Appendix 2. A brief introduction to measuring multifractality

Multifractality, as its name suggests, is the case of multiple fractalities. One way to understand fractality is to revisit (briefly!) some fundamental concepts of probability. We begin with a straightforward dice-rolling example to show how standard deviation can reveal temporal correlations, a statistical signature of the echoing across time scales noted above. Temporal correlations exemplify fractality.

Dice roll example

Let us say that we roll two six-sided dice, over and over again, taking the sum of the two numbers that roll face-up on the dice. Let us assume that the dice are fairly constructed so that, for each die, there is an even probability of each face, that is, for the numbers 1, 2, 3, 4, 5, and 6. As we roll the two dice, pick them up, and roll again, there should be no sequence between the two-dice sum for each roll. If we took note of the sums as we rolled the dice, we might record the following sequence: 4, 7, 9, 3, 7, 10, 8, 6, … and so forth. These values would eventually converge around an average of 6.5, and the standard deviation of the two-dice sums around that mean would converge at some constant value determined by the hands of the dice roller, the edges of the dice, the surface where the tumbling dice roll, etc. By the thousandth or so dice roll, the standard deviation will not budge very much from what it had been after the hundredth or so dice roll.

If we construct what's called a "random walk" from these dice rolls, we get a different result. Random-walk variability grows slowly for independent random events. A random walk is summing up of progressive values of individual measured "steps." In this case, each steps in the random walk will be from each dice roll. So, our random walk for the two-dice rolls from above would start out with 4 because that is the first "step." Our random walk would continue with 4+7 = 11, 4+7+9=20, and so forth. The random walk for the first eight dice rolls shown above would be 4, 11, 20, 23, 30, 40, 48, and 54. Because each number on the dice faces is positive, the random walk of the dice-roll sums will always increase, and so, whereas the series of individual dice-roll sums eventually settles on a stable value, the random walk for the dice-roll sums has a standard deviation that always increases. If the dice and the rolls are completely fair, if there's no effect of one dice roll on the next, then we can be fairly sure that, once the series of individual dice-rolls settle on its stable standard deviation, the standard deviation of the random walk will increase slowly, at a rate defined by the square root of roll number. So, for instance, the standard deviation of the random walk at 100 rolls will be double what it was at 25 rolls, and the standard deviation of the random walk at 400 will be double what it was at 100 rolls. That is, for fair dice with fair rolling, when there is no temporal correlation from one dice roll to the next, the standard deviation grows slowly, taking ever more rolls to double.

Faster growth of variability in temporally correlated events

What if your dice are loaded? What if someone has perfected the art of rolling the dice to manipulate the outcome? What if, essentially, there is a distinct relationship across time from one dice roll to the next? In that case, the random walk will grow much more erratically. If the dice-roll sums were temporally correlated, then the random walk would show us a standard deviation increasing much more quickly than before. We might see a standard deviation at 50 rolls double that of what is was at 25 rolls, and again doubling at 100 rolls and once more at 200 rolls. The temporally correlated dice-roll sums would exhibit a random walk with much more rapid growth of variability than we had seen with the random walk of the uncorrelated dice-roll sums. This more rapid growth of variability is often a signature of temporal correlations.

Estimating temporal correlations from standard deviation

Fractal analyses are essentially ways to assess the standard deviation for measured random-walk series. We take a measurement, construct the random walk by taking the cumulative sums as for the dice-rolls. To diagnose a measurement as "fractal," we need to be sure that the fast growth of random-walk standard deviation has to do specifically with temporal correlations. Fast growth of random-walk standard deviation might simply be due to nonstationary (i.e. "explosive") growth: that is, you may have perfected the art of rolling the same number, and as you roll more and more, you might find yourself preferring to roll 12 every time or a 2 every time. Rolling 12 every time and rolling 2 every time would reflect the same temporal correlations, but rolling 12 every time would cause standard deviation to positively explode whereas rolling 2 every time would provide only minimal increase. We want a way to assess temporal correlations, the dependence among consecutive dice-roll sums, without being fooled by trends in the data.

The same physiological, behavioral, and cognitive measurements that inform physical therapy can often be nonstationary. So, judicious use of fractal methods requires adequate removal of trends. Different data may require different methods particularly as different processes as well as different task constraints may produce different trends. [56,57]

Reference list given in: Cavanaugh J et al. Multifractality, interactivity, and the adaptive capacity of the human movement system: a perspective for advancing the conceptual basis of neurologic physical therapy. *J Neurol Phys Ther*. 2017; Oct 40 (4).