SUPPORTING INFORMATION

Weinberger et. al. Innovations and the growth of human population

Part I: Mathematical Analysis of the Basic Model

In the main text we introduce the basic model (Eqs. 1 and 3 in the main text) that accounts for the dynamics of human population biomass ϕ and habitat quality ω as:

$$(1) \qquad \begin{cases} \phi' = \big(f\,\omega - m\big)\phi(1-\phi), \\ \omega' = \lambda(1-\omega) - (B + Ef\,\omega)\phi, \end{cases}$$

Fixed points are found by solving the previous set of equations in the time-independent case. Possibles outcomes are classical and characterized by

- Extinction $\phi^* = 0$ and $\omega^* = 1$. By performing a linear stability analysis, the eigenvalues of the Jacobian matrix are given by $f - m$ and $-\lambda$. The stability of the extinction scenario holds true only when $f < m$ meaning than fecundity is smaller than mortality.
- Habitat Limitation $\omega^* = m/f$. Only attainable when $f > m$, population size is then limited by the habitat quality at large times. It holds that
$$\phi^* = G(\lambda) = \min\left\{1, \frac{\lambda(1-m/f)}{B+Em}\right\}.$$
- Space Limitation $\phi^* = 1$. The habitat quality is sufficient and human biomass "fills" its habitat. It follows that
$$\omega^* = H(\lambda) := \frac{\lambda - B}{\lambda + Ef},$$
and eigenvalues are $-(\lambda + Ef)$ and $-(f\,H(\lambda) - m)$, therefore stability holds true only when
$$H(\lambda) > m/f.$$

Set $\lambda^* = (B + Em)/(1 - m/f)$, since $H(\cdot)$ is a strictly increasing function, then the following dichotomy holds:

- For any value $\lambda > \lambda^*$ the Space Limitation equilibrium is stable
- For any value $\lambda < \lambda^*$ the Habitat Limitation equilibrium kicks in.

Part II: Extended model

The basic model now includes the impact of technological innovation upon the supply and potential deterioration of ecosystem services, and keeps track of the technological stock μ. Under the simplifications explained in the main text, the dynamics are given by

$$(2) \qquad \begin{cases} \phi' = \big(f\,\omega - m\big)\phi(1-\phi), \\ \omega' = c_1\mu^{\alpha}(1-\omega) - c_2\mu^{\beta}\phi, \\ \mu' = \rho\mu\phi - l(\mu - \varepsilon) \end{cases}$$

Notice that for the sake of simplicity, the equation for ω does not include the term $Ef\omega$ that accounts for the maintenance energy of a unit biomass since this is very small as compared to the extra metabolic energy required for individuals in a social group engaged in social learning dynamics and CCE. The set of fixed points associated to the extended model (2) is more complicated than in the basic model, and stability conditions cannot be trivially listed. Linear stability of any fixed point (ϕ, ω, μ) is determined by the sign of the larger eigenvalue of

$$\mathbf{J}(\phi,\omega,\mu) := \begin{bmatrix} (f\,\omega - m)(1-2\phi) & f\,\phi(1-\phi) & 0 \\ -c_2\mu^\beta & -c_1\mu^\alpha & \alpha c_1 \mu^{\alpha-1}(1-\omega) - \beta c_2 \mu^{\beta-1}\phi \\ \rho\mu & 0 & \rho\phi - l \end{bmatrix}.$$

The steady states are those points (ϕ, ω, μ) such that the left hand side of (2) becomes zero, i.e.

$$\begin{cases} 0 = \big(f\,\omega - m\big)\phi(1-\phi), \\ 0 = c_1\mu^\alpha(1-\omega) - c_2\mu^\beta\phi, \\ 0 = \rho\mu\phi - l(\mu - \varepsilon), \end{cases}$$

and from the first equation we notice that the solutions are restrained to the three following cases

$$\phi^* = 0 \text{ [extinction]}, \qquad \phi^* = 1 \text{ [space limitation]}, \qquad \omega^* = m/f \text{ [habitat limitation]}.$$

Indeed, if $(\phi^*, \omega^*, \mu^*)$ is such that

$$(f\omega^* - m)\phi^*(1-\phi^*) = 0,$$

then at least one of the factors must be exactly zero from where we obtain the only candidates to steady states are under the form listed above.

Remark 1: In the following and unless otherwise specified, we assume that $l > \rho$. The case $l < \rho$ will be analyzed separately in the final part of these notes. Condition $\rho < l$ means that technologies with great impact on further technological innovation should have small lifetimes.

Extinction $\phi^* = 0$

The other two variables are given by the solution of the system

$$\begin{cases} 0 = c_1|\mu^*|^\alpha(1-\omega^*) \\ 0 = -l(\mu^* - \varepsilon), \end{cases}$$

meaning that $\omega^* = 1$ and $\mu^* = \varepsilon$. The eigenvalues of the Jacobian matrix are $(f - m)$, $-c_1\varepsilon^\alpha$ and $-l$. Once again, stability of the extinction scenario is given by the condition $f < m$.

Space Limitation $\phi = 1$

The system that give us the variables ω^* and μ^* is

$$\begin{cases} 0 = c_1|\mu^*|^\alpha(1-\omega^*) - c_2|\mu^*|^\beta, \\ 0 = \rho\mu^* - l(\mu^* - \varepsilon) \end{cases}$$

and the solution is simply

$$\mu^* = \frac{l\varepsilon}{l-\rho}, \qquad \omega^* = 1 - \frac{c_2|\mu^*|^\beta}{c_1|\mu^*|^\alpha} = 1 - c_0^{-1}|\mu^*|^{-\theta} = 1 - \frac{1}{c_0}\left(\frac{l-\rho}{l\varepsilon}\right)^\theta,$$

with $\theta = \alpha - \beta$ and $c_0 = c_1/c_2$. Since μ^* must be nonnegative, it is necessary that $\rho < l$. While this condition ensures the existence of a fixed, positive μ^* equilibrium point, it also forces a limit upon technological stock, constraining the possibility for infinite and/or arbitrarily large values of μ, especially when ε or the minimum

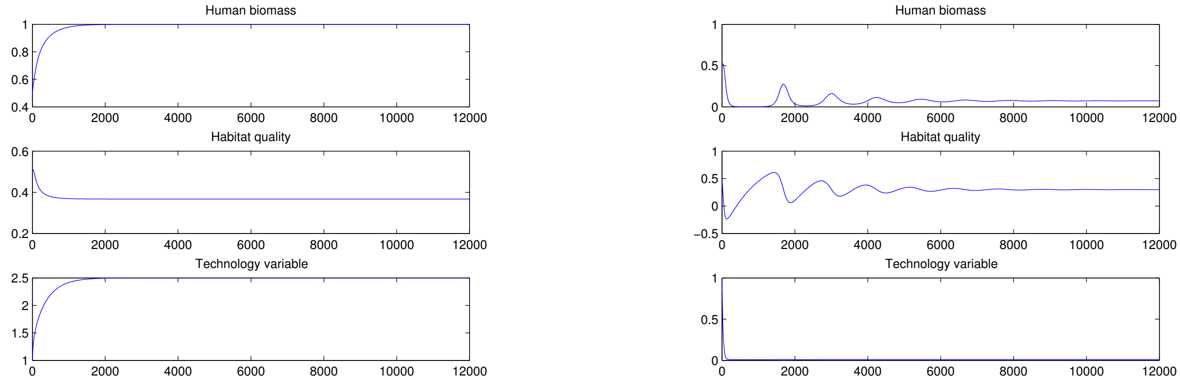


Figure S1: Space limitation scenario with $\theta = 0.5$. Parameters: $\beta = 0.9$, $\rho = 0.03$, $l = 0.05$, $f = 0.04$, $m = 0.012$ and $\varepsilon = 1$ (left panel) $\varepsilon = 0.01$ (right panel). Notice that in the right panel the stability condition $\frac{l\varepsilon}{l-\rho} = \mu^* > \exp\left[\frac{1}{\theta}\log\left(\frac{f}{c_0(f-m)}\right)\right]$ (see below) does not hold and we are not longer in the Space Limitation equilibrium but in the Habitat Limitation one and specifically in the regime HL1 in Figure 3 in the main text.

technological toolkit is small (Figure S1). Indeed, as μ increases the dynamics for the technology variable are such that

$$\mu' = \rho\mu\phi - l(\mu - \varepsilon) \leq (\rho - l)\mu + l\varepsilon < 0$$

implying that μ cannot keep increasing.

Jacobian matrix becomes

$$\mathbf{J}(1, \omega^*, \mu^*) := \begin{bmatrix} -(f\,\omega^* - m) & 0 & 0 \\ -c_2|\mu^*|^\beta & -c_1|\mu^*|^\alpha & \alpha c_1|\mu^*|^{\alpha-1}(1-\omega^*) - \beta c_2|\mu^*|^{\beta-1} \\ \rho\mu^* & 0 & \rho - l \end{bmatrix},$$

and the eigenvalues are

$$-(f\,\omega^* - m), \qquad -c_1|\mu^*|^\alpha < 0, \qquad \rho - l < 0 \text{ [see remark 1]}.$$

Since stable conditions require negative eigenvalues, and knowing that that both $-c_1|\mu^*|^\alpha$ and $\rho - l$ are negative, this imply that the stability condition is given by:

$$1 - c_0^{-1}|\mu^*|^{-\theta} = \omega^* > \frac{m}{f} \quad \Rightarrow \quad c_0\left(1 - \frac{m}{f}\right) > |\mu^*|^{-\theta},$$

meaning that the steady state of the habitat quality variable ω^* is larger than m/f. This can be obtained in the following subcases:

- Positive technological impact $\theta > 0$ (Figure 1 and see also SL1 Figure 3 in the main text). To have the stability condition, the steady value for the technology variable μ^* must be such that

$$\frac{l\varepsilon}{l-\rho} = \mu^* > \exp\left[\frac{1}{\theta}\log\left(\frac{f}{c_0(f-m)}\right)\right],$$

  getting a bound by below for μ^*. Considering the parameters given in Figure 1, this condition implies that $\mu^* > 1.36$, which is satisfied for $\varepsilon = 1$ where $\mu^* = 2.5$ but not for $\varepsilon = 0.01$ since $\mu^* = 0.025$. The equilibrium in this later case has shifted to the Habitat limitation one (HL1, see Figure 3 in the main text and below).

- Negative technological impact $\theta < 0$ (Figure 2, see SL2 in Figure 3). For convenience, let $\Theta = -\theta > 0$ such that the steady value for μ^* writes

$$c_0\left(1 - \frac{m}{f}\right) > |\mu^*|^{-\theta} = |\mu^*|^\Theta \quad \Leftrightarrow \quad \exp\left[\frac{1}{\Theta}\log\left(\frac{c_0(f-m)}{f}\right)\right] > \mu^* = \frac{l\varepsilon}{l-\rho},$$

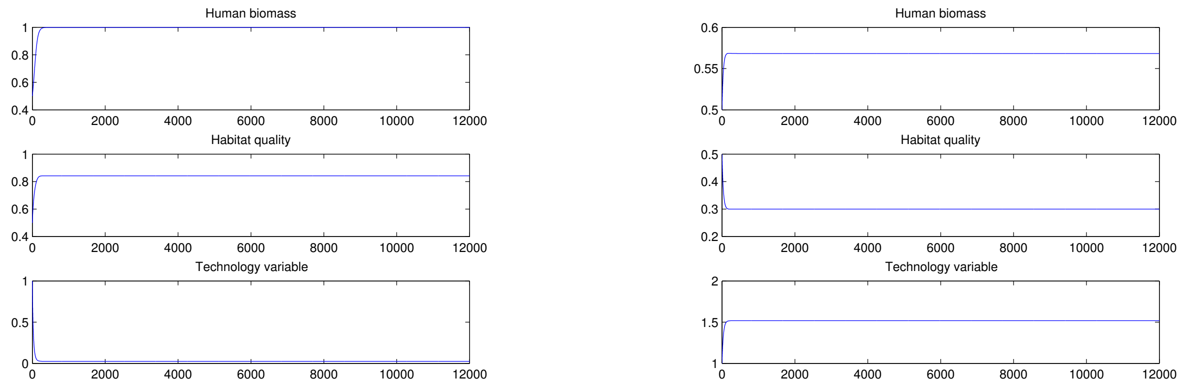


Figure S2: Space limitation scenario with $\theta = -0.5$. Parameters: $\rho = 0.03$, $l = 0.05$, $f = 0.04$, $m = 0.012$ and $\varepsilon = 1$ (right) $\varepsilon = 0.01$ (left). Notice that in the right panel the stability condition $\frac{l\varepsilon}{l-\rho} = \mu^* < \exp\left[\frac{1}{\Theta}\log\left(\frac{c_0(f-m)}{f}\right)\right]$ (see above) does not hold and we are not longer in the Space Limitation equilibrium but in the Habitat Limitation one and specifically in the regime HL2 in Figure 3 in the main text.

getting now an upper bound for steady state μ. Everything else being equal, for large values of ε the SL2 equilibrium becomes unstable and the dynamics transitions to the HL2 equilibrium (see Figure 3 in the main text and below).

- Zero technology impact $\theta = 0$. We have $1 - \frac{m}{f} > c_0^{-1} = \frac{c_2}{c_1}$, in particular, c_2 must be smaller than $c_1(1 - m/f)$.

Habitat Limitation

In this scenario habitat quality takes value $\omega^* := m/f$, thus restricting the maximal stable population size ϕ^* and the technology variable μ^*. According to the value of θ different sub-scenarios are possible. Solutions to the steady values of ϕ and μ *can not be trivially listed* and we only present the following formulas

$$\phi^* = c_0|\mu^*|^\theta\left(1 - \frac{m}{f}\right), \qquad \mu^* = \frac{l\varepsilon}{l - \rho\phi^*}.$$

It might seem that it suffices to know μ^* to have ϕ^*, but at the same time we need to know ϕ^* to characterise μ^*, thus solutions are not explicit. However, from the second formula, we do have that the solution ϕ^* is also given by

$$\phi^* = \frac{l}{\rho}\left(1 - \frac{\varepsilon}{\mu^*}\right),$$

then all admisible values μ^* are given by the solutions to the nonlinear equation

$$\frac{l}{\rho}\left(1 - \frac{\varepsilon}{\mu^*}\right) = c_0|\mu^*|^\theta\left(1 - \frac{m}{f}\right),$$

that can be seen as the points of intersection between the curves of the left and right side of the equation.

For $x > 0$, we can define the function

$$f_1(x) := \frac{l}{\rho}\left(1 - \frac{\varepsilon}{x}\right),$$

which is strictly increasing from $-\infty$ to l/ρ. On the other hand, the function

$$f_2(x) := c_0 x^\theta\left(1 - \frac{m}{f}\right)$$

is always positive and has different shapes depending on the value θ, in particular

($\theta > 0$) in this case $f_2(\mu)$ is a strictly increasing function that starts from 0 and diverges to infinity. The existence of a fixed point is given by the particular values of the parameters.

($\theta < 0$) meaning that $f_2(\mu)$ is a strictly decreasing function going from $+\infty$ to 0. We conclude that for any set of parameters there is only one ϕ^* and only one μ^* fixed point when $\omega^* = m/f$.

($\theta = 0$) then $f_2(\mu)$ is constant and equal to $c_0(1 - m/f)$. We have solutions μ^* and ϕ^* to the steady state system, when $\omega^* = m/f$ if and only if $c_0 < l/\rho(1 - m/f)$.

Nonetheless, in any of the previous subcases, we need to impose that $0 < \phi^* < 1$ (otherwise we are in the extinction or in the space limitation case) it follows that

$$0 < \phi^* = \frac{l}{\rho}\left(1 - \frac{\varepsilon}{\mu^*}\right) < 1 \quad \Rightarrow \quad \varepsilon < \mu^* < \frac{\varepsilon l}{l - \rho},$$

implying that the steady state of μ^* is bounded.

Infinite technology analysis

Conditions for obtaining infinite and/or arbitrarily large technologies ($\rho > l$ and $\rho\phi^* > l$) do not allow reaching stable equilibria; however, if we make μ diverge to infinity, under convergence of ϕ and ω, long-term dynamics can be analysed by reinterpreting the Basic Model. In this final subsection we explain roughly how to proceed in this case.

Since the linear stability analysis is difficult to perform explicitly for a 3 dimensional ODE system (it involves the Routh-Hurwitz stability criterion), we only focus in a dimensional reduction of the model. To have a notion of the stability of the system we assume first that μ is a parameter and then take the limit as μ goes to infinity. The Extended Model is then reduced to

$$(3) \qquad \begin{cases} \phi' = \big(f\,\omega - m\big)\phi(1 - \phi), \\ \omega' = c_1\mu^\alpha(1 - \omega) - c_2\mu^\beta\phi, \end{cases}$$

which is nothing but the Basic Model with λ and B rewritten as functions on the variable μ. Again, two non trivial steady states emerge:

Space Limitation

The steady state is $\phi^* = 1$ and

$$\omega^* = 1 - c_0^{-1}\mu^{-\theta},$$

then taking $\mu \to \infty$:

($\theta > 0$) habitat quality ω^* goes to 1, and the carrying capacity $\phi^* = 1$ is stable.

($\theta < 0$) habitat quality ω^* goes to 0 and $\phi^* = 1$ becomes unstable.

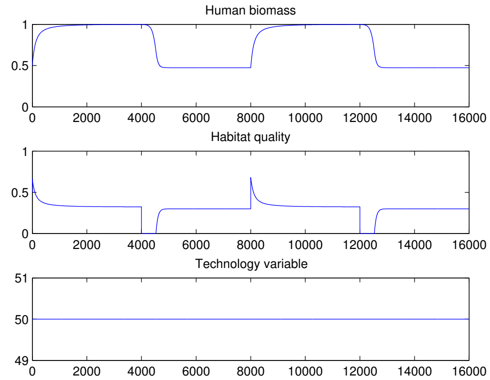
($\theta = 0$) once again carrying capacity $\phi^* = 1$ is stable if and only if $1 - c_0^{-1} > m/f$.

Habitat Limitation

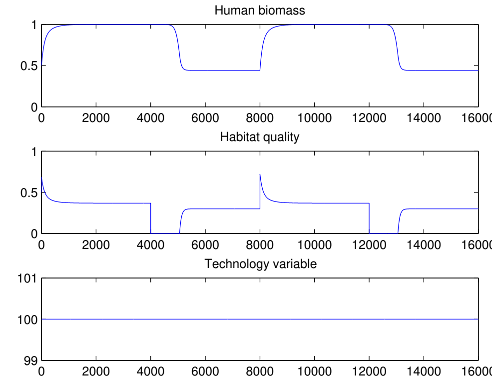
We have now $\omega^* = m/f$ and

$$\phi^* = \min\left\{1, c_0\mu^\theta(1 - \omega^*)\right\}.$$

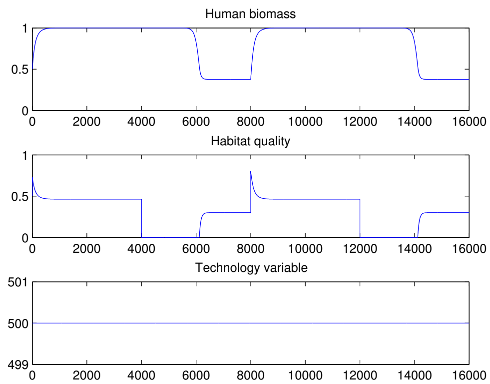
Population can go to the carrying capacity if and only if $\theta > 0$ or in words, technological costs are smaller than the benefits they accrue in terms of energy and material flows of ecosystem services. However, as soon as $\phi^* = 1$ we fall into the Space Limitation case previously studied.



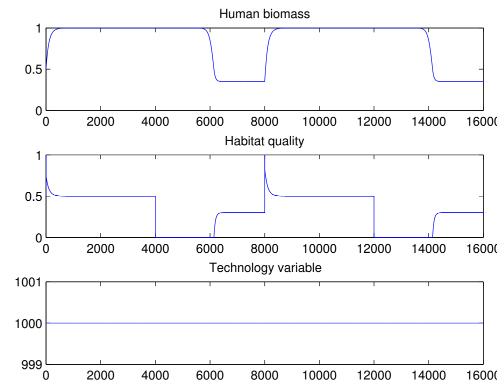
$\mu_0 = 50$



$\mu_0 = 100$



$\mu_0 = 500$



$\mu_0 = 1000$

Figure S3: Simulations for the reduced two dimensional model with $\mu(t) = \mu_0^{\theta(t)}$. Parameters: $\theta_0 = 0.1$, $\tau = 4000$, $\rho = 0.09$, $l = 0.05$, $f = 0.04$, $m = 0.012$ and $\varepsilon = 0.01$.

Further explorations

In this final section we explore numerically the case when $l < \rho$, more precisely, we simulate the system of the previous section for $\mu(t) = \mu_0$ fixed and focus on the changes of ϕ and ω as μ_0 takes large values. In particular, we are concerned with the effects on the dynamics of sudden changes in the sign of θ and the resilience to change of the solutions.

Consider a time horizon τ, that $\theta(t = 0) = \theta_0$ is positive at time 0 and that it remains constant for $t \in [0, \tau)$. At $t = \tau$ we change the sign of θ and let $\theta(t)$ be constant and equal to $-\theta_0$ for $t \in [\tau, 2\tau)$. Iterating this procedure adequately we will find that the solutions of the system behave as periodic functions whose shapes are determined by the function

$$\mu(t) = \begin{cases} \mu_0^{\theta_0} & \text{if } t \in [2k, 2k+1), \text{ some } k \in \mathbb{N} \\ \mu_0^{-\theta_0} & \text{if } t \in [2k-1, 2k), \text{ some } k \in \mathbb{N} \end{cases}$$

This situation is depicted in Figure S3 where we have used the parameter values explained in the main text. It is interesting to remark that for μ_0 large, the population ϕ takes more time to be sensitive to the change of θ and they remain close to the saturation point $\phi^* = 1$ for a larger time interval after the critical times $(2k-1)\tau$, $k \in \mathbb{N}$. Also, the recuperation of the population and convergence towards $\phi^* = 1$ after any time $2k\tau$, $k \in \mathbb{N}$ is faster. However, the opposite occurs for habitat quality, as technological stock increases, oscillations in habitat quality becomes larger and remain destructed or extinguished for longer time lapses.