S. Supplementary Text to “Contextuality in Canonical Systems of Random Variables” by Ehtibar N. Dzhafarov, Víctor H. Cervantes, and Janne V. Kujala (Phil. Trans. Roy. Soc. A xxx, 10.1098/rsta.xxxx.xxxx)

Theorem S.1 (Section 4, Remark 4.2)**.** *The rank of the system of linear equations (4.4)-(4.6)-(4.7) is* $2k-1+\binom{k}{2}$.

Proof of Theorem S.1. This system of linear equations can be written as

$$\mathbf{M}\times\mathbf{X}=\mathbf{P},$$

where

$$\mathbf{P}^{T}=\left(\begin{array}{c}\overbrace{p_1,\ldots,p_k}^{k},\overbrace{q_1,\ldots,q_k}^{k},\overbrace{\min\left(p_1,q_1\right),\ldots,\min\left(p_k,q_k\right)}^{k},\\ \\ \overbrace{\min\left(p_1+p_2,q_1+q_2\right),\ldots,\min\left(p_{k-1}+p_k,q_{k-1}+q_k\right)}^{\binom{k}{2}}\end{array}\right),$$

$$\mathbf{X}^{T}=\left\{x_{ij}:i,j\in\{1,\ldots,k\}\right\},$$

and $\mathbf{M}$ is a Boolean matrix. The $\left(k+k+k+\binom{k}{2}\right)$ rows of matrix $\mathbf{M}$ correspond to the elements of $\mathbf{P}$ and can be labeled as

$$\left(\overbrace{\mathbf{r}_{1\cdot},\ldots,\mathbf{r}_{k\cdot}}^{k},\overbrace{\mathbf{r}_{\cdot 1},\ldots,\mathbf{r}_{\cdot k}}^{k},\overbrace{\mathbf{r}_{11},\ldots,\mathbf{r}_{kk}}^{k},\overbrace{\mathbf{r}_{12},\ldots,\mathbf{r}_{k-1,k}}^{\binom{k}{2}}\right),$$

whereas the k^2 columns of $\mathbf{M}$ correspond to the elements of $\mathbf{X}$ and can be labeled as

$$\left\{\mathbf{c}_{ij}:i,j\in\{1,\ldots,k\}\right\}.$$

Thus, if $k=4$, the matrix $\mathbf{M}$ is

r \ c	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
1·	1	1	1	1												
2·					1	1	1	1								
3·									1	1	1	1				
4·													1	1	1	1
·1	1				1				1				1			
·2		1				1				1				1		
·3			1				1				1				1	
·4				1				1				1				1
11	1															
22						1										
33											1					
44																1
12	1	1			1	1										
13	1		1						1		1					
14	1			1									1			1
23						1	1			1	1					
24						1		1						1		1
34											1	1			1	1

rsta.royalsocietypublishing.org Phil. Trans. R. Soc. A 0000000

We will continue to illustrate the steps of the proof using this matrix. We begin by adding to $\mathbf{M}$ the row $\mathbf{r}_{all}$ with all cells equal to 1, and denote the new matrix $\mathbf{M}'$.

r \ **c**	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
1·	1	1	1	1												
2·					1	1	1	1								
3·									1	1	1	1				
4·													1	1	1	1
·1	1				1				1				1			
·2		1				1				1				1		
·3			1				1				1				1	
·4				1				1				1				1
11	1															
22						1										
33											1					
44																1
12	1	1			1	1										
13	1		1						1		1					
14	1			1									1			1
23						1	1			1	1					
24						1		1						1		1
34											1	1			1	1
all	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

This does not change the rank of the matrix since $\mathbf{r}_{all}$ is the sum of all $\mathbf{r}_{\cdot i}$. Then we observe that the rows $\mathbf{r}_{k\cdot}$, $\mathbf{r}_{\cdot k}$, and all $\mathbf{r}_{ik}$ with $i < k$ can be deleted as they are linear combinations of the remaining rows of $\mathbf{M}'$. Indeed, it can be checked directly that

$$\mathbf{r}_{k\cdot} = \mathbf{r}_{all} - \sum_{i=1}^{k-1} \mathbf{r}_{i\cdot},$$

$$\mathbf{r}_{\cdot k} = \mathbf{r}_{all} - \sum_{i=1}^{k-1} \mathbf{r}_{\cdot i},$$

$$(\mathbf{r}_{ik} - \mathbf{r}_{ii} - \mathbf{r}_{kk}) = (\mathbf{r}_{i\cdot} - \mathbf{r}_{ii}) + (\mathbf{r}_{\cdot i} - \mathbf{r}_{ii}) - \sum_{l<i} (\mathbf{r}_{li} - \mathbf{r}_{ll} - \mathbf{r}_{ii}) - \sum_{l>i}^{l<k} (\mathbf{r}_{il} - \mathbf{r}_{ii} - \mathbf{r}_{ll}),$$

for all $i < k$. Moreover, one can also delete $\mathbf{r}_{kk}$, because

$$\sum_{i<j<k} (\mathbf{r}_{ij} - \mathbf{r}_{ii} - \mathbf{r}_{jj}) + \sum_{i<k} (\mathbf{r}_{ik} - \mathbf{r}_{ii} - \mathbf{r}_{kk}) + \sum_{i<k} \mathbf{r}_{ii} + \mathbf{r}_{kk} = \mathbf{r}_{all}.$$

rsta.royalsocietypublishing.org Phil. Trans. R. Soc. A 0000000

Let the resulting matrix be $\mathbf{M}''$:

$\mathbf{r}$ \ $\mathbf{c}$	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
1·	1	1	1	1												
2·					1	1	1	1								
3·									1	1	1	1				
·1	1				1				1				1			
·2		1				1				1				1		
·3			1				1				1				1	
11	1															
22						1										
33											1					
12	1	1			1	1										
13	1		1						1		1					
23						1	1			1	1					
all	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

This matrix contains

$$3k + \overbrace{\binom{k}{2}}^{initial} - \underbrace{3}_{\mathbf{r}_{k\cdot},\mathbf{r}_{\cdot k},\mathbf{r}_{kk}} - \overbrace{(k-1)}^{all\ \mathbf{r}_{ik}, i<k} + \underbrace{1}_{\mathbf{r}_{all}} = 2k - 1 + \binom{k}{2}$$

rows. We prove that this matrix is of full row rank. Consider equation

$$\sum_{all\ \mathbf{r}\ \text{in}\ \mathbf{M}''} \alpha_{\mathbf{r}} \mathbf{r} = 0.$$

We use the following principle: if a row $\mathbf{r}$ intersects a columns whose only nonzero entry is in the row $\mathbf{r}$, then $\alpha_{\mathbf{r}} = 0$, and we can delete the row $\mathbf{r}$ from the matrix, decreasing the row rank of the matrix by 1. The following statements can be directly verified.

$\mathbf{r}_{all}$ can be deleted because column $\mathbf{c}_{kk}$ has its only 1 in $\mathbf{r}_{all}$.

$\mathbf{r}$ \ $\mathbf{c}$	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
1·	1	1	1	1												
2·					1	1	1	1								
3·									1	1	1	1				
·1	1				1				1				1			
·2		1				1				1				1		
·3			1				1				1				1	
11	1															
22						1										
33											1					
12	1	1			1	1										
13	1		1						1		1					
23						1	1			1	1					

Then each of $\mathbf{r}_{\cdot i}$can be deleted because the column $\mathbf{c}_{ki}$ has its only 1 in $\mathbf{r}_{\cdot i}$ $(i = 1, \ldots, k-1)$.

r \ c	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
1·	1	1	1	1												
2·					1	1	1	1								
3·									1	1	1	1				
11	1															
22						1										
33											1					
12	1	1			1	1										
13	1		1						1		1					
23						1	1			1	1					

Then each of $\mathbf{r}_{i\cdot}$ can be deleted because the column $\mathbf{c}_{ik}$ has its only 1 in $\mathbf{r}_{i\cdot}$ $(i = 1, \ldots, k-1)$.

r \ c	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
11	1															
22						1										
33											1					
12	1	1			1	1										
13	1		1						1		1					
23						1	1			1	1					

Then each of $\mathbf{r}_{ij}$ can be deleted because the column $\mathbf{c}_{ji}$ has its only 1 in $\mathbf{r}_{ij}$ $(i, j \in \{1, \ldots, k-1\}, i < j)$.

r \ c	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
11	1															
22						1										
33											1					

This leaves only $\mathbf{r}_{11}, \ldots, \mathbf{r}_{(k-1)(k-1)}$ that are obviously linearly independent. □

Theorem (Section 4, Theorem 4.3). *In a maximally-connected coupling S of $\mathcal{D}$ with $k > 5$, the distributions of the 1-splits and 2-splits uniquely determine the probabilities of all higher-order splits. Specifically, for any $2 < m \leq k/2$, and any $W = \{i_1, \ldots, i_m\} \subset \{1, \ldots, k\}$, the probability that the corresponding m-split equals 1 is*

$$\begin{array}{l} \min\left(p_{i_1} + p_{i_2} + \ldots + p_{i_m}, q_{i_1} + q_{i_2} + \ldots + q_{i_m}\right) = \sum_{j=1}^{m} \min\left(p_{i_j}, q_{i_j}\right) \\ + \sum_{j=1}^{m-1} \sum_{j'=j+1}^{m} \left[\min\left(p_{i_j} + p_{i_{j'}}, q_{i_j} + q_{i_{j'}}\right) - \min\left(p_{i_j}, q_{i_j}\right) - \min\left(p_{i_{j'}}, q_{i_{j'}}\right)\right]. \end{array} \quad \text{(S.1)}$$

Proof of Theorem 4.3. From (4.6) and (4.7),

$$\begin{array}{rcll} r_{12} + r_{21} & = & \min\left(p_1 + p_2, q_1 + q_2\right) - \min\left(p_1, q_1\right) - \min\left(p_2, q_2\right) & \\ \vdots & \vdots & \vdots & \\ r_{ij} + r_{ji} & = & \min\left(p_i + p_j, q_i + q_j\right) - \min\left(p_i, q_i\right) - \min\left(p_j, q_j\right) & (i < j). \\ \vdots & \vdots & \vdots & \\ r_{(k-1)k} + r_{k(k-1)} & = & \min\left(p_{k-1} + p_k, q_{k-1} + q_k\right) - \min\left(p_{k-1}, q_{k-1}\right) - \min\left(p_k, q_k\right) & \end{array}$$

Consider an m-split with $2 < m \leq k/2$, and assume without loss of generality that $W = (1, \ldots, m)$. We have

$$\sum_{i=1}^{m}\sum_{j=1}^{m} r_{ij} = \min\left(p_1 + \ldots + p_m, q_1 + \ldots + q_m\right). \tag{S.2}$$

The left-hand-side sum can be presented as

$$\sum_{i=1}^{m} r_{ii} + \sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\left(r_{ij} + r_{ji}\right)$$

$$= \sum_{i=1}^{m} \min\left(p_i, q_i\right) + \sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\left[\min\left(p_i + p_j, q_i + q_j\right) - \min\left(p_i, q_i\right) - \min\left(p_j, q_j\right)\right],$$

whence we get (4.8). □

Example S.2 (showing that the relation (4.8) may be violated, see Section 4.)**.** If

$R_1^1 =$	1	2	3	4	0	0
prob. mass $p =$	.6	.1	.1	.2	0	0

,

$R_1^2 =$	1	2	3	4	0	0
prob. mass $q =$	.2	.3	.4	.1	0	0

,

then

$$\overbrace{\min\left(p_1 + p_2 + p_3, q_1 + q_2 + q_3\right)}^{.8}$$

$$\neq \left.\begin{array}{rl} \min\left(p_1, q_1\right) & .2 \\ + \min\left(p_2, q_2\right) & .1 \\ + \min\left(p_3, q_3\right) & .1 \\ + \min\left(p_1 + p_2, q_1 + q_2\right) - \min\left(p_1, q_1\right) - \min\left(p_2, q_2\right) & .5 - .2 - .1 \\ + \min\left(p_1 + p_3, q_1 + q_3\right) - \min\left(p_1, q_1\right) - \min\left(p_3, q_3\right) & .6 - .2 - .1 \\ + \min\left(p_2 + p_3, q_2 + q_3\right) - \min\left(p_2, q_2\right) - \min\left(p_3, q_3\right) & .2 - .1 - .1 \end{array}\right\} = .5$$

□

Theorem (Section 4, Theorem 4.4)**.** *A maximally-connected coupling for a 1-2 system is unique if it exists. In this coupling, the only pairs of ij in (4.3) that may have nonzero probabilities assigned to them are the diagonal states $\{11, 22, \ldots, kk\}$ and either the states $\{i1, i2, \ldots, ik\}$ for a single fixed i or the states $\{1j, 2j, \ldots, kj\}$ for a single fixed j ($i, j = 1, \ldots, k$).*

Proof of Theorem 4.4. (The matrices illustrating the proof are shown for $k > 6$ but the theorem is valid for all $k > 1$.) If the only nonzero entries in the matrix are in the main diagonal, the theorem is trivially true. Assume therefore that $r_{ij} > 0$ for some $i \neq j$. Without loss of generality, we can assume that $r_{12} > 0$ and $p_1 + p_2 \leq q_1 + q_2$. Indeed, if some $r_{ij} > 0$, we can always rename the values so that $i = 1$ and $j = 2$; and if $p_1 + p_2 > q_1 + q_2$, then we can simply rename all ps into qs and vice versa. In the following we will use the expression "r_{ij} is p-minimized" if $p_i + p_j \leq q_i + q_j$, and "r_{ij} is q-minimized" if $p_i + p_j \geq q_i + q_j$ (in both cases, $i \neq j$).

We have (the empty cells are those whose value is to be determined later)

rsta.royalsocietypublishing.org Phil. Trans. R. Soc. A 0000000

	1	2	3	4	5	6	...	
1	r_{11}	$r_{12}>0$						p_1
2	r_{21}	r_{22}						p_2
3			r_{33}					
4				r_{44}				
5					r_{55}			
6						r_{66}		
$\vdots$							$\vdots$	$\vdots$
	q_1	q_2					...	

From (4.6)-(4.7), $r_{11}+r_{12}+r_{21}+r_{22}=\min\{p_1+p_2 q_1+q_2\}$, and since r_{12} is p-minimized, $r_{11}+r_{12}+r_{21}+r_{22}=p_1+p_2$. This means

	1	2	3	4	5	6	...	
1	r_{11}	$r_{12}>0$	0	0	0	0	**0**	$p_1=r_{11}+r_{12}$
2	r_{21}	r_{22}	0	0	0	0	**0**	$p_2=r_{21}+r_{22}$
3			r_{33}					
4				r_{44}				
5					r_{55}			
6						r_{66}		
$\vdots$							$\vdots$	$\vdots$
	$q_1\geq r_{11}+r_{21}$	$q_2\geq r_{12}+r_{22}$					...	

We also should have

	1	2	3	4	5	6	...	
1	r_{11}	$r_{12}>0$	0	0	0	0	**0**	$p_1=r_{11}+r_{12}$
2	0	r_{22}	0	0	0	0	**0**	$p_2=r_{22}$
3	0		r_{33}					
4	0			r_{44}				
5	0				r_{55}			
6	0					r_{66}		
$\vdots$	**0**						$\vdots$	$\vdots$
	$q_1=r_{11}$	$q_2\geq r_{12}+r_{22}$					...	

because $r_{11}=\min\{p_1,q_1\}$ and $r_{11}<p_1$.

Generalizing, we have established the following rules:

(R1) If $r_{ij}>0$ and it is p-minimized, then all non-diagonal elements in the rows i and j are zero except for r_{ij}, and all non-diagonal elements in the column i are zero.

(R2) (By symmetry, on exchanging ps and qs) If $r_{ij}>0$ and it is q-minimized, then all non-diagonal elements in the columns i and j are zero except for r_{ij}, and all non-diagonal elements in the row j are zero.

Returning to our special arrangement of the rows and columns, let us prove now that all r_{1j} with $j>2$ are q-minimized. Assume the contrary, and with no loss of generality, let $r_{15}=0$ be p-minimized. This would mean that

$$r_{15}+r_{51}=p_1+p_5-r_{11}-r_{55}=r_{12}+p_5-r_{55}=0,$$

which could only be true if $r_{12}=0$, which it is not.

	1	2	3	4	5	6	...	
1	r_{11}	$r_{12} > 0$	0 $q{-}min$	0 $q{-}min$	0 $q{-}min$	0 $q{-}min$	**0** $q{-}min$	$p_1 = r_{11} + r_{12}$
2	0	r_{22}	0	0	0	0	**0**	$p_2 = r_{22}$
3	0		r_{33}					
4	0			r_{44}				
5	0				r_{55}			p_5
6	0					r_{66}		
$\vdots$	**0**						$\vdots\vdots$	$\vdots$
	$q_1 = r_{11}$	$q_2 \geq r_{12} + r_{22}$					...	

Generalizing, we have established two additional rules:
(R3) If r_{ij} and $r_{ij'}$ are both p-minimized (for pairwise distinct i, j, j'), then they are both zero (because if one of them is not, say $r_{ij} > 0$, then $r_{ij'} = 0$ and it must be q-minimized).

(R4) (By symmetry, on exchanging ps and qs) If r_{ij} and $r_{i'j}$ are both q-minimized (for pairwise distinct i, i', j), then they are both zero.

Returning to our special arrangement of the rows and columns, it follows that nowhere in the matrix can we have $r_{ij} > 0$ $(i > 2)$ which is q-minimized. Indeed, if $j > 2$, then this would have contradicted R4 (because the zeros in the first row are all q-minimized), and if $j = 2$, it would have contradicted R2 (because $r_{12} > 0$).

Let us prove now that if $j > 2$ and $i > 2$ and $i \neq j$, then there is no $r_{ij} > 0$ that is p-minimized. Assume the contrary: $r_{ij} > 0$ and q-minimized, and consider r_{2i}, r_{i2}. With no loss of generality, let (i, j)=$(4, 6)$. In accordance with R1, we fill in the 4th and the 6th rows with zeros, and we fill in the 4th column with zeros too:

	1	2	3	4	5	6	...	
1	r_{11}	$r_{12} > 0$	0	0	0	0	**0**	$p_1 = r_{11} + r_{12}$
2	0	r_{22}	0	0	0	0	**0**	$p_2 = r_{22}$
3	0		r_{33}	0				
4	0	0	0	r_{44}	0	$r_{46} > 0$	**0**	$p_4 = r_{44} + r_{46}$
5	0			0	r_{55}			
6	0	0	0	$r_{64} = 0$	0	r_{66}	**0**	$p_6 = r_{66}$
$\vdots$	**0**			**0**			$\vdots\vdots$	$\vdots$
	$q_1 = r_{11}$	$q_2 \geq r_{12} + r_{22}$		$q_4 = r_{44}$		$q_6 \geq r_{46} + r_{66}$	...	

Then r_{24}, r_{42} are both zero, whence $\min(p_2 + p_4, q_2 + q_4)$ must equal $r_{22} + r_{44}$ to be a maximal coupling. But

$$\min(p_2 + p_4, q_2 + q_4) = \min(r_{22} + r_{44} + r_{46}, r_{12} + r_{22} + r_{44} + x) > r_{22} + r_{44},$$

since both r_{12} and r_{46} are positive, a contradiction.

We come to the conclusion that the only positive non-diagonal elements in the matrix can be in the column 2 (and they are all p-minimized).

	1	2	3	4	5	6	...	
1	r_{11}	$r_{12} > 0$	0	0	0	0	**0**	$p_1 = r_{11} + r_{12}$
2	0	r_{22}	0	0	0	0	**0**	$p_2 = r_{22}$
3	0	$r_{32} \geq 0$	r_{33}	0	0	0	**0**	$p_3 = r_{32} + r_{33}$
4	0	$r_{42} \geq 0$	0	r_{44}	0	0	**0**	$p_4 = r_{42} + r_{44}$
5	0	$r_{52} \geq 0$	0	0	r_{55}	0	**0**	$p_5 = r_{52} + r_{55}$
6	0	$r_{62} \geq 0$	0	0	0	r_{66}	**0**	$p_6 = r_{62} + r_{66}$
⋮	**0**	⋮	**0**	**0**	**0**	**0**	⋮⋮⋮	⋮
	$q_1 = r_{11}$	$q_2 \geq r_{12} + r_{22}$	$q_3 = r_{33}$	$q_4 = r_{44}$	$q_5 = r_{55}$	$q_6 = r_{66}$	...	

Generalizing, let $r_{ij} > 0$ and $i \neq j$. Then, if r_{ij} is p-minimized, all non-diagonal elements of the matrix outside column j are zero (and the non-diagonal elements in the jth column are p-minimized); if r_{ij} is q-minimized, then all non-diagonal elements of the matrix outside row i are zero (and the non-diagonal elements in the ith row are q-minimized).

It is easy to check that such a construction is always internally consistent. □

Corollary (Section 4, Corollary 4.5)**.** *The 1-2 system for the original rvs* R_1^1, R_1^2 *has a maximally-connected coupling if and only if either* $p_i > q_i$ *for no more than one* i *(this single possible* i *being the single fixed* i *in the formulation of the theorem), or* $p_j < q_j$ *for no more than one* j *(this single possible* j *being the single fixed* j *in the formulation of the theorem),* $i, j \in \{1, \ldots, k\}$.

Proof of Corollary 4.5. The "only if" part is obvious. To demonstrate the "if" part, consider (without loss of generality) the arrangement

	1	2	3	4	5	6	...	
1							...	$p_1 \geq q_1$
2							...	p_2
3							...	$p_3 \geq q_3$
4							...	$p_4 \geq q_4$
5							...	$p_5 \geq q_5$
6							...	$p_6 \geq q_6$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮⋮⋮	⋮
	q_1	$q_2 \geq p_2$	q_3	q_4	q_5	q_6	...	

and fill it in as

	1	2	3	4	5	6	...	
1	q_1	$p_1 - q_1$	0	0	0	0	**0**	$p_1 \geq q_1$
2	0	p_2	0	0	0	0	**0**	p_2
3	0	$p_3 - q_3$	q_3	0	0	0	**0**	$p_3 \geq q_3$
4	0	$p_4 - q_4$	0	q_4	0	0	**0**	$p_4 \geq q_4$
5	0	$p_5 - q_5$	0	0	q_5	0	**0**	$p_5 \geq q_5$
6	0	$p_6 - q_6$	0	0	0	q_6	**0**	$p_6 \geq q_6$
⋮	**0**	⋮	**0**	**0**	**0**	**0**	⋮⋮⋮	⋮
	q_1	$q_2 \geq p_2$	q_3	q_4	q_5	q_6	...	

with the empty cells filled in with zeros. Check that (a) all rows sum to the marginals; (b) the second column sums to

$$\sum_{i=1}^{k} p_i - \left(\sum_{i=1}^{k} q_i - q_2\right) = q_2;$$

(c) the rest of the columns sum to the marginals; (d) all r_{ii} are $\min(p_i, q_i)$; and (e) for all pairs r_{ij} $(i \neq j)$ the sums $r_{ii} + r_{ij} + r_{ji} + r_{jj}$ equal $\min\left(p_i + p_j, q_i + q_j\right)$. The latter is proved by considering first all $j \neq 2$, where it is obvious, and then $j = 2$ where the computation is, for $i \neq 2$,

$$r_{ii} + r_{i2} + r_{2i} + r_{22} = q_i + (p_i - q_i) + 0 + p_2 = p_i + p_2,$$

as it should be because the values in the second column are to be p-minimized. □

Theorem (Section 4, Theorem 4.6). *The system $\mathcal{D}$ is noncontextual if and only if its 1-2 subsystem is noncontextual, i.e., if and only if one of the* R_1^1 and R_1^2 *nominally dominates the other.*

Proof of Theorem 4.6. The "only if" part is Theorem 4.1. All we need to proof the "if " part is to check that the relation (4.8) holds. Assume the arrangement is as in the previous corollary. Consider first any set $i_1, \ldots, i_m$ that does not include 2:

$$\min\left(p_{i_1} + p_{i_2} + \ldots + p_{i_m}, q_{i_1} + q_{i_2} + \ldots + q_{i_m}\right) = q_{i_1} + q_{i_2} + \ldots + q_{i_m},$$

$$\sum_{j=1}^{m} \min\left(p_{i_j}, q_{i_j}\right) = q_{i_1} + q_{i_2} + \ldots + q_{i_m},$$

$$\min\left(p_{i_j} + p_{i_{j'}}, q_{i_j} + q_{i_{j'}}\right) - \min\left(p_{i_j}, q_{i_j}\right) - \min\left(p_{i_{j'}}, q_{i_{j'}}\right) = 0.$$

So, (4.8) holds. If one of the indices (let it be i_1) is 2, then

$$q_2 + q_{i_2} + \ldots + q_{i_m} = \left(p_2 + \sum_{x \neq 2} (p_x - q_x)\right) + q_{i_2} + \ldots + q_{i_m} > p_2 + p_{i_2} + \ldots + p_{i_m},$$

so

$$\min\left(p_2 + p_{i_2} + \ldots + p_{i_m}, q_2 + q_{i_2} + \ldots + q_{i_m}\right) = p_2 + p_{i_2} + \ldots + p_{i_m}.$$

We also have

$$\sum_{j=1}^{m} \min\left(p_{i_j}, q_{i_j}\right) = p_2 + q_{i_2} + \ldots + q_{i_m},$$

and for any $j \neq 2, j' \neq 2$,

$$\min\left(p_{i_j} + p_{i_{j'}}, q_{i_j} + q_{i_{j'}}\right) - \min\left(p_{i_j}, q_{i_j}\right) - \min\left(p_{i_{j'}}, q_{i_{j'}}\right) = 0,$$

$$\min\left(p_2 + p_{i_j}, q_2 + q_{i_j}\right) - \min\left(p_2, q_2\right) - \min\left(p_{i_j}, q_{i_j}\right) = p_{i_j} - q_{i_j}.$$

Since index $i_1 = 2$ is paired with each of $i_2, \ldots, i_m$ only once, the right-hand side in (4.8) is

$$p_2 + q_{i_2} + (p_{i_2} - q_{i_2}) + \ldots + q_{i_m} + (p_{i_m} - q_{i_m}) = p_2 + p_{i_2} + \ldots + p_{i_m}.$$

□