4}^{6} m_i) l_2 l_4 c_{34} + (m_5 + m_6) l_2 l_5 c_{345} + m_6 l_2 l_6 c_{3456}$,

$H_{25} = H_{35} + (m_5 + m_6) l_2 l_5 c_{345} + m_6 l_2 l_6 c_{3456}$,

$H_{26} = H_{36} + m_6 l_2 l_6 c_{3456}$,

$c_{\tau 2} = c_{\tau 3} + (\sum_{i=2}^{6} m_i) l_2(l_1 a_1 s_2) - (\sum_{i=3}^{6} m_i) l_2 l_3 a_3 s_3 - (\sum_{i=4}^{6} m_i) l_2 l_4 a_4 s_{34} - (m_5 + m_6) l_2 l_5 a_5 s_{345} + m_6 l_2 l_6 a_6 s_{3456}$,

$g_{\tau 2} = g_{\tau 3} + (\sum_{i=2}^{6} m_i) l_2 g c_{12}$;

$H_{31} = H_{41} + (\sum_{i=3}^{6} m_i) l_3(l_1 c_{23} + l_2 c_3 + l_3) + (\sum_{i=4}^{6} m_i) l_3 l_4 c_4 + (m_5 + m_6) l_3 l_5 c_{45} + m_6 l_3 l_6 c_{456}$,

$H_{32} = H_{42} + (\sum_{i=3}^{6} m_i) l_3(l_2 c_3 + l_3) + (\sum_{i=4}^{6} m_i) l_3 l_4 c_4 + (m_5 + m_6) l_3 l_5 c_{45} + m_6 l_3 l_6 c_{456}$,

$H_{33} = H_{43} + (\sum_{i=3}^{6} m_i) l_3(l_3) + (\sum_{i=4}^{6} m_i) l_3 l_4 c_4 + (m_5 + m_6) l_3 l_5 c_{45} + m_6 l_3 l_6 c_{456}$,

$H_{34} = H_{44} + (\sum_{i=4}^{6} m_i) l_3 l_4 c_4 + (m_5 + m_6) l_3 l_5 c_{45} + m_6 l_3 l_6 c_{456}$,

$H_{35} = H_{45} + (m_5 + m_6) l_3 l_5 c_{45} + m_6 l_3 l_6 c_{456}$,

$H_{36} = H_{46} + m_6 l_3 l_6 c_{456}$,

$c_{\tau 3} = c_{\tau 4} + (\sum_{i=3}^{6} m_i) l_3(l_1 a_1 s_{23} + l_2 a_2 s_3) - (\sum_{i=4}^{6} m_i) l_3 l_4 a_4 s_4 - (m_5 + m_6) l_3 l_5 a_5 s_{45} + m_6 l_3 l_6 a_6 s_{456}$,

$g_{\tau 3} = g_{\tau 4} + (\sum_{i=3}^{6} m_i) l_3 g c_{123}$;

$H_{41} = H_{51} + (m_4 + m_5 + m_6) l_4(l_1 c_{234} + l_2 c_{34} + l_3 c_4 + l_4) + (m_5 + m_6) l_4 l_5 c_5 + m_6 l_4 l_6 c_{56}$,

$H_{42} = H_{52} + (m_4 + m_5 + m_6) l_4(l_2 c_{34} + l_3 c_4 + l_4) + (m_5 + m_6) l_4 l_5 c_5 + m_6 l_4 l_6 c_{56}$,

$H_{43} = H_{53} + (m_4 + m_5 + m_6) l_4(l_3 c_4 + l_4) + (m_5 + m_6) l_4 l_5 c_5 + m_6 l_4 l_6 c_{56}$,

$H_{44} = H_{54} + (m_4 + m_5 + m_6) l_4(l_4) + (m_5 + m_6) l_4 l_5 c_5 + m_6 l_4 l_6 c_{56}$,

$H_{45} = H_{55} + (m_5 + m_6) l_4 l_5 c_5 + m_6 l_4 l_6 c_{56}$,

$H_{46} = H_{56} + m_6 l_4 l_6 c_{56}$,

$c_{\tau 4} = c_{\tau 5} + (m_4 + m_5 + m_6) l_4(l_1 a_1 s_{234} + l_2 a_2 s_{34} + l_3 a_3 s_4) - (m_5 + m_6) l_4 l_5 a_5 s_5 + m_6 l_4 l_6 a_6 s_{56}$,

$g_{\tau 4} = g_{\tau 5} + (m_4 + m_5 + m_6) l_4 g c_{1234}$;

$H_{51} = H_{61} + (m_5 + m_6) l_5(l_1 c_{2345} + l_2 c_{345} + l_3 c_{45} + l_4 c_5 + l_5) + m_6 l_5 l_6 c_6$,

$H_{52} = H_{62} + (m_5 + m_6) l_5(l_2 c_{345} + l_3 c_{45} + l_4 c_5 + l_5) + m_6 l_5 l_6 c_6$,

$H_{53} = H_{63} + (m_5 + m_6) l_5(l_3 c_{45} + l_4 c_5 + l_5) + m_6 l_5 l_6 c_6$,

$H_{54} = H_{64} + (m_5 + m_6) l_5(l_4 c_5 + l_5) + m_6 l_5 l_6 c_6$,

$H_{55} = H_{65} + (m_5 + m_6) l_5(l_5) + m_6 l_5 l_6 c_6$,

$H_{56} = H_{66} + m_6 l_5 l_6 c_6$,

$c_{\tau 5} = c_{\tau 6} + (m_5 + m_6) l_5(l_1 a_1 s_{2345} + l_2 a_2 s_{345} + l_3 a_3 s_{45} + l_4 a_4 s_5) - m_5 l_5 l_6 a_6 s_6$,

$g_{\tau 5} = g_{\tau 6} + (m_5 + m_6) l_5 g c_{12345}$;

$H_{61} = m_6 l_6(l_1 c_{23456} + l_2 c_{3456} + l_3 c_{456} + l_4 c_{56} + l_5 c_6 + l_6)$,

$H_{62} = m_6 l_6(l_2 c_{3456} + l_3 c_{456} + l_4 c_{56} + l_5 c_6 + l_6)$,

$H_{63} = m_6 l_6(l_3 c_{456} + l_4 c_{56} + l_5 c_6 + l_6)$,

$H_{64} = m_6 l_6(l_4 c_{56} + l_5 c_6 + l_6)$,

$H_{65} = m_6 l_6(l_5 c_6 + l_6)$,

$H_{66} = m_6 l_6(l_6)$,

$c_{\tau 6} = m_6 l_6(l_6)(l_1 a_1 s_{23456} + l_2 a_2 s_{3456} + l_3 a_3 s_{456} + l_4 a_5 s_{56} + l_5 a_5 s_6)$,

$g_{\tau 6} = m_6 l_6 g c_{123456}$;

in which, $m_i$ denotes the mass of the $i$th link of the manipulator; $l_i$ denotes the length of the $i$th link; and $a_i = (\sum_{j=1}^{i} \dot{\theta}_j)^2$ and $b_i = \sum_{j=1}^{i} \ddot{\theta}_j$, with $i = 1, 2, \cdots, 6$. Besides, $g$ denotes the gravitational acceleration. Additionally, $c_i = \cos(\theta_i)$, $s_i = \sin(\theta_i)$, $c_{ijk\cdots} = \cos(\theta_i + \theta_j + \theta_k + \theta_{\cdots})$ and $s_{ijk\cdots} = \sin(\theta_i + \theta_j + \theta_k + \theta_{\cdots})$. Specifically speaking, for the manipulator used in this paper, $m_1 = 7.887$ kg, $m_2 = 5.730$ kg, $m_3 = 3.198$ kg, $m_4 = 3.020$ kg, $m_5 = 2.773$ kg, $m_6 = 0.337$ kg, $l_1 = 0.301$ m, $l_2 = 0.290$ m, $l_3 = 0.230$ m, $l_4 = 0.225$ m, $l_5 = 0.214$ m and $l_6 = 0.103$ m.

## REFERENCES


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