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Supplementary material for 'An adaptive two-sample test for high-dimensional means'

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SUMMARY

This supplementary material contains the proof of the main results in Section 1, additional simulation studies in Sections 2 and 3, the complete results of the real data analysis in Section 4, and example R code for simulations in Section 5.

1. PROOF OF MAIN RESULTS

The proofs in the supplementary material assume that the covariance matrices for the two groups of sample are Σ_1 and Σ_2 respectively; it is easy to simplify the results under the common covariance matrix $\Sigma = (\sigma_{ij})_{p\times p}$ assumption as presented in the main section. We write $\Sigma_k = (\sigma_{k,ij})_{p\times p}$ for $k = 1$ and 2.

We need the following assumptions, which extend conditions C1–C3 in Section 3.2 of the main file to the case when Σ_1 and Σ_2 are different.

C1 Covariance assumption. There exists some constant B such that

$$B^{-1} \le \lambda_{\min}(\Sigma_1), \lambda_{\min}(\Sigma_2), \lambda_{\max}(\Sigma_1), \text{ and } \lambda_{\max}(\Sigma_2) \le B,$$

where $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalues of a matrix A. In addition, the correlations are bounded away from -1 and 1, i.e.,

$$\max_{k=1,2;1\le i\ne j\le p} |\sigma_{k,ij}| / (\sigma_{k,ii}\sigma_{k,jj})^{1/2} < 1 - \eta$$

for some $\eta > 0$.

C2 Mixing assumption. For a set of multivariate random vectors $Z = \{Z^{(j)} : j \geq 1\}$ and integers $a < b$, let $\mathcal{Z}_a^b$ be the σ-algebra generated by $\{Z^{(j)} : j \in [a, b]\}$. For each $s \geq 1$, define the α-mixing coefficient $\alpha_Z(s) = \sup_{t \geq 1}\{|\mathrm{pr}(A \cap B) - \mathrm{pr}(A)\mathrm{pr}(B)| : A \in \mathcal{Z}_1^t, B \in \mathcal{Z}_{t+s}^{\infty}\}$. We assume $\{(X_{ki}^{(j)}, i = 1, \ldots, n_k) : j \geq 1\}$ is α-mixing for $k = 1, 2$, and $\alpha_X(s) \leq M\delta^s$, where $\delta \in (0, 1)$ and M is some constant.

C3 Moment assumption. We assume $\log p/n^{1/4} = o(1)$. In addition, there exists a positive constant M and $h \in [-M, M]$,

$$\max_{k=1,2;1\leq i\leq p} \mathrm{E}\left\{e^{h(X_{k1}^{(i)} - \mu_k^{(i)})^2}\right\} < \infty.$$

Remark 1. The covariance assumption C1 and the moment assumption C3 follow from Cai et al. (2014) and they are needed to establish the weak convergence of the supremum type test statistic, $L(\infty)$. When $\gamma < \infty$, the asymptotic normality can be established under weaker assumptions on the eigenvalues and correlations. However, in order to establish weak convergence of $L(\gamma)$ for $\gamma > 2$, stronger moment assumptions may still be needed than those in Chen & Qin (2010), whose test statistic is similar to $L(2)$. Condition C2 follows from Chen et al. (2014) and the mixing assumption imposes the weak dependence structure of the data. Such condition is commonly used in time series and spatial statistics. A similar mixing assumption has also been adopted in Zhong et al. (2013). Alternatively, we may consider the weak dependence structure introduced in Bai & Saranadasa (1996) and Chen & Qin (2010), where a factor type model for X is assumed. Similar asymptotic behavior is expected for the proposed adaptive test with certain regularity conditions and we will pursue this in our future research. Since the variables (i.e. single nucleotide polymorphisms) in the motivating genome-wide association studies have a local dependency structure with their correlations often decaying to zero as their physical distances (on chromosomes) increase, we focus on the mixing type weak dependence in this paper.

We need the following lemmas to calculate the mean and variance functions in the propositions.

LEMMA 1. *For $k \in \{1, 2\}$ and $1 \leq i \leq p$, we have:*

(i) if a is even and $a = 2d$,

$$\mathrm{E}\left\{\left(\bar{X}_k^{(i)} - \mu_k^{(i)}\right)^a\right\} = \frac{a!}{d!2^d} n_k^{-d} \sigma_{k,ii}^d + o\left(n_k^{-d}\right)$$

(ii) if $a \geq 3$ is odd and $a = 2d + 1$

$$\mathrm{E}\left\{\left(\bar{X}_k^{(i)} - \mu_k^{(i)}\right)^a\right\} = \frac{a!}{(d-1)!3!2^{d-1}} n_i^{-(d+1)} m_{ki} \sigma_{k,ii}^{d-1} + o\left(n_k^{-(d+1)}\right),$$

where m_{ki} is the third central moment of ith marginal random variable from group k, i.e., $m_{ki} = \mathrm{E}\left\{(X_k^{(i)} - \mu_k^{(i)})^3\right\}$. Note that in this case, $\mathrm{E}\left\{(\bar{X}_k^{(i)} - \mu_k^{(i)})^a\right\} = o(n_k^{-a/2})$.

LEMMA 2. *For $k \in \{1, 2\}$ and $1 \leq i, j \leq p$, consider integers $h, l \geq 1$. If $h + l$ is an even number with $h + l = 2c$,*

$$\mathrm{E}\left\{\left(\bar{X}_k^{(i)} - \mu_k^{(i)}\right)^h \left(\bar{X}_k^{(j)} - \mu_k^{(j)}\right)^l\right\}$$
$$= \frac{1}{n_k^c} \sum_{\substack{2c_1+c_3=h \\ 2c_2+c_3=l}} \frac{h!l!}{c_1!c_2!c_3!2^{c_1+c_2}} \sigma_{k,ii}^{c_1} \sigma_{k,jj}^{c_2} \sigma_{k,ij}^{c_3} + o\left(n_k^{-c}\right);$$

if $h+l$ *is an odd number with* $h+l=2c+1$

$$\begin{aligned}
&\mathrm{E}\left\{\left(\bar{X}_k^{(i)}-\mu_k^{(i)}\right)^h\left(\bar{X}_k^{(j)}-\mu_k^{(j)}\right)^l\right\}\\
&=\frac{1}{n_k^{c+1}}\sum_{\substack{a+b=3\\2c_1+c_3=h-a\\2c_2+c_3=l-b}}\frac{h!l!}{c_1!c_2!a!b!c_3!2^{c_1+c_2}}m_{k,i^aj^b}\sigma_{k,ii}^{c_1}\sigma_{k,jj}^{c_2}\sigma_{k,ij}^{c_3}+o\left(n_k^{-c-1}\right),
\end{aligned}$$

where $m_{k,i^aj^b}=\mathrm{E}\left\{(X_k^{(i)}-\mu_k^{(i)})^a(X_k^{(j)}-\mu_k^{(j)})^b\right\}$ *for* $a+b=3$.

Proof of Lemma 1. For even a with $a=2d$ and $k=1,2$,

$$\begin{aligned}
\mathrm{E}\left\{\left(\bar{X}_k^{(i)}-\mu_k^{(i)}\right)^a\right\}&=\frac{1}{n_k^a}\mathrm{E}\left[\left\{\sum_{j=1}^{n_k}\left(X_{kj}^{(i)}-\mu_k^{(i)}\right)\right\}^a\right]\\
&=\frac{1}{n_k^a}\sum_{\substack{t\geq1;l_1,\ldots,l_t>0\\l_1+\ldots+l_t=a}}\binom{n_k}{t}\frac{a!}{l_1!\ldots l_t!}\prod_{s=1}^{t}\mathrm{E}\left\{\left(X_{k1}^{(i)}-\mu_k^{(i)}\right)^{l_s}\right\}\\
&\sim\frac{1}{n_k^a}\sum_{\substack{t\geq1;l_1,\ldots,l_t>0\\l_1+\ldots+l_t=a}}\frac{n_k^ta!}{t!l_1!\ldots l_t!}\prod_{s=1}^{t}\mathrm{E}\left\{\left(X_{k1}^{(i)}-\mu_k^{(i)}\right)^{l_s}\right\}\\
&=\sum_{\substack{t\geq1;l_1,\ldots,l_t>0\\l_1+\ldots+l_t=a}}\frac{a!}{n_k^{a-t}t!l_1!\ldots l_t!}\prod_{s=1}^{t}\mathrm{E}\left\{\left(X_{k1}^{(i)}-\mu_k^{(i)}\right)^{l_s}\right\}\\
&=\sum_{\substack{t=d\\l_1=\ldots=l_t=2}}\frac{a!}{n_k^{a-t}t!l_1!\ldots l_t!}\sigma_{k,ii}^d+o\left(n_k^{-d}\right)\\
&=\frac{a!}{d!2^d}n_k^{-d}\sigma_{k,ii}^d+o\left(n_k^{-d}\right).
\end{aligned}$$

For odd a with $a=2d+1$ and $k=1,2$, we can obtain the conclusion from a similar argument:

$$\begin{aligned}
\mathrm{E}\left\{\left(\bar{X}_k^{(i)}-\mu_k^{(i)}\right)^a\right\}&\sim\sum_{\substack{t\geq1;l_1,\ldots,l_t>0\\l_1+\ldots+l_t=a}}\frac{a!}{n_k^{a-t}t!l_1!\ldots l_t!}\prod_{s=1}^{t}\mathrm{E}\left\{\left(X_{k1}^{(i)}-\mu_k^{(i)}\right)^{l_s}\right\}\\
&=\sum_{\substack{t=d\\\text{one } l_s \text{ is } 3\\\text{others are } 2}}\frac{a!}{n_k^{a-t}t!l_1!\ldots l_t!}m_{ki}\sigma_{k,ii}^{d-1}+o\left(n_k^{-d-1}\right)\\
&=d\times\frac{a!}{d!3!2^{d-1}}n_k^{-(d+1)}m_{ki}\sigma_{k,ii}^{d-1}+o\left(n_k^{-d-1}\right)\\
&=\frac{a!}{(d-1)!3!2^{d-1}}n_k^{-(d+1)}m_{ki}\sigma_{k,ii}^{d-1}+o\left(n_k^{-d-1}\right).
\end{aligned}$$

This completes the proof of Lemma 1. □

Proof of Lemma 2. The proof is similar to that of Lemma 1. In particular, if $h+l$ is an even number with $h+l=2c$

$$
\begin{aligned}
& E\left\{\left(\bar{X}_k^{(i)}-\mu_k^{(i)}\right)^h\left(\bar{X}_k^{(j)}-\mu_k^{(j)}\right)^l\right\} \\
&= E\left[\frac{1}{n_k^{h+l}}\left\{\sum_{s=1}^{n_k}(X_{ks}^{(i)}-\mu_k^{(i)})\right\}^h\left\{\sum_{t=1}^{n_k}(X_{kt}^{(j)}-\mu_k^{(j)})\right\}^l\right] \\
&= \frac{1}{n_k^{h+l}}\sum_{\substack{2c_1+c_3=h\\2c_2+c_3=l}}\binom{h}{c_3}\binom{l}{c_3}\frac{n_k^{c_1+c_2+c_3}(2c_1)!(2c_2)!c_3!}{c_1!c_2!2^{c_1+c_2}} \\
&\quad\times\left\{E(X_{k1}^{(i)}-\mu_k^{(i)})^2\right\}^{c_1}\left\{E(X_{k1}^{(j)}-\mu_k^{(j)})^2\right\}^{c_2}\left\{E(X_{k1}^{(i)}-\mu_k^{(i)})(X_{k1}^{(j)}-\mu_k^{(j)})\right\}^{c_3} \\
&\quad+o\left(\frac{1}{n_k^c}\right) \\
&= \frac{1}{n_k^c}\sum_{\substack{2c_1+c_3=h\\2c_2+c_3=l}}\binom{h}{c_3}\binom{l}{c_3}\frac{(2c_1)!(2c_2)!c_3!}{c_1!c_2!2^{c_1+c_2}}\sigma_{k,ii}^{c_1}\sigma_{k,jj}^{c_2}\sigma_{k,ij}^{c_3}+o\left(\frac{1}{n_k^c}\right) \\
&= \frac{1}{n_k^c}\sum_{\substack{2c_1+c_3=h\\2c_2+c_3=l}}\frac{h!l!}{c_1!c_2!c_3!2^{c_1+c_2}}\sigma_{k,ii}^{c_1}\sigma_{k,jj}^{c_2}\sigma_{k,ij}^{c_3}+o\left(\frac{1}{n_k^c}\right).
\end{aligned}
$$

If $h+l$ is an odd number with $h+l=2c+1$, similarly, we have

$$
\begin{aligned}
& E\left\{\left(\bar{X}_k^{(i)}-\mu_k^{(i)}\right)^h\left(\bar{X}_k^{(j)}-\mu_k^{(j)}\right)^l\right\} \\
&= E\left[\frac{1}{n_k^{h+l}}\left\{\sum_{s=1}^{n_k}(X_{ks}^{(i)}-\mu_k^{(i)})\right\}^h\left\{\sum_{t=1}^{n_k}(X_{kt}^{(j)}-\mu_k^{(j)})\right\}^l\right] \\
&= \frac{1}{n_k^{c+1}}\sum_{\substack{a+b=3\\2c_1+c_3=h-a\\2c_2+c_3=l-b}}\binom{h}{a}\binom{l}{b}\binom{h-a}{c_3}\binom{l-b}{c_3} \\
&\qquad\times\frac{(2c_1)!(2c_2)!c_3!}{c_1!c_2!2^{c_1+c_2}}m_{k,i^aj^b}\sigma_{k,ii}^{c_1}\sigma_{k,jj}^{c_2}\sigma_{k,ij}^{c_3}+o\left(\frac{1}{n_k^{c+1}}\right) \\
&= \frac{1}{n_k^{c+1}}\sum_{\substack{a+b=3\\2c_1+c_3=h-a\\2c_2+c_3=l-b}}\frac{h!l!}{c_1!c_2!a!b!c_3!2^{c_1+c_2}}m_{k,i^aj^b}\sigma_{k,ii}^{c_1}\sigma_{k,jj}^{c_2}\sigma_{k,ij}^{c_3}+o\left(\frac{1}{n_k^{c+1}}\right).
\end{aligned}
$$

This completes the proof. □

1·1. *Approximations of the mean, variance, and covariance for the sum-of-powers tests under $H_0 : \mu_1 = \mu_2$*

In this section, we prove Propositions 1, 2 and 3 in the main file.

Proof of Proposition 1. It is trivial to find $\mu(1) = 0$. We focus on $\gamma \geq 2$. Under the null hypothesis,

$$\begin{aligned}
\mu(\gamma) &= \mathrm{E}\left[\sum_{i=1}^{p}\left\{\left(\bar{X}_1^{(i)} - \mu_1^{(i)}\right) - \left(\bar{X}_2^{(i)} - \mu_2^{(i)}\right)\right\}^{\gamma}\right] \\
&= \sum_{i=1}^{p}\sum_{a=0}^{\gamma}\binom{\gamma}{a}(-1)^{\gamma-a}\,\mathrm{E}\left\{\left(\bar{X}_1^{(i)} - \mu_1^{(i)}\right)^{a}\right\}\mathrm{E}\left\{\left(\bar{X}_2^{(i)} - \mu_2^{(i)}\right)^{\gamma-a}\right\}.
\end{aligned}$$

For even $\gamma \geq 2$, by Lemma 1,

$$\begin{aligned}
\mu(\gamma) &= \sum_{i=1}^{p}\sum_{a=0}^{\gamma}\binom{\gamma}{a}(-1)^{\gamma-a}\,\mathrm{E}\left\{\left(\bar{X}_1^{(i)} - \mu_1^{(i)}\right)^{a}\right\}\mathrm{E}\left\{\left(\bar{X}_2^{(i)} - \mu_2^{(i)}\right)^{\gamma-a}\right\} \\
&= \sum_{i=1}^{p}\left[\sum_{d=0}^{\gamma/2}\binom{\gamma}{2d}\mathrm{E}\left\{\left(\bar{X}_1^{(i)} - \mu_1^{(i)}\right)^{2d}\right\}E\left\{\left(\bar{X}_2^{(i)} - \mu_2^{(i)}\right)^{\gamma-2d}\right\}\right] \\
&\quad - \sum_{i=1}^{p}\left[\sum_{d=0}^{\gamma/2-1}\binom{\gamma}{2d+1}\mathrm{E}\left\{\left(\bar{X}_1^{(i)} - \mu_1^{(i)}\right)^{2d+1}\right\}E\left\{\left(\bar{X}_2^{(i)} - \mu_2^{(i)}\right)^{\gamma-2d-1}\right\}\right] \\
&= \sum_{i=1}^{p}\left\{\sum_{d=0}^{\gamma/2}\binom{\gamma}{2d}\frac{(2d)!\sigma_{1,ii}^{d}}{d!2^{d}n_1^{d}}\frac{(\gamma-2d)!\sigma_{2,ii}^{\gamma/2-d}}{(\gamma/2-d)!2^{\gamma/2-d}n_2^{(\gamma/2-d)}} + o\left(\frac{1}{n^{\gamma/2}}\right)\right\} \\
&\quad - \sum_{i=1}^{p}\left\{\sum_{d=0}^{\gamma/2-1}\binom{\gamma}{2d+1}\frac{(2d+1)!m_{1i}\sigma_{1,ii}^{d-1}}{(d-1)!3!2^{d-1}n_1^{d+1}}\frac{(\gamma-2d-1)!m_{2i}\sigma_{2,ii}^{\gamma/2-d}}{(\gamma/2-d-2)!3!2^{\gamma/2-d-2}n_2^{(\gamma/2-d)}} + o\left(\frac{1}{n^{\gamma/2+1}}\right)\right\} \\
&= \frac{\gamma!}{2^{\gamma/2}}\sum_{d=0}^{\gamma/2}\frac{\sum_{i=1}^{p}\sigma_{1,ii}^{d}\sigma_{2,ii}^{\gamma/2-d}}{d!(\gamma/2-d)!n_1^{d}n_2^{\gamma/2-d}} + o(pn^{-\gamma/2}).
\end{aligned}$$

For odd $\gamma \geq 3$, let $c = \lfloor \gamma/2 \rfloor$, and we have from Lemma 1,

$$
\begin{aligned}
\mu(\gamma) &= \sum_{i=1}^{p} \sum_{a=0}^{\gamma} \binom{\gamma}{a} (-1)^{\gamma-a} \, \mathrm{E}\left\{ \left(\bar{X}_1^{(i)} - \mu_1^{(i)} \right)^{a} \right\} \mathrm{E}\left\{ \left(\bar{X}_2^{(i)} - \mu_2^{(i)} \right)^{\gamma-a} \right\} \\
&= \sum_{i=1}^{p} \Bigg[\sum_{d=1}^{c} \binom{\gamma}{2d+1} \mathrm{E}\left\{ \left(\bar{X}_1^{(i)} - \mu_1^{(i)} \right)^{2d+1} \right\} \mathrm{E}\left\{ \left(\bar{X}_2^{(i)} - \mu_2^{(i)} \right)^{\gamma-1-2d} \right\} \\
&\qquad - \sum_{d=1}^{c} \binom{\gamma}{2d+1} \mathrm{E}\left\{ \left(\bar{X}_1^{(i)} - \mu_1^{(i)} \right)^{\gamma-1-2d} \right\} \mathrm{E}\left\{ \left(\bar{X}_2^{(i)} - \mu_2^{(i)} \right)^{2d+1} \right\} \Bigg] \\
&= \sum_{i=1}^{p} \sum_{d=1}^{c} \binom{\gamma}{2d+1} \frac{(2d+1)!(\gamma-1-2d)!}{(d-1)!(c-d)!3!2^{c-1}} \left(\frac{m_{1i} \sigma_{1,ii}^{d-1} \sigma_{2,ii}^{c-d}}{n_1^{d+1} n_2^{c-d}} - \frac{\sigma_{1,ii}^{c-d} m_{2i} \sigma_{2,ii}^{d-1}}{n_1^{c-d} n_2^{d+1}} \right) \\
&\quad + o(pn^{-c-1}) \\
&= \sum_{i=1}^{p} \sum_{d=1}^{c} \frac{\gamma!}{(d-1)!(c-d)!3!2^{c-1}} \left(\frac{m_{1i} \sigma_{1,ii}^{d-1} \sigma_{2,ii}^{c-d}}{n_1^{d+1} n_2^{c-d}} - \frac{\sigma_{1,ii}^{c-d} m_{2i} \sigma_{2,ii}^{d-1}}{n_1^{c-d} n_2^{d+1}} \right) + o(pn^{-c-1}).
\end{aligned}
$$

The approximation for $\mu^{(i)}(\gamma) = \mathrm{E}\left(\bar{X}_1^{(i)} - \bar{X}_2^{(i)} \right)^{\gamma}$ can be directly obtained from the above derivations. When $\Sigma_1 = \Sigma_2$, the results become those in Proposition 1. This completes the proof. $\square$

Proof of Proposition 2. It is trivial to find $\sigma^2(1)$ as

$$
\sigma^2(1) = \frac{1}{n_1} 1_p{}^{\mathrm{T}} \Sigma_1 1_p + \frac{1}{n_2} 1_p{}^{\mathrm{T}} \Sigma_2 1_p.
$$

For $\gamma \geq 2$, we write

$$
\begin{aligned}
\sigma^2(\gamma) &= \mathrm{E}\Bigg[\Bigg\{ \sum_{i=1}^{p} \left(\bar{X}_1^{(i)} - \bar{X}_2^{(i)} \right)^{\gamma} \Bigg\}^2 \Bigg] - [\mathrm{E}\{L(\gamma)\}]^2 \\
&= \mu(2\gamma) - \sum_{i=1}^{p} \{\mu^{(i)}(\gamma)\}^2 + \mathrm{E}\Bigg\{ \sum_{i \neq j} \left(\bar{X}_1^{(i)} - \bar{X}_2^{(i)} \right)^{\gamma} \left(\bar{X}_1^{(j)} - \bar{X}_2^{(j)} \right)^{\gamma} \Bigg\} \\
&\quad - \sum_{i \neq j} \mu^{(i)}(\gamma) \mu^{(j)}(\gamma). \qquad (1)
\end{aligned}
$$

The first two terms in equation (1), $\mu(2\gamma)$ and $\sum_{i=1}^{p}\{\mu^{(i)}(\gamma)\}^2$, can be obtained from Proposition 1. We next focus on the last two terms. Under the null hypothesis, we have

$$
\begin{aligned}
&\sum_{i\neq j} \mathrm{E}\left\{\left(\bar{X}_1^{(i)}-\bar{X}_2^{(i)}\right)^{\gamma}\left(\bar{X}_1^{(j)}-\bar{X}_2^{(j)}\right)^{\gamma}\right\}\\
&=\sum_{i\neq j} \mathrm{E}\Bigg\{\sum_{h=0}^{\gamma}\sum_{l=0}^{\gamma}\binom{\gamma}{h}\binom{\gamma}{l}(-1)^{h+l}\left(\bar{X}_1^{(i)}-\mu_1^{(i)}\right)^{h}\left(\bar{X}_2^{(i)}-\mu_2^{(i)}\right)^{\gamma-h}\\
&\qquad\times\left(\bar{X}_1^{(j)}-\mu_1^{(j)}\right)^{l}\left(\bar{X}_2^{(j)}-\mu_2^{(j)}\right)^{\gamma-l}\Bigg\}\\
&=\sum_{i\neq j}\sum_{h=0}^{\gamma}\sum_{l=0}^{\gamma}\binom{\gamma}{h}\binom{\gamma}{l}(-1)^{h+l}\,\mathrm{E}\left\{\left(\bar{X}_1^{(i)}-\mu_1^{(i)}\right)^{h}\left(\bar{X}_1^{(j)}-\mu_1^{(j)}\right)^{l}\right\}\\
&\qquad\times\mathrm{E}\left\{\left(\bar{X}_2^{(i)}-\mu_2^{(i)}\right)^{\gamma-h}\left(\bar{X}_2^{(j)}-\mu_2^{(j)}\right)^{\gamma-l}\right\}
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&\sum_{i\neq j}\mu^{(i)}(\gamma)\mu^{(j)}(\gamma)\\
&=\sum_{i\neq j}\sum_{h=0}^{\gamma}\sum_{l=0}^{\gamma}\binom{\gamma}{h}\binom{\gamma}{l}(-1)^{h+l}\,\mathrm{E}\left\{\left(\bar{X}_1^{(i)}-\mu_1^{(i)}\right)^{h}\right\}\mathrm{E}\left\{\left(\bar{X}_1^{(j)}-\mu_1^{(j)}\right)^{l}\right\}\\
&\qquad\times\mathrm{E}\left\{\left(\bar{X}_2^{(i)}-\mu_2^{(i)}\right)^{\gamma-h}\right\}\mathrm{E}\left\{\left(\bar{X}_2^{(j)}-\mu_2^{(j)}\right)^{\gamma-l}\right\}.
\end{aligned}
$$

We write

$$
\begin{aligned}
&\sum_{i\neq j} \mathrm{E}\left\{\left(\bar{X}_1^{(i)}-\bar{X}_2^{(i)}\right)^{\gamma}\left(\bar{X}_1^{(j)}-\bar{X}_2^{(j)}\right)^{\gamma}\right\}-\sum_{i\neq j}\mu^{(i)}(\gamma)\mu^{(j)}(\gamma)\\
&=\sum_{i\neq j}\sum_{h+l\ is\ even}\binom{\gamma}{h}\binom{\gamma}{l}(-1)^{h+l}\\
&\quad\times\Bigg[\mathrm{E}\left\{\left(\bar{X}_1^{(i)}-\mu_1^{(i)}\right)^{h}\left(\bar{X}_1^{(j)}-\mu_1^{(j)}\right)^{l}\right\}\mathrm{E}\left\{\left(\bar{X}_2^{(i)}-\mu_2^{(i)}\right)^{\gamma-h}\left(\bar{X}_2^{(j)}-\mu_2^{(j)}\right)^{\gamma-l}\right\}\\
&\qquad-\mathrm{E}\left\{\left(\bar{X}_1^{(i)}-\mu_1^{(i)}\right)^{h}\right\}\mathrm{E}\left\{\left(\bar{X}_1^{(j)}-\mu_1^{(j)}\right)^{l}\right\}\\
&\qquad\quad\times\mathrm{E}\left\{\left(\bar{X}_2^{(i)}-\mu_2^{(i)}\right)^{\gamma-h}\right\}\mathrm{E}\left\{\left(\bar{X}_2^{(j)}-\mu_2^{(j)}\right)^{\gamma-l}\right\}\Bigg]\\
&\quad+\sum_{i\neq j}\sum_{h+l\ is\ odd}\cdots.
\end{aligned}
$$

Under the proposed conditions C1–C3, by Lemma 2, the above display can be further expressed as

$$
\begin{aligned}
&\sum_{i\neq j} \mathrm{E}\left\{\left(\bar{X}_1^{(i)}-\bar{X}_2^{(i)}\right)^{\gamma}\left(\bar{X}_1^{(j)}-\bar{X}_2^{(j)}\right)^{\gamma}\right\}-\sum_{i\neq j}\mu^{(i)}(\gamma)\mu^{(j)}(\gamma)\\
&=\sum_{i\neq j}\sum_{\substack{2c_1+c_3+2d_1+d_3=\gamma\\ 2c_2+c_3+2d_2+d_3=\gamma\\ c_1,c_2,c_3,d_1,d_2,d_3\geq 0}}\binom{\gamma}{2c_1+c_3}\binom{\gamma}{2c_2+c_3}\\
&\qquad\times\left\{\frac{1}{n_1^{c_1+c_2+c_3}}\frac{(2c_1+c_3)!(2c_2+c_3)!}{c_1!c_2!c_3!2^{c_1+c_2}}\sigma_{1,ii}^{c_1}\sigma_{1,jj}^{c_2}\sigma_{1,ij}^{c_3}+o\left(n_1^{-(c_1+c_2+c_3)}\right)\right\}\\
&\qquad\quad\times\left\{\frac{1}{n_2^{d_1+d_2+d_3}}\frac{(2d_1+d_3)!(2d_2+d_3)!}{d_1!d_2!d_3!2^{d_1+d_2}}\sigma_{2,ii}^{d_1}\sigma_{2,jj}^{d_2}\sigma_{2,ij}^{d_3}+o\left(n_2^{-(d_1+d_2+d_3)}\right)\right\}\\
&\quad-\sum_{i\neq j}\sum_{\substack{2c_1+2d_1=\gamma\\ 2c_2+2d_2=\gamma\\ c_1,c_2,d_1,d_2\geq 0}}\binom{\gamma}{2c_1}\binom{\gamma}{2c_2}\left\{\frac{1}{n_1^{c_1+c_2}}\frac{(2c_1)!(2c_2)!}{c_1!c_2!2^{c_1+c_2}}\sigma_{1,ii}^{c_1}\sigma_{1,jj}^{c_2}+o\left(n_1^{-(c_1+c_2)}\right)\right\}\\
&\qquad\quad\times\left\{\frac{1}{n_2^{d_1+d_2}}\frac{(2d_1)!(2d_2)!}{d_1!d_2!2^{d_1+d_2}}\sigma_{2,ii}^{d_1}\sigma_{2,jj}^{d_2}+o\left(n_2^{-(d_1+d_2)}\right)\right\}\\
&\quad+o(pn^{-\gamma})\\
&=\sum_{\substack{2c_1+c_3+2d_1+d_3=\gamma\\ 2c_2+c_3+2d_2+d_3=\gamma\\ c_1,c_2,c_3,d_1,d_2,d_3\geq 0,c_3+d_3>0}}\frac{(\gamma!)^2\sum_{i\neq j}\sigma_{1,ii}^{c_1}\sigma_{1,jj}^{c_2}\sigma_{1,ij}^{c_3}\sigma_{2,ii}^{d_1}\sigma_{2,jj}^{d_2}\sigma_{2,ij}^{d_3}}{n_1^{c_1+c_2+c_3}n_2^{d_1+d_2+d_3}c_1!c_2!c_3!d_1!d_2!d_3!2^{c_1+c_2+d_1+d_2}}\\
&\quad+o(pn^{-\gamma}),
\end{aligned}
$$

where the terms with $h+l$ odd is ignorable under the assumed strong mixing conditions.

Consequently, from equation (1), we have

$$
\begin{aligned}
\sigma^2(\gamma)=\;&\mu(2\gamma)-\sum_{i=1}^{p}\{\mu^{(i)}(\gamma)\}^2\\
&+\sum_{\substack{2c_1+c_3+2d_1+d_3=\gamma\\ 2c_2+c_3+2d_2+d_3=\gamma\\ c_1,c_2,c_3,d_1,d_2,d_3\geq 0,c_3+d_3>0}}\frac{(\gamma!)^2\sum_{i\neq j}\sigma_{1,ii}^{c_1}\sigma_{1,jj}^{c_2}\sigma_{1,ij}^{c_3}\sigma_{2,ii}^{d_1}\sigma_{2,jj}^{d_2}\sigma_{2,ij}^{d_3}}{n_1^{c_1+c_2+c_3}n_2^{d_1+d_2+d_3}c_1!c_2!c_3!d_1!d_2!d_3!2^{c_1+c_2+d_1+d_2}}\\
&+o(pn^{-\gamma}).
\end{aligned}
$$

For the first two terms, from the proof of Proposition 1, we have if γ is even,

$$
\begin{aligned}
&\mu(2\gamma)-\sum_{i=1}^{p}\{\mu^{(i)}(\gamma)\}^2\\
&\sim\frac{(2\gamma)!}{2^{\gamma}}\sum_{i=1}^{p}\sum_{d=0}^{\gamma}\frac{\sigma_{1,ii}^{d}\sigma_{2,ii}^{\gamma-d}}{d!(\gamma-d)!n_1^{d}n_2^{\gamma-d}}-\frac{(\gamma!)^2}{2^{\gamma}}\sum_{i=1}^{p}\left\{\sum_{d=0}^{\gamma/2}\frac{\sigma_{1,ii}^{d}\sigma_{2,ii}^{\gamma/2-d}}{d!(\gamma/2-d)!n_1^{d}n_2^{\gamma/2-d}}\right\}^2;
\end{aligned}
$$

on the other hand, if γ is odd, we have

$$\mu(2\gamma) - \sum_{i=1}^{p} \{\mu^{(i)}(\gamma)\}^2 \sim \frac{(2\gamma)!}{2^\gamma} \sum_{i=1}^{p} \sum_{d=0}^{\gamma} \frac{\sigma_{1,ii}^{d} \sigma_{2,ii}^{\gamma-d}}{d!(\gamma-d)! n_1^d n_2^{\gamma-d}}.$$

When $\Sigma_1 = \Sigma_2$, we have the results in Proposition 2. $\square$

Proof of Proposition 3. We study the covariance between $L(s)$ and $L(t)$ for finite integers $s, t \in \Gamma$. Similar to the proof of Proposition 2, under the null hypothesis, we have

$$\begin{aligned}
&\operatorname{cov}\{L(t), L(s)\} \\
&= \mathrm{E}\{L(t)L(s)\} - \mathrm{E}\{L(t)\}\,\mathrm{E}\{L(s)\} \\
&= \mathrm{E}\Bigg\{ \sum_{i=1}^{p} \left(\bar{X}_1^{(i)} - \bar{X}_2^{(i)}\right)^{t+s} \Bigg\} + \mathrm{E}\Bigg\{ \sum_{i \neq j} \left(\bar{X}_1^{(i)} - \bar{X}_2^{(i)}\right)^{t} \left(\bar{X}_1^{(j)} - \bar{X}_2^{(j)}\right)^{s} \Bigg\} - \mu(t)\mu(s) \\
&= \mu(t+s) - \sum_{i=1}^{p} \mu^{(i)}(t)\mu^{(i)}(s) + \sum_{i \neq j} \mathrm{E}\Bigg\{ \left(\bar{X}_1^{(i)} - \bar{X}_2^{(i)}\right)^{t} \left(\bar{X}_1^{(j)} - \bar{X}_2^{(j)}\right)^{s} \Bigg\} \\
&\quad - \sum_{i \neq j} \mu^{(i)}(t)\mu^{(j)}(s)
\end{aligned}$$

We focus on the last two terms and have

$$\begin{aligned}
&\mathrm{E}\left\{ \sum_{i \neq j} \left(\bar{X}_1^{(i)} - \bar{X}_2^{(i)}\right)^{t} \left(\bar{X}_1^{(j)} - \bar{X}_2^{(j)}\right)^{s} \right\} - \sum_{i \neq j} \mu^{(i)}(t)\mu^{(j)}(s) \\
&= \sum_{i \neq j} \sum_{h=0}^{t} \sum_{l=0}^{s} \binom{t}{h} \binom{s}{l} (-1)^{t-h+s-l} \\
&\quad \times \Bigg[\mathrm{E}\left\{ \left(\bar{X}_1^{(i)} - \mu_1^{(i)}\right)^{h} \left(\bar{X}_1^{(j)} - \mu_1^{(j)}\right)^{l} \right\} \mathrm{E}\left\{ \left(\bar{X}_2^{(i)} - \mu_2^{(i)}\right)^{t-h} \left(\bar{X}_2^{(j)} - \mu_2^{(j)}\right)^{s-l} \right\} \\
&\qquad - \mathrm{E}\left\{ \left(\bar{X}_1^{(i)} - \mu_1^{(i)}\right)^{h} \right\} \mathrm{E}\left\{ \left(\bar{X}_1^{(j)} - \mu_1^{(j)}\right)^{l} \right\} \\
&\qquad\quad \times \mathrm{E}\left\{ \left(\bar{X}_2^{(i)} - \mu_2^{(i)}\right)^{t-h} \right\} \mathrm{E}\left\{ \left(\bar{X}_2^{(j)} - \mu_2^{(j)}\right)^{s-l} \right\} \Bigg] \\
&= \sum_{i \neq j} \sum_{h+l \text{ is even}} \cdots + \sum_{i \neq j} \sum_{h+l \text{ is odd}} \cdots
\end{aligned}$$

If $s+t$ is even, from a similar argument as in the proof of Proposition 2, we have

$$
\begin{aligned}
&\mathrm{E}\left\{\sum_{i\neq j}\left(\bar{X}_1^{(i)}-\bar{X}_2^{(i)}\right)^t\left(\bar{X}_1^{(j)}-\bar{X}_2^{(j)}\right)^s\right\}-\sum_{i\neq j}\mu^{(i)}(t)\mu^{(j)}(s)\\
=&\sum_{i\neq j}\sum_{\substack{2c_1+c_3+2d_1+d_3=t\\ 2c_2+c_3+2d_2+d_3=s\\ c_1,c_2,c_3,d_1,d_2,d_3\geq 0,c_3+d_3>0}}\binom{t}{2c_1+c_3}\binom{s}{2c_2+c_3}\binom{2c_1+c_3}{c_3}\binom{2c_2+c_3}{c_3}\\
&\times\binom{t-2c_1-c_3}{d_3}\binom{s-2c_2-c_3}{d_3}\frac{(2c_1)!(2c_2)!c_3!}{c_1!c_2!2^{c_1+c_2}}\frac{(2d_1)!(2d_2)!d_3!}{c_1!c_2!2^{d_1+d_2}}\\
&\times\frac{1}{n_1^{c_1+c_2+c_3}n_2^{d_1+d_2+d_3}}\sigma_{1,ii}^{c_1}\sigma_{1,jj}^{c_2}\sigma_{1,ij}^{c_3}\sigma_{2,ii}^{d_1}\sigma_{2,jj}^{d_2}\sigma_{2,ij}^{d_3}\\
&+o(pn^{-(s+t)/2})\\
=&\sum_{\substack{2c_1+c_3+2d_1+d_3=t\\ 2c_2+c_3+2d_2+d_3=s\\ c_1,c_2,c_3,d_1,d_2,d_3\geq 0,c_3+d_3>0}}\frac{t!s!\sum_{i\neq j}\sigma_{1,ii}^{c_1}\sigma_{1,jj}^{c_2}\sigma_{1,ij}^{c_3}\sigma_{2,ii}^{d_1}\sigma_{2,jj}^{d_2}\sigma_{2,ij}^{d_3}}{n_1^{c_1+c_2+c_3}n_2^{d_1+d_2+d_3}c_1!c_2!c_3!d_1!d_2!d_3!2^{c_1+c_2+d_1+d_2}}\\
&+o(pn^{-(s+t)/2})
\end{aligned}
$$

Therefore, we obtain the expression for $\operatorname{cov}\{L(t),L(t)\}$ in Proposition 3 in the case of even $s+t$. If $s+t$ is odd, by Lemma 2, similarly we have

$$
\begin{aligned}
&\mathrm{E}\left\{\sum_{i\neq j}\left(\bar{X}_1^{(i)}-\bar{X}_2^{(i)}\right)^t\left(\bar{X}_1^{(j)}-\bar{X}_2^{(j)}\right)^s\right\}-\sum_{i\neq j}\mu^{(i)}(t)\mu^{(j)}(s)\\
=&\sum_{i\neq j}\sum_{\substack{a+b=3\\ 2c_1+c_3+2d_1+d_3=t-a\\ 2c_2+c_3+2d_2+d_3=s-b\\ a,b>0\ or\ c_3+d_3>0}}\binom{t}{a}\binom{s}{b}\binom{t-a}{2c_1+c_3}\binom{2c_1+c_3}{c_3}\binom{s-b}{2c_2+c_3}\binom{2c_2+c_3}{c_3}\\
&\times\binom{t-a-2c_1-c_3}{d_3}\binom{s-b-2c_2-c_3}{d_3}\frac{(2c_1)!(2c_2)!c_3!}{c_1!c_2!2^{c_1+c_2}}\frac{(2d_1)!(2d_2)!d_3!}{d_1!d_2!2^{d_1+d_2}}\\
&\times\left(\frac{m_{1,i^aj^b}\sigma_{1,ii}^{c_1}\sigma_{1,jj}^{c_2}\sigma_{1,ij}^{c_3}\sigma_{2,ii}^{d_1}\sigma_{2,jj}^{d_2}\sigma_{2,ij}^{d_3}}{n_1^{c_1+c_2+c_3+2}n_2^{d_1+d_2+d_3}}-\frac{m_{2,i^aj^b}\sigma_{1,ii}^{c_1}\sigma_{1,jj}^{c_2}\sigma_{1,ij}^{c_3}\sigma_{2,ii}^{d_1}\sigma_{2,jj}^{d_2}\sigma_{2,ij}^{d_3}}{n_1^{c_1+c_2+c_3}n_2^{d_1+d_2+d_3+2}}\right)\\
&+o(pn^{-(s+t+1)/2})\\
=&\sum_{\substack{a+b=3\\ 2c_1+c_3+2d_1+d_3=t-a\\ 2c_2+c_3+2d_2+d_3=s-b\\ a,b>0\ or\ c_3+d_3>0}}\frac{t!s!}{a!b!c_1!c_2!c_3!d_1!d_2!d_3!2^{c_1+c_2+d_1+d_2}}\\
&\times\sum_{i\neq j}\left(\frac{m_{1,i^aj^b}\sigma_{1,ii}^{c_1}\sigma_{1,jj}^{c_2}\sigma_{1,ij}^{c_3}\sigma_{2,ii}^{d_1}\sigma_{2,jj}^{d_2}\sigma_{2,ij}^{d_3}}{n_1^{c_1+c_2+c_3+2}n_2^{d_1+d_2+d_3}}-\frac{m_{2,i^aj^b}\sigma_{1,ii}^{c_1}\sigma_{1,jj}^{c_2}\sigma_{1,ij}^{c_3}\sigma_{2,ii}^{d_1}\sigma_{2,jj}^{d_2}\sigma_{2,ij}^{d_3}}{n_1^{c_1+c_2+c_3}n_2^{d_1+d_2+d_3+2}}\right)\\
&+o(pn^{-(s+t+1)/2}).
\end{aligned}
$$

Under the mixing condition, note that the first term is $O\left\{p(n_1+n_2)^{-(t+s+1)/2}\right\}$. Since $t+s$ is odd, by Proposition 1,

$$\mu(t+s) = O\left\{p(n_1+n_2)^{-(s+t+1)/2}\right\}.$$

Also, $\sum \mu^{(i)}(t)\mu^{(i)}(s) = O\left\{p(n_1+n_2)^{-(s+t+1)/2}\right\}$. These results imply

$$\operatorname{cov}\left\{L(t), L(s)\right\} = O\left\{p(n_1+n_2)^{-(s+t+1)/2}\right\} = o\left\{pn^{-(s+t)/2}\right\}.$$

Take $\Sigma_1 = \Sigma_2$ and we have the same asymptotic approximations in Proposition 3. This completes the proof. □

1·2. *Approximations of the mean, variance, and covariance for sum-of-powers tests under $H_1 : \mu_1 \neq \mu_2$*

Proof of Proposition 4. We denote $\delta_i = \mu_1^{(i)} - \mu_2^{(i)}$ for $i = 1, \ldots, p$. Under the alternative, it is trivial to find $\mu_A(1) = \sum_{i=1}^p \delta_i$. We focus on $\gamma \geq 2$. The mean function of $L(\gamma)$ under H_1 equals

$$\begin{aligned}
\mu_A(\gamma) &= \mathrm{E}\left[\sum_{i=1}^p \left\{\left(\bar{X}_1^{(i)} - \mu_1^{(i)}\right) - \left(\bar{X}_2^{(i)} - \mu_2^{(i)}\right) + \delta_i\right\}^\gamma\right] \\
&= \sum_{i=1}^p \sum_{\substack{0\leq a\leq\gamma \\ 0\leq b\leq\gamma-a}} \binom{\gamma}{a}\binom{\gamma-a}{b}(-1)^b\delta_i^{\gamma-a-b}\,\mathrm{E}\left\{\left(\bar{X}_1^{(i)} - \mu_1^{(i)}\right)^a\right\}\mathrm{E}\left\{\left(\bar{X}_2^{(i)} - \mu_2^{(i)}\right)^b\right\}.
\end{aligned}$$

By Lemma 1 and the proof of Proposition 1,

$$\begin{aligned}
\mu_A(\gamma) &= \sum_{i=1}^p \sum_{\substack{a+b+c=\gamma \\ 0\leq a\leq\gamma-c}} \binom{\gamma}{c}\binom{\gamma-c}{a}(-1)^b\delta_i^c\,\mathrm{E}\left\{\left(\bar{X}_1^{(i)} - \mu_1^{(i)}\right)^a\right\}\mathrm{E}\left\{\left(\bar{X}_2^{(i)} - \mu_2^{(i)}\right)^b\right\} \\
&= \sum_{i=1}^p \sum_{c=0}^\gamma \binom{\gamma}{c}(-1)^b\delta_i^c \sum_{\substack{a+b+c=\gamma \\ 0\leq a\leq\gamma-c}} \binom{\gamma-c}{a}\mathrm{E}\left\{\left(\bar{X}_1^{(i)} - \mu_1^{(i)}\right)^a\right\}\mathrm{E}\left\{\left(\bar{X}_2^{(i)} - \mu_2^{(i)}\right)^b\right\} \\
&= \mu(\gamma) + \sum_{i=1}^p \sum_{c=1}^\gamma \binom{\gamma}{c}\delta_i^c\mu^{(i)}(\gamma-c),
\end{aligned}$$

where approximations for $\mu^{(i)}(\gamma-c)$ are given in Proposition 1.

In particular, when $\gamma = 2$, since $\mu^{(i)}(0) = 1$ and $\mu^{(i)}(1) = 0$, we obtain

$$\mu_A(2) = \mu(2) + \sum_{i=1}^p \delta_i^2;$$

when $\gamma = 3$, note that $\mu^{(i)}(2) = \sigma_{1,ii}/n_1 + \sigma_{2,ii}/n_2$ and then

$$\mu_A(3) = \mu(3) + \sum_{i=1}^p \delta_i^3 + 3\sum_{i=1}^p \delta_i\mu^{(i)}(2)$$

$$= \mu(3) + \sum_{i=1}^{p} \delta_i^3 + 3\sum_{i=1}^{p} \delta_i \left(\frac{\sigma_{1,ii}}{n_1} + \frac{\sigma_{2,ii}}{n_2} \right);$$

when $\gamma = 4$, similarly

$$\begin{aligned} \mu_A(4) &= \mu(4) + \sum_{i=1}^{p} \delta_i^4 + 4\sum_{i=1}^{p} \delta_i^3 \mu^{(i)}(1) + 6\sum_{i=1}^{p} \delta_i^2 \mu^{(i)}(2) + 4\sum_{i=1}^{p} \delta_i \mu^{(i)}(3) \\ &= \mu(4) + \sum_{i=1}^{p} \delta_i^4 + 6\sum_{i=1}^{p} \delta_i^2 \left(\frac{\sigma_{1,ii}}{n_1} + \frac{\sigma_{2,ii}}{n_2} \right) + 4\sum_{i=1}^{p} \delta_i \left(\frac{m_{1i}}{n_1^2} - \frac{m_{2i}}{n_2^2} \right). \end{aligned}$$

In particular, for Xs' following normal distribution, $m_{1i} = m_{2i} = 0$ and

$$\mu_A(4) = \mu(4) + \sum_{i=1}^{p} \delta_i^4 + 6\sum_{i=1}^{p} \delta_i^2 \left(\frac{\sigma_{1,ii}}{n_1} + \frac{\sigma_{2,ii}}{n_2} \right).$$

Take $\Sigma_1 = \Sigma_2$ and we have the conclusion in Proposition 4. □

Proof of Proposition 5. We write $\mu_A^{(i)}(\gamma) = \mathrm{E}\left(\bar{X}_1^{(i)} - \bar{X}_2^{(i)}\right)^\gamma$ under H_1. We have

$$\begin{aligned} \sigma_A^2(\gamma) &= \mathrm{E}\left[\left\{ \sum_{i=1}^{p} \left(\bar{X}_1^{(i)} - \bar{X}_2^{(i)} \right)^\gamma \right\}^2 \right] - \left[\mathrm{E}\left\{ L(\gamma) \right\} \right]^2 \\ &= \mu_A(2\gamma) - \sum_{i=1}^{p} \{\mu_A^{(i)}(\gamma)\}^2 + \mathrm{E}\left\{ \sum_{i \neq j} \left(\bar{X}_1^{(i)} - \bar{X}_2^{(i)} \right)^\gamma \left(\bar{X}_1^{(j)} - \bar{X}_2^{(j)} \right)^\gamma \right\} \\ &\quad - \sum_{i \neq j} \mu_A^{(i)}(\gamma) \mu_A^{(j)}(\gamma). \end{aligned} \tag{2}$$

When $\gamma = 1$,

$$\begin{aligned} \sigma_A^2(1) &= \mu_A(2) - \sum_{i=1}^{p} \{\mu_A^{(i)}(1)\}^2 + \sum_{i \neq j} \mathrm{E}\left\{ \left(\bar{X}_1^{(i)} - \bar{X}_2^{(i)} \right) \left(\bar{X}_1^{(j)} - \bar{X}_2^{(j)} \right) - \mu_A^{(i)}(1) \mu_A^{(j)}(1) \right\} \\ &= \frac{1}{n_1} 1_p{}^{\mathrm{T}} \Sigma_1 1_p + \frac{1}{n_2} 1_p{}^{\mathrm{T}} \Sigma_2 1_p. \end{aligned}$$

where the last equation follows from $\mu_A^{(i)}(1) = \delta_i$,

$$\mu_A(2) = \mu(2) + \sum_{i=1}^{p} \delta_i^2 = \sum_{i=1}^{p} \left(\frac{\sigma_{1,ii}}{n_1} + \frac{\sigma_{2,ii}}{n_2} \right) + \sum_{i=1}^{p} \delta_i^2$$

and

$$\mathrm{E}\left\{ \left(\bar{X}_1^{(i)} - \bar{X}_2^{(i)} \right) \left(\bar{X}_1^{(j)} - \bar{X}_2^{(j)} \right) - \mu_A^{(i)}(1) \mu_A^{(j)}(1) \right\} = \frac{\sigma_{1,ij}}{n_1} + \frac{\sigma_{2,ij}}{n_2}.$$

For $\gamma \geq 2$, we focus on the last two terms in (2), and under the alternative hypothesis, we have

$$\begin{aligned}
&\sum_{i\neq j} \mathrm{E}\left\{ \left(\bar{X}_1^{(i)} - \bar{X}_2^{(i)}\right)^{\gamma} \left(\bar{X}_1^{(j)} - \bar{X}_2^{(j)}\right)^{\gamma} \right\} \\
&= \sum_{i\neq j} \sum_{h=0}^{\gamma} \sum_{l=0}^{\gamma} \binom{\gamma}{h} \binom{\gamma}{l} \delta_i^h \delta_j^l \\
&\quad \times \mathrm{E}\left\{ \left(\bar{X}_1^{(i)} - \mu_1^{(i)} + \mu_2^{(i)} - \bar{X}_2^{(i)}\right)^{\gamma-h} \left(\bar{X}_1^{(j)} - \mu_1^{(j)} + \mu_2^{(j)} - \bar{X}_2^{(j)}\right)^{\gamma-l} \right\}
\end{aligned}$$

Then from the proof of Proposition 2, we have

$$\begin{aligned}
&\sum_{i\neq j} \mathrm{E}\left\{ \left(\bar{X}_1^{(i)} - \bar{X}_2^{(i)}\right)^{\gamma} \left(\bar{X}_1^{(j)} - \bar{X}_2^{(j)}\right)^{\gamma} \right\} - \sum_{i\neq j} \mu_A^{(i)}(\gamma)\mu_A^{(j)}(\gamma) \\
&\sim \ \sum_{i\neq j} \sum_{h=0}^{\gamma} \sum_{l=0}^{\gamma} \binom{\gamma}{h} \binom{\gamma}{l} \delta_i^h \delta_j^l r_{ij}(\gamma-h, \gamma-l),
\end{aligned}$$

where $r_{ij}(\cdot,\cdot)$ is defined as: if $s+t$ is even

$$r_{ij}(s,t) = \sum_{\substack{2c_1+c_3+2d_1+d_3=t \\ 2c_2+c_3+2d_2+d_3=s \\ c_1,c_2,c_3,d_1,d_2,d_3\geq 0, c_3+d_3>0}} \frac{t!s!\sigma_{1,ii}^{c_1}\sigma_{1,jj}^{c_2}\sigma_{1,ij}^{c_3}\sigma_{2,ii}^{d_1}\sigma_{2,jj}^{d_2}\sigma_{2,ij}^{d_3}}{n_1^{c_1+c_2+c_3} n_2^{d_1+d_2+d_3} c_1!c_2!c_3!d_1!d_2!d_3!2^{c_1+c_2+d_1+d_2}};$$

if $s+t$ is odd

$$\begin{aligned}
r_{ij}(s,t) = &\sum_{\substack{a+b=3 \\ 2c_1+c_3+2d_1+d_3=t-a \\ 2c_2+c_3+2d_2+d_3=s-b \\ a,b>0 \ or \ c_3+d_3>0}} \frac{t!s!}{a!b!c_1!c_2!c_3!d_1!d_2!d_3!2^{c_1+c_2+d_1+d_2}} \\
&\times \left(\frac{m_{1,i^a j^b}\sigma_{1,ii}^{c_1}\sigma_{1,jj}^{c_2}\sigma_{1,ij}^{c_3}\sigma_{2,ii}^{d_1}\sigma_{2,jj}^{d_2}\sigma_{2,ij}^{d_3}}{n_1^{c_1+c_2+c_3+2} n_2^{d_1+d_2+d_3}} - \frac{m_{2,i^a j^b}\sigma_{1,ii}^{c_1}\sigma_{1,jj}^{c_2}\sigma_{1,ij}^{c_3}\sigma_{2,ii}^{d_1}\sigma_{2,jj}^{d_2}\sigma_{2,ij}^{d_3}}{n_1^{c_1+c_2+c_3} n_2^{d_1+d_2+d_3+2}} \right).
\end{aligned}$$

Note that when $\Sigma_1 = \Sigma_2$, $r_{ij}(s,t)$ becomes that defined in Proposition 5.

In particular, when $\gamma = 2$, we have

$$\begin{aligned}
\sigma_A^2(2) &= \mu_A(4) - \sum_{i=1}^{p} \{\mu_A^{(i)}(2)\}^2 + \sum_{i\neq j} \mathrm{E}\left\{ \left(\bar{X}_1^{(i)} - \bar{X}_2^{(i)}\right)^2 \left(\bar{X}_1^{(j)} - \bar{X}_2^{(j)}\right)^2 - \mu_A^{(i)}(2)\mu_A^{(j)}(2) \right\} \\
&= \mu_A(4) - \sum_{i=1}^{p} \{\mu_A^{(i)}(2)\}^2 + \sum_{i\neq j} \left\{ \delta_i^2\delta_j^2 r_{ij}(0,0) + r_{ij}(2,2) + \delta_i^2 \mu^{(j)}(2) r_{ij}(2,0) \right. \\
&\quad \left. + \delta_j^2 \mu^{(i)}(2) r_{ij}(0,2) + 2\delta_i r_{ij}(1,2) + 2\delta_j r_{ij}(2,1) + 4\delta_i\delta_j r_{ij}(1,1) \right\} \\
&\sim \mu_A(4) - \sum_{i=1}^{p} \{\mu_A^{(i)}(2)\}^2 + \sum_{i\neq j} \left\{ r_{ij}(2,2) + 4\delta_i\delta_j r_{ij}(1,1) \right\}.
\end{aligned}$$

Note that

$$r_{ij}(1,1) = \frac{\sigma_{1,ij}}{n_1} + \frac{\sigma_{2,ij}}{n_2}$$

and

$$\sigma^2(2) \sim \mu(4) - \sum_{i=1}^{p}\{\mu^{(i)}(2)\}^2 + \sum_{i\neq j} r_{ij}(2,2).$$

Thus, from the approximations for $\mu_A(4)$, $\mu_A^{(i)}(2)$, and $r_{ij}(1,1)$, we have

$$\begin{aligned}\sigma_A^2(2) &\sim \mu(4) + \sum_{i=1}^{p}\delta_i^4 + 6\sum_{i=1}^{p}\delta_i^2\left(\frac{\sigma_{1,ii}}{n_1}+\frac{\sigma_{2,ii}}{n_2}\right) + 4\sum_{i=1}^{p}\delta_i\left(\frac{m_{1i}}{n_1^2}-\frac{m_{2i}}{n_2^2}\right)\\ &\quad - \sum_{i=1}^{p}\{\mu^{(i)}(2)+\delta_i^2\}^2 + \sum_{i\neq j}\Big\{r_{ij}(2,2) + 4\delta_i\delta_j r_{ij}(1,1)\Big\}.\\ &\sim \sigma^2(2) + 4\sum_{i,j}\delta_i\delta_j\Big(\frac{\sigma_{1,ij}}{n_1}+\frac{\sigma_{2,ij}}{n_2}\Big).\end{aligned}$$

When $\Sigma_1 = \Sigma_2$, we have

$$\sigma_A^2(2) \sim \sigma^2(2) + 4\Big(\frac{1}{n_1}+\frac{1}{n_2}\Big)\sum_{i,j}\delta_i\delta_j\sigma_{ij}.$$

Similarly to the derivation for the asymptotic variance, under the alternative hypothesis, we have

$$\begin{aligned}&\mathrm{cov}_A\{L(t),L(s)\}\\ &= \mathrm{E}\{L(t)L(s)\} - \mathrm{E}\{L(t)\}\,\mathrm{E}\{L(s)\}\\ &= \mathrm{E}\Bigg\{\sum_{i=1}^{p}\Big(\bar X_1^{(i)}-\bar X_2^{(i)}\Big)^{t+s}\Bigg\} + \mathrm{E}\Bigg\{\sum_{i\neq j}\Big(\bar X_1^{(i)}-\bar X_2^{(i)}\Big)^{t}\Big(\bar X_1^{(j)}-\bar X_2^{(j)}\Big)^{s}\Bigg\} - \mu_A(t)\mu_A(s)\\ &= \mu_A(t+s) - \sum_{i=1}^{p}\mu_A^{(i)}(t)\mu_A^{(i)}(s) + \sum_{i\neq j}\mathrm{E}\Bigg\{\Big(\bar X_1^{(i)}-\bar X_2^{(i)}\Big)^{t}\Big(\bar X_1^{(j)}-\bar X_2^{(j)}\Big)^{s}\Bigg\}\\ &\quad - \sum_{i\neq j}\mu_A^{(i)}(t)\mu_A^{(j)}(s)\end{aligned}$$

and from the proof of Proposition 2, we have

$$\begin{aligned}&\sum_{i\neq j}\mathrm{E}\left\{\Big(\bar X_1^{(i)}-\bar X_2^{(i)}\Big)^{t}\Big(\bar X_1^{(j)}-\bar X_2^{(j)}\Big)^{s}\right\} - \sum_{i\neq j}\mu_A^{(i)}(t)\mu_A^{(j)}(s)\\ &\sim \sum_{i\neq j}\sum_{h=0}^{t}\sum_{l=0}^{s}\binom{t}{h}\binom{s}{l}\delta_i^h\delta_j^l r_{ij}(t-h,s-l).\end{aligned}$$

1·3. *Proof of Theorems*

We focus on the proof of Theorem 1. The proof of Theorem 2 follows from the same argument as part (i) and therefore is omitted.

Proof of Theorem 1.(i) For finite $\gamma \in \Gamma$, we first show the limiting distribution for each $L(\gamma)$. The Cramér-Wold Theorem can be used to show the joint distribution. For notational convenience, we write

$$L^{(i)}(\gamma) = \left(\bar{X}_1^{(i)} - \bar{X}_2^{(i)}\right)^{\gamma} \quad \text{and} \quad \mu^{(i)}(\gamma) = \mathrm{E}\left\{L^{(i)}(\gamma)\right\},$$

for $1 \le i \le p$. Note that $\mu(\gamma) = \sum_{i=1}^{p} \mu^{(i)}(\gamma)$. We use Bernstein's block method on page 316 in Ibragimov & Linnik (1971); see also Chen et al. (2014). Partition the sequence

$$\sigma^{-1}(\gamma)\left\{L^{(i)}(\gamma) - \mu^{(i)}(\gamma)\right\}, 1 \le i \le p,$$

into r blocks, where each block contains b variables such that $rb \le p < (r+1)b$. Further, for each $1 \le j \le r$, we partition the jth block into two sub-blocks with a larger one A_{j1} and a smaller one A_{j2}. Suppose each A_{j1} has b_1 variables and each A_{j2} has $b_2 = b - b_1$ variables. We require $r \to \infty, b_1 \to \infty, b_2 \to \infty, rb_1/p \to 1$ and $rb_2/p \to 0$ as $p \to \infty$. We write

$$A_{j1}(\gamma) = \sum_{i=1}^{b_1}\left[L^{\{(j-1)b+i\}}(\gamma) - \mu^{\{(j-1)b+i\}}(\gamma)\right];$$
$$A_{j2}(\gamma) = \sum_{i=1}^{b_2}\left[L^{\{(j-1)b+b_1+i\}}(\gamma) - \mu^{\{(j-1)b+b_1+i\}}(\gamma)\right].$$

Further define

$$\mathcal{L}_1 = \sigma^{-1}(\gamma)\sum_{j=1}^{r} A_{j1}(\gamma);$$
$$\mathcal{L}_2 = \sigma^{-1}(\gamma)\sum_{j=1}^{r} A_{j2}(\gamma);$$
$$\mathcal{L}_3 = \sigma^{-1}(\gamma)\sum_{i=rb+1}^{p}\left\{L^{(i)}(\gamma) - \mu^{(i)}(\gamma)\right\}.$$

We have decomposition

$$\sigma^{-1}(\gamma)\left\{L(\gamma) - \mu(\gamma)\right\} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3.$$

The Bernstein's block method makes A_{i1}'s "almost" independent, thus the study of $\mathcal{L}_1$ may be related to the well-studied cases of sums of independent random variables. Also, since b_2 is small compared with b_1, the sum $\mathcal{L}_2$ and $\mathcal{L}_3$ will be small compared with the total sum of variables in the sequence, i.e., $\sigma^{-1}(\gamma)\left\{L(\gamma) - \mu(\gamma)\right\}$. We next show

$$\sigma^{-1}(\gamma)\left\{L(\gamma) - \mu(\gamma)\right\} = \mathcal{L}_1 + o_p(1).$$

As $\mathrm{E}\left(\mathcal{L}_2\right) = \mathrm{E}\left(\mathcal{L}_3\right) = 0$, it is sufficient to prove that $\mathrm{var}\left(\mathcal{L}_2\right) = \mathrm{var}\left(\mathcal{L}_3\right) = o(1)$. Consider $\mathrm{var}\left[\mathcal{L}_2\right]$ and we have

$$\begin{aligned}\mathrm{var}\left(\mathcal{L}_2\right) &= \sigma^{-2}(\gamma)\,\mathrm{var}\left\{\sum_{j=1}^{r} A_{j2}(\gamma)\right\} \\ &\le \sigma^{-2}(\gamma)\sum_{j_1=1}^{r}\sum_{j_2=1}^{r}\sum_{i_1=1}^{b_2}\sum_{i_2=1}^{b_2}\left|\mathrm{cov}\left\{L^{(j_1 b+b_1+i_1)}(\gamma), L^{(j_2 b+b_1+i_2)}(\gamma)\right\}\right|.\end{aligned}$$

We have the following α-mixing inequality (see, e.g., Guyon, 1995) that for any $\epsilon > 0$

$$\mathrm{cov}\{n^{\gamma/2}L^{(i)}(\gamma), n^{\gamma/2}L^{(j)}(\gamma)\} \le 8\alpha_X(|i-j|)^{\frac{\epsilon}{2+\epsilon}} \max_i [\mathrm{E}\{n^{\gamma/2}L^{(i)}(\gamma)\}^{2+\epsilon}]^{\frac{2}{2+\epsilon}}.$$

Then take $\epsilon = 1$ and from Proposition 1 we have

$$\begin{aligned}
&\left|\mathrm{cov}\left\{L^{(j_1b+b_1+i_1)}(\gamma), L^{(j_2b+b_1+i_2)}(\gamma)\right\}\right| \\
&= n^{-\gamma}\left|\mathrm{cov}\left\{n^{\gamma/2}L^{(j_1b+b_1+i_1)}(\gamma), n^{\gamma/2}L^{(j_2b+b_1+i_2)}(\gamma)\right\}\right| \\
&\le Bn^{-\gamma}\alpha_X\{|(j_1b+b_1+i_1)-(j_2b+b_1+i_2)|\}^{\frac{1}{3}} \\
&\le Bn^{-\gamma}M\delta^{|j_1b+i_1-j_2b-i_2|/3}
\end{aligned}$$

where B is some big constant. The above result implies that

$$\begin{aligned}
\mathrm{var}\,(\mathcal{L}_2) &\le \sigma^{-2}(\gamma)\sum_{j_1=1}^{r}\sum_{j_2=1}^{r}\sum_{i_1=1}^{b_2}\sum_{i_2=1}^{b_2}\left|\mathrm{cov}\left\{L^{(j_1b+b_1+i_1)}(\gamma), L^{(j_2b+b_1+i_2)}(\gamma)\right\}\right| \\
&\le \sigma^{-2}(\gamma)\sum_{j_1=1}^{r}\sum_{j_2=1}^{r}\sum_{i_1=1}^{b_2}\sum_{i_2=1}^{b_2} n^{-\gamma}BM\delta^{|j_1b+i_1-j_2b-i_2|/3} \\
&= O(1)\frac{n^{\gamma}}{p}rb_2n^{-\gamma} \;=\; O(1)\frac{rb_2}{p},
\end{aligned}$$

which goes to 0 as $p\to\infty$. This implies $\mathcal{L}_2 = o_p(1)$. Similarly, we can show that $\mathcal{L}_3 = o_p(1)$ under the strong mixing assumption. Therefore, we only need to focus on $\mathcal{L}_1$. Based on the similar arguments on page 338 in Ibragimov & Linnik (1971), we have for properly chosen r and b_2

$$\left|\mathrm{E}\left\{\exp\left(it\mathcal{L}_1\right)\right\} - \mathrm{E}^r\left[\exp\left\{it\sigma(\gamma)^{-1}A_{1,1}(\gamma)\right\}\right]\right| \le 16r\alpha_X(b_2)\to 0.$$

This implies that there exist independent random variables $\{\xi_j; j=1,\cdots,r\}$ such that ξ_j and $A_{j1}(\gamma)$ are identically distributed and $\mathcal{L}_1$ has the same asymptotic distribution as $\sigma^{-1}(\gamma)\sum_{j=1}^r\xi_j$. Then, we only need to show that the central limit theorem holds for $\sigma^{-1}(\gamma)\sum_{j=1}^r\xi_j$. This can be done by checking the Lyapunov condition. In particular, from the moment bounds in Theorem 1 in Kim (1994), the strong mixing assumption implies

$$\begin{aligned}
\mathrm{E}\left\{\sigma^{-1}(\gamma)A_{1,1}(\gamma)\right\}^4 &= \sigma^{-4}(\gamma)n^{-2\gamma}\,\mathrm{E}\left[\sum_{i=1}^{b_1}n^{\gamma/2}\left\{L^{(i)}(\gamma)-\mu^{(i)}(\gamma)\right\}\right]^4 \\
&= O(1)\sigma^{-4}(\gamma)n^{-2\gamma}b_1^2\left\{B_1 + B_2\sum_{i=1}^{b_1} i^{l-1}\alpha(i)^{\epsilon/(4+\epsilon)}\right\} \\
&= O(1)\frac{n^{2\gamma}}{p^2}n^{-2\gamma}b_1^2 \;=\; O\left(\frac{b_1^2}{p^2}\right),
\end{aligned}$$

where B_1 and B_2 are constants. Thus we have $\sum_{j=1}^r \sigma^{-4}(\gamma)\,\mathrm{E}\,\xi_j^4 = O\left(rb_1^2p^{-2}\right) = o(1)$ and the Lyapunov condition holds.

Thus, for any finite $\gamma\in\Gamma$, we have proved the asymptotic normal distribution of $L(\gamma)$. For any linear combination of $L(\gamma)$'s with respect to different γ, a similar argument as above gives the asymptotic normal distribution. Then the Cramér-Wold Theorem implies the asymptotic joint distribution of $\{L(\gamma);\gamma\in\Gamma\}$.

(ii) The conclusion follows directly from the proof of Theorem 6 in Cai et al. (2014). In particular, let $n = \frac{n_1 n_2}{n_1+n_2}$ and $\zeta_n = 2h^{-1/2}\sqrt{\log(p+n)}$, where h is defined in the moment assumption to ensure $\max_{\substack{k=1,2\\1\le j\le p}} \mathrm{E}\{e^{h(X_{k1}^{(j)}-\mu_k^{(j)})^2}\} < \infty$. We write

$$Z_{k,i}^{(j)} = \frac{X_{k,i}^{(j)}}{\sqrt{\sigma_{k,jj}}} I\left(\frac{\left|X_{k,i}^{(j)}\right|}{\sqrt{\sigma_{k,ii}}} \le \zeta_{n_k}\right)$$

for $k = 1, 2$ and $1 \le i \le n_k$. Let

$$\bar{L}^{(j)} = \frac{1}{n_1}\sum_{i=1}^{n_1} Z_{1i}^{(j)} - \frac{1}{n_2}\sum_{i=1}^{n_2} Z_{2i}^{(j)}, \qquad 1 \le j \le p.$$

From the proof of Theorem 6 in Cai et al. (2014),

$$\mathrm{pr}\left\{\frac{n_1 n_2}{n_1+n_2}\max_{1\le j\le p}\left(\bar{L}^{(j)}\right)^2 - a_p \le x\right\} \to \exp\left\{-\frac{1}{\sqrt{\pi}}\exp\left(-\frac{x}{2}\right)\right\}, \tag{3}$$

where $a_p = 2\log p - \log(\log p)$.
We write $L^{(j)} = (\bar{X}_1^{(j)}/\sqrt{\sigma_{1,jj}} - \bar{X}_2^{(j)}/\sqrt{\sigma_{2,jj}})$. Note that $L(\infty) = \max_{1\le j\le p}\left(L^{(j)}\right)^2$. Next we show that

$$\mathrm{pr}\left\{n\max_{1\le j\le p}\left(\bar{L}^{(j)}\right)^2 - a_p > x\right\} \sim \mathrm{pr}\left\{nL(\infty) - a_p > x\right\}.$$

We have under the null hypothesis

$$\begin{aligned}
&\mathrm{pr}\left\{\max_{1\le j\le p}\sqrt{n}\left|\bar{L}^{(j)} - L^{(j)}\right| > (\log p)^{-1}\right\}\\
&\le \mathrm{pr}\left\{\max_{1\le j\le p}\max_{\substack{k=1,2\\1\le i\le n_k}}(X_{k,i}^{(j)} - \mu_k^{(j)})/\sqrt{\sigma_{k,jj}} > \zeta_n\right\}\\
&\le np\max_{1\le j\le p}\mathrm{pr}\left\{e^{h\left(X_{k,1}^{(j)}-\mu_k^{(j)}\right)^2/\sigma_{k,jj}} > e^{h\zeta_n^2}\right\}\\
&= O\left(p^{-1} + n^{-1}\right).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\mathrm{pr}\left\{n\max_{1\le j\le p}\left(\bar{L}^{(j)}\right)^2 - a_p > x\right\}\\
&= \mathrm{pr}\left[nL(\infty) + n\left\{\max_{1\le j\le p}\left(\bar{L}^{(j)}\right)^2 - \max_{1\le j\le p}\left(L^{(j)}\right)^2\right\} - a_p > x\right]\\
&= \mathrm{pr}\Bigg[nL(\infty) + n\left\{\max_{1\le j\le p}\left(\bar{L}^{(j)}\right)^2 - \max_{1\le j\le p}\left(L^{(j)}\right)^2\right\} - a_p > x;\\
&\qquad \max_{1\le j\le p}\sqrt{n}\left|\bar{L}^{(j)} - L^{(j)}\right| < (\log p)^{-1}\Bigg] + o(1).
\end{aligned}$$

The above probability is bounded above/below by

$$\text{pr}\left\{nL(\infty) \pm \left(\frac{2}{\log p}\sqrt{n}\max_{1\le j\le p}\left|L^{(j)}\right| + \frac{1}{(\log p)^2}\right) - a_p > x\right\}.$$

Note that $\text{pr}\left\{\max_{1\le j\le p}\sqrt{n}L^{(j)} > (\log p)^{1/2+\epsilon}\right\} = o(1)$. Then we have

$$\text{pr}\left\{n\max_{1\le j\le p}\left(\bar{L}^{(j)}\right)^2 - a_p > x\right\} \sim \text{pr}\left\{nL(\infty) - a_p > x + o(1)\right\},$$

which completes the proof for asymptotic distribution of $L(\infty)$.

(iii) The proof of the asymptotic independence follows from a similar argument as that in Hsing (1995). Consider the sequence of random variables $\tilde{L}^{(j)}(\gamma)$ defined on the conditional probability measure $\widetilde{\text{pr}}$, given the event $nL(\infty) < a_p + y$ such that

$$\widetilde{\text{pr}}\left\{\tilde{L}^{(j)}(\gamma) \le x_j, 1 \le j \le p\right\} = \text{pr}\left\{L^{(j)}(\gamma) \le x_j, 1 \le j \le p \;\middle|\; \frac{n_1 n_2}{n_1+n_2}L(\infty) < a_p + y\right\}.$$

To show the asymptotic independence, we only need prove the asymptotic normality of $\sigma^{-1}(\gamma)\{\tilde{L}(\gamma) - \mu(\gamma)\} = \sigma^{-1}(\gamma)\sum_{i=1}^{p}\{\tilde{L}^{(j)}(\gamma) - \mu^{(i)}\}$ as in proof (i).
As the proof in (i), partition the sequence

$$\sigma^{-1}(\gamma)\left\{\tilde{L}^{(i)}(\gamma) - \mu^{(i)}(\gamma)\right\}, \text{ for } 1 \le i \le p,$$

into r blocks, where each block contains b variables such that $rb \le p < (r+1)b$. For each of the r blocks, we consider dividing it into two sub-blocks; a larger sub-block $\tilde{A}_{j1}$ with the first b_1 variables and a smaller sub-block $\tilde{A}_{j2}$ with the last $b_2 = b - b_1$ variables. We write

$$\tilde{A}_{j1}(\gamma) = \sum_{i=1}^{b_1}\left\{\tilde{L}^{((j-1)b+i)}(\gamma) - \mu^{((j-1)b+i)}(\gamma)\right\}, \qquad 1 \le j \le r;$$

$$\tilde{A}_{j2}(\gamma) = \sum_{i=1}^{b_2}\left\{\tilde{L}^{((j-1)b+b_1+i)}(\gamma) - \mu^{((j-1)b+b_1+i)}(\gamma)\right\}, \qquad 1 \le j \le r.$$

Therefore, we can write

$$\sigma^{-1}(\gamma)\left\{\tilde{L}(\gamma) - \mu(\gamma)\right\} = \tilde{\mathcal{L}}_1 + \tilde{\mathcal{L}}_2 + \tilde{\mathcal{L}}_3,$$

where

$$\tilde{\mathcal{L}}_1 = \sigma^{-1}(\gamma)\sum_{j=1}^{r}\tilde{A}_{j1}(\gamma);$$

$$\tilde{\mathcal{L}}_2 = \sigma^{-1}(\gamma)\sum_{j=1}^{r}\tilde{A}_{j2}(\gamma);$$

$$\tilde{\mathcal{L}}_3 = \sigma^{-1}(\gamma)\sum_{j=rb+1}^{p}\left\{\tilde{L}^{(j)}(\gamma) - \mu^{(j)}(\gamma)\right\}.$$

Our goal is to show that $\sigma(\gamma)^{-1}\left\{\tilde{L}(\gamma)-\mu(\gamma)\right\}$ satisfies the central limit theorem. We first show that $\widetilde{\mathrm{E}}\left(\tilde{\mathcal{L}}_2^2\right)=\widetilde{\mathrm{E}}\left(\tilde{\mathcal{L}}_3^2\right)=o(1)$, where $\widetilde{\mathrm{E}}$ is the expectation under the conditional probability measure $\widetilde{\mathrm{pr}}$. In particular,

$$\begin{aligned}
\widetilde{\mathrm{E}}\left(\tilde{\mathcal{L}}_2^2\right) &= \sigma^{-2}(\gamma)\widetilde{\mathrm{E}}\Bigg\{\Bigg(\sum_{j=1}^{r}\tilde{A}_{j2}(\gamma)\Bigg)^2\Bigg\} \\
&\leq \sigma^{-2}(\gamma)\left(\sum_{i\neq j}\left[\widetilde{\mathrm{E}}\left\{\tilde{A}_{i2}^2(\gamma)\right\}\right]^{1/2}\left[\widetilde{\mathrm{E}}\left\{\tilde{A}_{j2}^2(\gamma)\right\}\right]^{1/2}+\sum_{j=1}^{p}\widetilde{\mathrm{E}}\left\{\tilde{A}_{j2}^2(\gamma)\right\}\right) \\
&\leq \sigma^{-2}(\gamma)\left[\mathrm{pr}\left\{\frac{n_1n_2}{n_1+n_2}L(\infty)<a_p+y\right\}\right]^{-1} \\
&\quad\times\left(\sum_{i\neq j}\left[\mathrm{E}\left\{A_{i2}^2(\gamma)\right\}\right]^{1/2}\left[\mathrm{E}\left\{A_{j2}^2(\gamma)\right\}\right]^{1/2}+\sum_{j=1}^{p}\mathrm{E}\left\{A_{j2}^2(\gamma)\right\}\right),
\end{aligned}$$

where in the last step we use the fact that

$$\widetilde{\mathrm{E}}\left\{\tilde{A}_{j2}^2(\gamma)\right\}=\frac{\mathrm{E}\left\{A_{j2}^2(\gamma)\mid\frac{n_1n_2}{n_1+n_2}L(\infty)<a_p+y\right\}}{\mathrm{pr}\left\{\frac{n_1n_2}{n_1+n_2}L(\infty)<a_p+y\right\}}\leq\frac{\mathrm{E}\left\{A_{j2}^2(\gamma)\right\}}{\mathrm{pr}\left\{\frac{n_1n_2}{n_1+n_2}L(\infty)<a_p+y\right\}}.$$

The above bound goes to 0 under the strong mixing assumption by choosing proper convergence rate of b_2; see Equation (18.4.8) of Ibragimov & Linnik (1971). Similarly, we can show that $\widetilde{\mathrm{E}}\left(\tilde{\mathcal{L}}_3^2\right)=o(1)$. Therefore, we only need to focus on $\tilde{\mathcal{L}}_1$ part.
We need the following result:

$$\tilde{\alpha}_X(l)\leq 4\frac{\mathrm{pr}\left\{L_{b_2}(\infty)>a_p+y\right\}+\alpha(l)}{\left[P\left\{\frac{n_1n_2}{n_1+n_2}L(\infty)<a_p+y\right\}\right]^3},$$

where $L_l=\max_{1\leq h\leq p-l;h\leq k\leq h+l}\left\{\frac{n_1n_2}{n_1+n_2}L^{(k)}(\infty)\right\}$. The proof follows from a similar argument as that of Lemma 2.2 in Hsing (1995). Then from a similar argument to that on page 338 in Ibragimov & Linnik (1971), we obtain

$$\begin{aligned}
&\left|\widetilde{\mathrm{E}}\left\{\exp\left(it\tilde{\mathcal{L}}_1\right)\right\}-\widetilde{E}^r\left[\exp\left\{it\sigma^{-1}(\gamma)\tilde{A}_{j1}(\gamma)\right\}\right]\right| \\
&\leq 16r\tilde{\alpha}_X(b_2) \\
&\leq 64r\frac{\mathrm{pr}\left\{L_{b_2}(\infty)>a_p+y\right\}+\alpha_X(b_2)}{\left[\mathrm{pr}\left\{\frac{n_1n_2}{n_1+n_2}L(\infty)<a_p+y\right\}\right]^3}.
\end{aligned}$$

For the chosen r,b, we have the above quantity goes to 0. Thus $\sigma(\gamma)^{-1}\{\tilde{L}(\gamma)-\mu(\gamma)\}$ has the same limiting distribution as $\sigma^{-1}(\gamma)\sum_{i=1}^{r}\tilde{\xi}_i$, where $\tilde{\xi}_i$'s are independent random variables, and $\tilde{\xi}_i$ and $\tilde{A}_{i1}(\gamma)$ are identically distributed under measure $\widetilde{\mathrm{pr}}$. Following a similar argument as in Hsing (1995), we can show that $\widetilde{\mathrm{E}}\{\sigma^{-1}(\gamma)\sum_{i=1}^{r}\tilde{\xi}_i\}\to 0$ and $\widetilde{\mathrm{E}}[\{\sigma^{-1}(\gamma)\sum_{i=1}^{r}\tilde{\xi}_i\}^2]\to$

1. Further check the Lyapunov condition that

$$\sigma^{-4}(\gamma)\sum_{i=1}^{r}\widetilde{\mathrm{E}}\left(\tilde{\xi}_i^4\right)\leq\sigma^{-4}(\gamma)\frac{1}{\mathrm{pr}\left\{\frac{n_1n_2}{n_1+n_2}L(\infty)<a_p+y\right\}}\sum_{i=1}^{r}\mathrm{E}\left(\xi_i^4\right)\to\ 0,$$

where ξ_i's are defined as those in proof (i) and the convergence result also follows from (i). This implies the asymptotic normality of the conditional distribution given event $\frac{n_1n_2}{n_1+n_2}L(\infty)<a_p+y$. Then we have the asymptotic independence of $[\{L(\gamma)-\mu(\gamma)\}/\sigma(\gamma)]_{\gamma\in\Gamma'}^{\mathrm{T}}$ and $L(\infty)$. □

1·4. *An asymptotic power study and selection of* Γ

To study the power performance of different test statistic $L(\gamma)$, we consider a special case when the signal strength is fixed at the same level, denoted by δ, and $n_1=n_2=n/2$. For notational convenience, we also assume $\sigma_i=1$. We study the dense and sparse signal cases separately.

Case 1 dense signal with $\beta<1/2$. From Theorem 2 in Section 3.3, the asymptotic power of $L(\gamma)$ with $\gamma<\infty$ is mainly determined by the term $\{\mu_A(\gamma)-\mu(\gamma)\}/\sigma_A(\gamma)$. In addition, the asymptotic power of $L(\gamma)$ goes to 1 if $n^{\gamma/2}\{\mu_A(\gamma)-\mu(\gamma)\}/p^{1/2}\to\infty$. This implies that for any finite γ, a sufficient condition for the asymptotic power of $L(\gamma)$ going to 1 is

$$\frac{\delta}{n^{-1/2}p^{(2\beta-1)/(2\gamma)}}\to\infty,\qquad\text{as } p,n\to\infty.\tag{4}$$

Note that $p^{(2\beta-1)/(2\gamma)}\to 0$ as $p\to\infty$. Therefore, to compare the asymptotic powers of $L(\gamma)$'s, we focus on the local alternative such that

$$n^{1/2}\delta\to 0,\qquad\text{as } p,n\to\infty.$$

Equivalently, we write

$$\delta=n^{-1/2}r^{1/2},\ \text{where } r\to 0 \text{ as } p,n\to\infty.$$

Here r indicates the signal strength.

Under this alternative, when γ is odd,

$$\begin{aligned}\mu_A(\gamma)-\mu(\gamma)&=\sum_{i=1}^{p}\sum_{c=1}^{\gamma}\binom{\gamma}{c}\delta_i^c\mu^{(i)}(\gamma-c),\\&\sim\gamma\sum_{i=1}^{p}\delta_i\mu^{(i)}(\gamma-1)\\&\sim\sum_{d=0}^{(\gamma-1)/2}\frac{\gamma!}{d!\{(\gamma-1)/2-d\}!}\times r^{1/2}p^{1-\beta}n^{-\gamma/2},\end{aligned}$$

where $a_{n,p}\sim b_{n,p}$ as $n,p\to\infty$ means that $a_{n,p}=\{1+o(1)\}b_{n,p}$. Similarly, when γ is even,

$$\mu_A(\gamma)-\mu(\gamma)\sim o(1)r^{1/2}p^{1-\beta}n^{-\gamma/2}.$$

Furthermore, under this local alternative,

$$\sigma_A^2(\gamma)\sim\sigma^2(\gamma)\sim c_\gamma pn^{-\gamma}$$

where c_γ is some constant depending on γ and is given in Proposition 2.

From the above results, we have that as $n, p \to \infty$,

$$\frac{\mu_A(\gamma) - \mu(\gamma)}{\sigma_A(\gamma)} \sim \frac{\sum_{d=0}^{(\gamma-1)/2} \frac{\gamma!}{d!\{(\gamma-1)/2-d\}!} \times r^{1/2} p^{1/2-\beta}}{c_\gamma^{1/2}} \text{ if } \gamma \text{ is odd,} \tag{5}$$

$$\frac{\mu_A(\gamma) - \mu(\gamma)}{\sigma_A(\gamma)} \sim o(1) r^{1/2} p^{1/2-\beta} \text{ if } \gamma \text{ is even.} \tag{6}$$

The above results imply that the asymptotic test power is not going to 1 if $r^{1/2}p^{1/2-\beta} < \infty$. Therefore, we focus on the local alternative when

$$r \to 0 \text{ and } r^{1/2} p^{1/2-\beta} \to \infty.$$

From approximations (5) and (6), $L(\gamma)$ with odd γ has asymptotic power going to 1 while that with even γ does not, i.e., under the considered alternative, $L(\gamma)$ with odd γ is more powerful than even γ. Therefore, we only need to focus on odd γ's and compare their power. To determine which odd γ gives an asymptotically more powerful test, we only need to find which γ maximizes

$$c_\gamma^{-1/2} \sum_{d=0}^{(\gamma-1)/2} \frac{\gamma!}{d!\{(\gamma-1)/2-d\}!}$$

To simplify our discussion, we first consider the simplest case when $\sigma_{ij} = 0$ for $i \neq j$. In this case,

$$c_\gamma^{-1/2} \sum_{d=0}^{(\gamma-1)/2} \frac{\gamma!}{d!\{(\gamma-1)/2-d\}!} = \frac{\sum_{d=0}^{(\gamma-1)/2} \frac{\gamma!}{d!\{(\gamma-1)/2-d\}!}}{\sqrt{\sum_{d=0}^{\gamma} \frac{(2\gamma)!}{d!(\gamma-d)!}}},$$

which has maximum when $\gamma = 1$. Therefore, $L(1)$ is asymptotically most powerful.

More generally, consider the case when $\sigma_{ij} \geq 0$, a similar calculation following Proposition 2 also gives that $L(1)$ is asymptotically more powerful under the considered alternative. On the other hand, due to the slow convergence of the asymptotic results which depends on $p^{1/2-\beta}$, the finite performance of $L(1)$ may not be as good as other $L(\gamma)$'s with $\gamma > 1$, especially when β is close to $1/2$ and p is not large enough.

We illustrate these results through simulations. For most of the simulation results in Section 2 of the supplementary material, we can see that $L(1)$ is most powerful when $\beta = 0.1$, i.e., the signals are very dense. We further study the impact of sample size and sparsity parameters in Tables 13–15, which present results when the sample sizes n_1 and n_2 vary from 25 to 200 and and the sparsity parameter is $\beta = 0.1$, 0.2, and 0.5. These results show that sample size has little impact on the selection of parameter γ in the sum-of-powers tests for fixed signal sparsity β and fixed dimension $p = 200$. On the other hand, when β becomes closer to 0.5, $L(1)$ becomes less powerful compared with other $L(\gamma)$ tests, which is mainly due to the slow convergence of the asymptotic results which depends on $p^{1/2-\beta}$ as discussed above. To validate this argument, we conduct a new study with $\beta = 0.2$, $n_1 = n_2 = 50$, and the dimension p from 200 to 1000. Table 16 presents the corresponding simulation results. We can see that when $p = 200$, $L(1)$ is not as powerful as $L(2)$ and $L(3)$; however, as p becomes larger, $L(1)$ becomes more powerful. These observations are consistent with the theoretical result that $L(1)$ is asymptotically most powerful when signals are dense.

Similarly, when the absolute value of the signal strength is fixed at the same level such that $|\delta_i| = \delta$ for $i \in \mathcal{S}_\beta$, and the signs are random with half positive and half negative. Then a similar argument as above would give $L(2)$ is asymptotically most powerful. Therefore, motivated by these results, we would recommend including small γ's such as $\{1,2\}$ in Γ due to the asymptotic optimality under the considered special case. On the other hand, we also recommend having medium γ's such as $\{3,\ldots,6\}$ in Γ to have better finite sample performance, especially when β is close to $1/2$.

Case 2 sparse signal with $\beta > 1/2$. Following the work in Cai et al. (2014), when the signal strength is fixed at the same level $(2r\log p)^{1/2}n^{-1/2}$ for certain constant r, the power of $L(\infty)$ converges to 1 under certain regularity conditions. And it was also shown that the rate $r(\log p)^{1/2}n^{-1/2}$ is minimax optimal for testing against sparse alternatives. On the other hand, under this alternative, $L(\gamma)$ with finite γ loses power since $\{\mu_A(\gamma)-\mu(\gamma)\}/\sigma_A(\gamma)$ is bounded. Therefore, for the sparse signal case, $L(\infty)$ is more powerful than $L(\gamma)$ with $\gamma < \infty$.

Combining the above results, we would recommend including small γ's such as $\{1,2\}$ and medium γ's such as $\{3,\ldots,6\}$ in Γ to achieve balance between the asymptotic and finite sample performances when the signals are dense; in addition, we also recommend including $\{\infty\}$ in Γ, which is more powerful when the signals are sparse. Our extensive simulation studies in the next section show such a choice of Γ gives stable and good performance.

2. SIMULATIONS: MULTIVARIATE NORMAL DATA

We considered simulation set-ups similar to those of Chen et al. (2014) to compare the proposed sum-of-powers and adaptive tests with several existing tests proposed by Bai & Saranadasa (1996), Chen & Qin (2010), Srivastava & Du (2008), Cai et al. (2014), and Chen et al. (2014). The candidate set of γ for the sum-of-powers tests was set as $\Gamma = \{1, \ldots, 6, \infty\}$. We generated two groups of random samples $\{X_{1i}\}_{i=1}^{n_1}$ and $\{X_{2j}\}_{j=1}^{n_2}$ with sample sizes $n_1 = n_2 = 50$ from two multivariate normal distributions with dimension $p = 200$, $X_{ki} \sim N(\mu_k, \Sigma)$ for $k = 1, 2$. Without loss of generality, we always set $\mu_1 = 0$. Under the null hypothesis, we had $\mu_2 = 0$;under the alternative hypothesis, we had $\lfloor p^{1-\beta} \rfloor$ non-zero elements in μ_2, where β controlled the sparsity with its value ranging from 0 to 1. In our simulations, we used $\beta = 0.1, 0.2, ..., 0.9$, covering very dense signals for an alternative hypothesis at $\beta = 0.1$, to dense and then to only moderately dense at $\beta = 0.2$ and $\beta = 0.5$, finally to moderately sparse and very sparse signals at $\beta = 0.7$ and 0.9 respectively. Also, we assumed that the non-zero elements of μ_2 were uniformly distributed among positions $\{1, \ldots, p\}$, and their values were constant $\{2r(1/n_1 + 1/n_2) \log p\}^{1/2}$, where r controlled the signal strength. Denote the common covariance matrix $\Sigma = D^{1/2} R D^{1/2}$, where R was the correlation matrix and the diagonal matrix D contained the variances. We considered four structures for the correlation matrix $R = (r_{ij})$:

(a) 1-band structure; that is $r_{ii} = 1$ for $i = 1, \ldots, p$, $r_{ij} = 0.4$ for $|i - j| = 1$, and $r_{ij} = 0$ for $|i - j| > 1$.
(b) Autoregressive structure with order 1; that is, $r_{ij} = 0.6^{|i-j|}$ for $i, j = 1, \ldots, p$.
(c) Overlapping block diagonal structure; as used in Rothman (2012) and Xue et al. (2012), we partitioned the p variables into $K = 20$ blocks of an equal size, say J_k for $k = 1, \ldots, 20$. Denote i_k as the maximum index in J_k and

$$r_{ij} = 0.6 I_{\{i=j\}} + 0.4 \sum_{k=1}^{K} I_{\{i \in J_k, j \in J_k\}} + 0.4 \sum_{k=1}^{K-1} (I_{\{i=i_k, j \in J_{k+1}\}} + I_{\{i \in J_{k+1}, j=i_k\}}).$$

(d) Compound symmetric structure; that is, $r_{ij} = 1 I_{\{i=j\}} + 0.4 I_{\{i \neq j\}}$.

We also considered two cases for D:

(i) Equal variances; that is, $D = I_p$, an identity matrix;
(ii) Unequal variances; that is, $D = \mathrm{diag}(d_{11}, \ldots, d_{pp})$ with $d_{ii} \sim U(0.1, 10)$ independently.

By default we applied the banding estimator (Bickel & Levina, 2008) to obtain the estimates for Σ; we also considered two other estimators, an L_1-penalized estimator (Xue et al., 2012) and the sample covariance estimator. Five-fold cross-validation was used to select the tuning parameters, i.e. the bandwidth $k \in \{0, 1, 2, ..., 50\}$ for the banding estimator (unless specified otherwise) and $\lambda \in \{0, 0.05, 0.1, ..., 1\}$ for the L_1-penalized estimator implemented in R package `hglasso`. We considered $p = 200$ and $p = 500$. For each setting, we generated 1,000 replicates to estimate the empirical type I error or power of each test. The p-values were calculated based on both the asymptotics results in the theorem and the permutation method (with $B = 1,000$). The nominal significance level was set at $\alpha = 0.05$.

2·1. *Results for $p = 200$*

Tables 1–4 present the results for $p = 200$ using the banding covariance estimator with its bandwidth selected by 5-fold cross validation. The main conclusions about the relative power performance of the various tests held as discussed in the text, regardless of what true covariance matrices were used. In particular, we note that the results were not sensitive to whether the marginal variances of the variables were the same or not; that is, whether $D = I_p$ or $D = \text{diag}(d_{ii})$ was used. It is remarkable that even if the true covariance matrix was compound symmetric, under which the mixing condition C2 was violated, the proposed asymptotic adaptive sum-of-powers tests still performed quite well.

Table 1. *Empirical test sizes and powers (%) based on 1,000 replicates of multivariate normal data with sample sizes $n_1 = n_2 = 50$, dimension $p = 200$, and 1-band covariance matrix.*

Equal variances ($\mathbf{D} = \mathbf{I}_p$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.02$	$r = 0.04$	$r = 0.6$	$r = 0.9$	$r = 1.8$	$r = 2.7$
SPU(1)	5 (4)	81 (79)	99 (98)	18 (18)	24 (24)	6 (6)	6 (6)
SPU(2)	5 (6)	28 (28)	60 (63)	67 (65)	90 (89)	20 (20)	33 (33)
SPU(3)	4 (5)	66 (67)	95 (94)	67 (69)	93 (92)	30 (34)	58 (61)
SPU(4)	5 (6)	21 (23)	51 (53)	81 (83)	98 (98)	62 (63)	87 (86)
SPU(5)	5 (6)	34 (41)	68 (72)	79 (80)	98 (97)	64 (67)	91 (90)
SPU(6)	4 (5)	14 (18)	30 (38)	80 (82)	98 (98)	72 (74)	94 (93)
SPU(∞)	7 (5)	12 (10)	18 (16)	69 (61)	92 (88)	77 (72)	96 (92)
aSPU	5 (6)	64 (69)	94 (96)	79 (80)	98 (97)	73 (67)	94 (90)
CLZ	9 (6)	32 (21)	62 (48)	87 (78)	99 (97)	49 (35)	71 (57)
CLX	7 (5)	12 (10)	18 (16)	69 (61)	92 (88)	77 (72)	96 (92)
BZ	5 (6)	29 (28)	61 (63)	67 (65)	90 (90)	21 (20)	34 (34)
CQ	5 (6)	29 (28)	61 (63)	67 (65)	90 (90)	21 (20)	34 (34)
SD	4 (6)	25 (28)	57 (63)	64 (64)	88 (88)	17 (20)	30 (34)

Unequal variances ($\mathbf{D} = \text{diag}(d_{ii})$ with $d_{ii} \sim U(0.1, 10)$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.10$	$r = 0.15$	$r = 2.5$	$r = 4.0$	$r = 3.5$	$r = 5$
SPU(1)	5 (5)	84 (84)	96 (95)	16 (16)	22 (22)	4 (6)	5 (5)
SPU(2)	5 (4)	24 (23)	40 (38)	48 (46)	79 (77)	8 (7)	10 (9)
SPU(3)	4 (4)	46 (46)	65 (64)	46 (47)	78 (78)	5 (6)	7 (7)
SPU(4)	5 (5)	13 (13)	20 (19)	50 (49)	82 (82)	8 (8)	11 (10)
SPU(5)	4 (5)	14 (16)	21 (24)	42 (45)	76 (78)	4 (6)	7 (8)
SPU(6)	5 (6)	8 (10)	10 (13)	37 (41)	72 (75)	6 (7)	8 (11)
SPU(∞)	7 (5)	34 (28)	55 (48)	79 (75)	98 (97)	86 (83)	97 (97)
aSPU	6 (5)	69 (70)	89 (90)	75 (69)	98 (97)	79 (72)	96 (93)
CLZ	9 (4)	71 (57)	92 (85)	85 (74)	99 (97)	57 (41)	75 (64)
CLX	7 (5)	34 (29)	55 (48)	79 (74)	98 (97)	86 (84)	97 (97)
BZ	5 (5)	25 (23)	40 (38)	49 (46)	80 (77)	8 (7)	11 (9)
CQ	5 (5)	25 (23)	40 (38)	49 (46)	80 (77)	8 (7)	11 (9)
SD	4 (4)	64 (66)	88 (90)	55 (56)	86 (87)	21 (22)	34 (35)

The size (type I error) or power outside and inside parentheses were calculated from asymptotics- and permutation-based p-values, respectively. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

Table 2. *Empirical test sizes and powers (%) based on 1,000 replicates of multivariate normal data with sample sizes $n_1 = n_2 = 50$, dimension $p = 200$, and AR1 covariance matrix.*

Equal variances ($\mathbf{D} = \mathbf{I}_p$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.04$	$r = 0.06$	$r = 0.6$	$r = 0.9$	$r = 1.5$	$r = 2.5$
SPU(1)	5 (5)	78 (76)	92 (91)	12 (12)	15 (14)	6 (5)	6 (5)
SPU(2)	5 (5)	47 (46)	69 (67)	50 (49)	76 (75)	12 (14)	20 (22)
SPU(3)	4 (4)	71 (70)	88 (89)	48 (50)	78 (76)	14 (18)	38 (40)
SPU(4)	5 (5)	38 (37)	61 (60)	71 (71)	95 (95)	40 (41)	78 (78)
SPU(5)	4 (5)	47 (49)	70 (72)	68 (70)	93 (93)	46 (48)	82 (83)
SPU(6)	4 (4)	26 (29)	42 (45)	72 (76)	95 (96)	54 (58)	89 (89)
SPU(∞)	6 (5)	18 (15)	25 (21)	67 (59)	90 (87)	63 (63)	92 (90)
aSPU	6 (5)	66 (66)	85 (85)	69 (68)	94 (93)	56 (54)	90 (87)
CLZ	12 (5)	56 (34)	77 (57)	82 (67)	98 (93)	40 (23)	65 (42)
CLX	6 (5)	18 (15)	25 (21)	67 (60)	90 (88)	63 (63)	92 (90)
BZ	6 (5)	48 (46)	70 (67)	52 (49)	77 (75)	13 (15)	21 (23)
CQ	6 (5)	48 (46)	70 (67)	52 (49)	77 (75)	13 (15)	21 (23)
SD	4 (5)	43 (45)	67 (68)	46 (49)	73 (74)	11 (14)	18 (22)

Unequal variances ($\mathbf{D} = \mathrm{diag}(d_{ii})$ with $d_{ii} \sim U(0.1, 10)$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.15$	$r = 0.25$	$r = 2.5$	$r = 3.5$	$r = 3.0$	$r = 4.5$
SPU(1)	5 (5)	71 (68)	91 (90)	11 (10)	13 (12)	5 (5)	5 (5)
SPU(2)	5 (5)	30 (28)	56 (53)	38 (35)	56 (55)	7 (6)	8 (7)
SPU(3)	4 (4)	43 (42)	67 (66)	32 (33)	53 (52)	5 (5)	5 (5)
SPU(4)	4 (4)	18 (17)	31 (29)	42 (41)	64 (63)	5 (5)	8 (6)
SPU(5)	4 (4)	17 (20)	30 (32)	35 (37)	57 (59)	4 (4)	5 (6)
SPU(6)	3 (4)	10 (12)	15 (18)	34 (37)	53 (59)	4 (5)	5 (6)
SPU(∞)	6 (5)	54 (48)	89 (86)	78 (74)	95 (93)	76 (74)	94 (93)
aSPU	6 (5)	64 (63)	91 (89)	73 (66)	92 (88)	66 (60)	91 (88)
CLZ	12 (5)	88 (72)	99 (97)	81 (63)	94 (85)	48 (26)	68 (45)
CLX	6 (5)	54 (50)	89 (86)	78 (74)	95 (93)	76 (74)	94 (93)
BZ	6 (4)	32 (28)	57 (54)	39 (35)	58 (55)	7 (6)	8 (7)
CQ	6 (4)	32 (28)	57 (54)	39 (35)	58 (55)	7 (6)	8 (7)
SD	4 (5)	71 (73)	97 (97)	40 (42)	61 (63)	12 (13)	20 (21)

The size (type I error) or power outside and inside parentheses were calculated from asymptotics- and permutation-based p-values, respectively. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

Table 3. *Empirical test sizes and powers (%) based on 1,000 replicates of multivariate normal data with sample sizes $n_1 = n_2 = 50$, dimension $p = 200$, and overlapping block diagonal covariance matrix.*

Equal variances ($\mathbf{D} = \mathbf{I}_p$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.04$	$r = 0.06$	$r = 0.6$	$r = 0.9$	$r = 1.8$	$r = 2.7$
SPU(1)	4 (5)	64 (63)	81 (80)	10 (9)	12 (12)	5 (5)	5 (5)
SPU(2)	6 (5)	40 (38)	62 (59)	42 (41)	68 (66)	13 (12)	20 (18)
SPU(3)	5 (5)	60 (60)	79 (79)	41 (42)	71 (71)	17 (18)	36 (38)
SPU(4)	6 (6)	34 (34)	54 (53)	68 (67)	93 (93)	50 (48)	80 (80)
SPU(5)	4 (5)	41 (42)	63 (64)	64 (66)	91 (92)	56 (58)	84 (85)
SPU(6)	5 (5)	25 (28)	38 (42)	71 (74)	94 (95)	67 (69)	91 (92)
SPU(∞)	7 (6)	19 (16)	26 (21)	67 (60)	91 (90)	77 (73)	95 (94)
aSPU	6 (6)	50 (52)	72 (73)	67 (65)	93 (91)	70 (64)	92 (90)
CLZ	16 (6)	56 (34)	75 (53)	82 (65)	98 (92)	48 (24)	66 (40)
CLX	7 (6)	19 (16)	26 (21)	67 (61)	91 (89)	77 (74)	95 (93)
BZ	6 (5)	42 (38)	64 (59)	45 (41)	71 (66)	15 (12)	22 (19)
CQ	6 (5)	42 (38)	64 (59)	45 (41)	71 (66)	15 (12)	22 (19)
SD	5 (5)	36 (37)	58 (59)	38 (41)	64 (66)	11 (12)	17 (18)

Unequal variances ($\mathbf{D} = \mathrm{diag}(d_{ii})$ with $d_{ii} \sim U(0.1, 10)$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.15$	$r = 0.25$	$r = 2.5$	$r = 3.5$	$r = 3.0$	$r = 4.5$
SPU(1)	4 (5)	56 (56)	78 (77)	10 (9)	11 (11)	5 (5)	4 (5)
SPU(2)	6 (5)	28 (25)	48 (46)	32 (29)	48 (47)	8 (7)	9 (8)
SPU(3)	5 (6)	34 (34)	56 (56)	28 (28)	45 (44)	6 (6)	7 (7)
SPU(4)	6 (5)	16 (16)	29 (27)	40 (39)	61 (61)	7 (6)	9 (8)
SPU(5)	5 (6)	17 (18)	27 (29)	31 (34)	54 (57)	5 (6)	6 (7)
SPU(6)	5 (6)	11 (12)	15 (18)	33 (37)	52 (58)	5 (7)	6 (8)
SPU(∞)	7 (6)	56 (50)	86 (83)	80 (74)	94 (93)	76 (73)	95 (93)
aSPU	7 (6)	55 (53)	86 (84)	72 (64)	92 (88)	66 (60)	90 (87)
CLZ	16 (6)	86 (67)	99 (95)	80 (58)	96 (83)	48 (24)	66 (40)
CLX	7 (6)	56 (51)	86 (84)	80 (74)	94 (93)	76 (73)	95 (93)
BZ	7 (5)	29 (26)	50 (46)	34 (30)	50 (47)	9 (7)	10 (8)
CQ	7 (5)	29 (26)	50 (46)	34 (30)	50 (47)	9 (7)	10 (8)
SD	5 (5)	64 (65)	93 (93)	33 (35)	51 (53)	11 (12)	17 (18)

The size (type I error) or power outside and inside parentheses were calculated from asymptotics- and permutation-based p-values, respectively. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

Table 4. *Empirical test sizes and powers (%) based on 1,000 replicates of multivariate normal data with sample sizes $n_1 = n_2 = 50$, dimension $p = 200$, and compound symmetric covariance matrix.*

Equal variances ($\mathbf{D} = \mathbf{I}_p$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.02$	$r = 0.04$	$r = 0.6$	$r = 0.9$	$r = 1.8$	$r = 2.7$
SPU(1)	6 (6)	10 (9)	11 (10)	7 (7)	7 (7)	7 (6)	7 (6)
SPU(2)	8 (6)	11 (10)	14 (12)	12 (9)	14 (11)	9 (7)	10 (8)
SPU(3)	7 (6)	10 (10)	13 (12)	13 (11)	18 (16)	9 (8)	11 (10)
SPU(4)	6 (6)	10 (9)	12 (12)	18 (18)	34 (35)	15 (14)	30 (30)
SPU(5)	5 (6)	9 (10)	11 (12)	23 (26)	42 (47)	24 (26)	48 (52)
SPU(6)	5 (6)	8 (10)	10 (13)	29 (36)	55 (64)	36 (45)	74 (81)
SPU(∞)	6 (6)	8 (8)	11 (11)	53 (55)	77 (79)	75 (77)	95 (96)
aSPU	6 (7)	8 (9)	11 (12)	41 (49)	65 (72)	68 (72)	92 (95)
CLZ	23 (6)	28 (10)	34 (12)	67 (14)	93 (30)	34 (7)	44 (8)
CLX	6 (7)	8 (9)	11 (11)	53 (56)	77 (79)	75 (77)	95 (96)
BZ	8 (6)	12 (10)	14 (12)	13 (9)	15 (11)	10 (7)	10 (8)
CQ	8 (6)	12 (10)	14 (12)	13 (9)	15 (11)	10 (7)	10 (8)
SD	3 (6)	5 (9)	7 (12)	5 (9)	6 (11)	4 (7)	4 (8)

Unequal variances ($\mathbf{D} = \mathrm{diag}(d_{ii})$ with $d_{ii} \sim U(0.1, 10)$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.2$	$r = 0.3$	$r = 2.5$	$r = 4.0$	$r = 3$	$r = 5$
SPU(1)	6 (6)	11 (11)	13 (13)	7 (7)	8 (7)	6 (6)	6 (6)
SPU(2)	8 (6)	14 (11)	17 (14)	11 (9)	13 (11)	8 (6)	8 (6)
SPU(3)	7 (6)	12 (11)	14 (13)	12 (10)	15 (14)	7 (6)	7 (6)
SPU(4)	6 (6)	11 (11)	13 (13)	13 (13)	24 (25)	6 (6)	7 (6)
SPU(5)	6 (6)	10 (10)	11 (13)	14 (16)	28 (32)	6 (6)	6 (6)
SPU(6)	5 (6)	8 (10)	10 (12)	14 (20)	32 (40)	5 (6)	5 (7)
SPU(∞)	6 (6)	55 (57)	81 (82)	69 (71)	94 (95)	75 (76)	97 (97)
aSPU	6 (7)	43 (50)	70 (76)	57 (64)	89 (92)	68 (72)	95 (96)
CLZ	22 (6)	64 (19)	86 (34)	64 (12)	91 (24)	34 (7)	47 (8)
CLX	6 (7)	55 (58)	81 (82)	69 (71)	94 (95)	75 (77)	97 (97)
BZ	8 (6)	14 (11)	17 (13)	12 (9)	14 (11)	8 (6)	9 (6)
CQ	8 (6)	14 (11)	17 (13)	12 (9)	14 (11)	8 (6)	9 (6)
SD	3 (6)	10 (17)	15 (25)	5 (9)	6 (11)	4 (7)	4 (8)

The size (type I error) or power outside and inside parentheses were calculated from asymptotics- and permutation-based p-values, respectively. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

2·2. *Results for $p = 500$ and 1000*

Tables 5–7 present the results for $p = 500$ using the banding estimator with its bandwidth selected by 5-fold cross validation. The main conclusions remained the same as those for $p = 200$.

Table 5. *Empirical test sizes and powers (%) based on 1,000 replicates of multivariate normal data with sample sizes $n_1 = n_2 = 50$, dimension $p = 500$, and 1-band covariance matrix.*

Equal variances ($\mathbf{D} = \mathbf{I}_p$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.01$	$r = 0.02$	$r = 0.4$	$r = 0.6$	$r = 1.5$	$r = 2.1$
SPU(1)	6 (6)	88 (87)	99 (99)	15 (14)	19 (18)	6 (6)	6 (6)
SPU(2)	6 (6)	24 (24)	54 (54)	48 (48)	76 (76)	14 (13)	18 (18)
SPU(3)	5 (5)	73 (74)	97 (96)	45 (47)	80 (81)	17 (18)	30 (31)
SPU(4)	5 (5)	17 (20)	41 (44)	64 (68)	94 (94)	44 (46)	68 (70)
SPU(5)	4 (6)	39 (42)	74 (77)	61 (64)	92 (93)	49 (53)	76 (78)
SPU(6)	4 (6)	11 (14)	24 (30)	61 (68)	94 (95)	60 (63)	84 (86)
SPU(∞)	9 (6)	12 (8)	15 (11)	49 (41)	80 (73)	69 (64)	88 (86)
aSPU	6 (6)	73 (76)	97 (98)	64 (63)	94 (94)	64 (58)	85 (83)
CLZ	13 (5)	31 (16)	57 (35)	78 (60)	97 (92)	46 (28)	61 (41)
CLX	9 (6)	12 (8)	15 (11)	49 (41)	80 (73)	69 (65)	88 (86)
BZ	6 (6)	25 (24)	56 (53)	49 (48)	77 (76)	14 (14)	19 (18)
CQ	6 (6)	25 (24)	56 (53)	49 (48)	77 (76)	14 (14)	19 (18)
SD	5 (6)	20 (24)	49 (53)	44 (48)	72 (75)	12 (14)	16 (19)

The size (type I error) or power outside and inside parentheses were calculated from asymptotics- and permutation-based p-values, respectively. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

Table 6. *Empirical test sizes and powers (%) based on 1,000 replicates of multivariate normal data with sample sizes $n_1 = n_2 = 50$, dimension $p = 500$, and the first-order autoregressive covariance matrix.*

Equal variances ($\mathbf{D} = \mathbf{I}_p$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.01$	$r = 0.02$	$r = 0.4$	$r = 0.6$	$r = 1.5$	$r = 2.1$
SPU(1)	7 (6)	85 (83)	98 (98)	12 (9)	13 (12)	7 (5)	7 (6)
SPU(2)	6 (6)	42 (42)	82 (82)	36 (35)	59 (58)	12 (12)	15 (15)
SPU(3)	5 (5)	78 (78)	97 (97)	30 (30)	57 (57)	12 (13)	20 (21)
SPU(4)	5 (6)	32 (33)	72 (73)	52 (54)	86 (87)	36 (38)	61 (62)
SPU(5)	4 (5)	53 (56)	86 (87)	48 (50)	85 (86)	41 (43)	71 (73)
SPU(6)	4 (6)	21 (25)	49 (56)	53 (58)	90 (92)	55 (58)	81 (83)
SPU(∞)	9 (6)	15 (11)	25 (19)	48 (40)	81 (73)	68 (64)	89 (86)
aSPU	6 (5)	72 (73)	97 (97)	54 (51)	88 (86)	62 (55)	85 (82)
CLZ	18 (6)	58 (30)	87 (68)	76 (51)	96 (86)	46 (22)	62 (34)
CLX	9 (6)	15 (11)	25 (20)	48 (40)	81 (73)	68 (64)	89 (86)
BZ	7 (6)	44 (43)	83 (81)	38 (36)	60 (59)	13 (12)	17 (14)
CQ	7 (6)	44 (43)	83 (81)	38 (36)	60 (59)	13 (12)	17 (14)
SD	6 (6)	39 (42)	78 (80)	32 (36)	55 (58)	10 (12)	13 (15)

The size (type I error) or power outside and inside parentheses were calculated from asymptotics- and permutation-based p-values, respectively. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

Table 7. *Empirical test sizes and powers (%) based on 1,000 replicates of multivariate normal data with sample sizes $n_1 = n_2 = 50$, dimension $p = 500$, and overlapping block diagonal covariance matrix.*

Equal variances ($\mathbf{D} = \mathbf{I}_p$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.01$	$r = 0.02$	$r = 0.4$	$r = 0.6$	$r = 1.5$	$r = 2.1$
SPU(1)	6 (6)	72 (71)	88 (87)	8 (7)	8 (8)	6 (6)	6 (5)
SPU(2)	5 (5)	49 (47)	76 (74)	30 (28)	45 (42)	8 (7)	9 (8)
SPU(3)	5 (5)	70 (70)	87 (87)	30 (31)	50 (51)	7 (7)	12 (12)
SPU(4)	6 (6)	44 (44)	69 (69)	64 (64)	89 (89)	23 (24)	43 (43)
SPU(5)	5 (5)	57 (59)	77 (80)	65 (68)	88 (89)	31 (33)	56 (57)
SPU(6)	5 (6)	33 (37)	57 (61)	77 (80)	95 (96)	48 (51)	73 (76)
SPU(∞)	10 (6)	23 (17)	33 (25)	78 (71)	92 (90)	69 (65)	89 (87)
aSPU	7 (6)	60 (61)	81 (82)	73 (68)	93 (91)	60 (54)	85 (79)
CLZ	23 (6)	78 (44)	91 (68)	93 (69)	99 (92)	45 (14)	56 (20)
CLX	10 (7)	23 (18)	33 (26)	78 (72)	92 (90)	69 (64)	89 (87)
BZ	7 (5)	52 (48)	78 (75)	33 (28)	48 (43)	9 (7)	10 (9)
CQ	7 (5)	52 (48)	78 (75)	33 (28)	48 (43)	9 (7)	10 (9)
SD	4 (5)	44 (47)	70 (74)	25 (28)	38 (43)	6 (7)	7 (8)

The size (type I error) or power outside and inside parentheses were calculated from asymptotics- and permutation-based p-values, respectively. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

2·3. *Using other covariance estimators*

Tables 8–12 present the results for $p = 200$ using two alternative covariance matrix estimators. First, we used the sample covariance matrix. It is somewhat surprising that the asymptotic adaptive sum-of-powers test performed well, including having well-controlled type I error rates. Among individual sum-of-powers tests, an asymptotic sum-of-powers test with an even integer γ was conservative with a test size smaller than the nominal level except for the extreme case with a compound symmetric covariance matrix, in which the asymptotic sum-of-powers tests might have slightly inflated type I error rates. Second, we applied the L_1-penalized covariance matrix estimator. It was more time-consuming but did not outperform the banding estimator, yielding the asymptotic adaptive sum-of-powers tests with possibly inflated type I error rates.

Table 8. *Empirical test sizes and powers (%) based on 1,000 replicates of multivariate normal data with sample sizes $n_1 = n_2 = 50$, dimension $p = 200$, and 1-band covariance matrix. The covariance matrix was estimated by the sample covariance matrix.*

Equal variances ($\mathbf{D} = \mathbf{I}_p$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.02$	$r = 0.04$	$r = 0.6$	$r = 0.9$	$r = 1.8$	$r = 2.7$
SPU(1)	5 (4)	82 (79)	98 (98)	18 (18)	24 (24)	7 (6)	7 (6)
SPU(2)	1 (6)	8 (28)	32 (63)	35 (65)	70 (89)	5 (20)	10 (33)
SPU(3)	5 (5)	66 (67)	95 (94)	68 (69)	94 (92)	31 (34)	58 (61)
SPU(4)	2 (6)	10 (23)	30 (53)	70 (83)	96 (98)	47 (63)	80 (86)
SPU(5)	5 (6)	34 (41)	68 (72)	80 (80)	98 (97)	64 (67)	91 (90)
SPU(6)	3 (5)	9 (18)	21 (38)	73 (82)	97 (98)	68 (74)	92 (93)
SPU(∞)	7 (5)	12 (10)	18 (16)	69 (61)	92 (88)	77 (72)	96 (92)
aSPU	4 (6)	64 (69)	94 (96)	75 (80)	97 (97)	71 (67)	93 (90)
CLZ	9 (6)	32 (21)	62 (48)	87 (78)	99 (97)	49 (35)	71 (57)
CLX	7 (5)	12 (10)	18 (16)	69 (61)	92 (88)	77 (72)	96 (92)
BZ	5 (6)	29 (28)	61 (63)	67 (65)	90 (90)	21 (20)	34 (34)
CQ	5 (6)	29 (28)	61 (63)	67 (65)	90 (90)	21 (20)	34 (34)
SD	4 (6)	25 (28)	57 (63)	64 (64)	88 (88)	17 (20)	30 (34)

Unequal variances ($\mathbf{D} = \mathrm{diag}(d_{ii})$ with $d_{ii} \sim U(0.1, 10)$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.10$	$r = 0.15$	$r = 2.5$	$r = 4.0$	$r = 3.5$	$r = 5$
SPU(1)	5 (5)	84 (84)	96 (95)	17 (16)	23 (22)	6 (6)	6 (5)
SPU(2)	1 (4)	8 (23)	19 (38)	26 (46)	57 (77)	2 (7)	2 (9)
SPU(3)	4 (4)	46 (46)	65 (64)	46 (47)	78 (78)	6 (6)	7 (7)
SPU(4)	3 (5)	8 (13)	12 (19)	40 (49)	75 (82)	4 (8)	7 (10)
SPU(5)	4 (5)	14 (16)	21 (24)	42 (45)	76 (78)	4 (6)	7 (8)
SPU(6)	4 (6)	7 (10)	8 (13)	35 (41)	68 (75)	4 (7)	7 (11)
SPU(∞)	7 (5)	34 (28)	55 (48)	79 (75)	98 (97)	86 (83)	97 (97)
aSPU	5 (5)	68 (70)	88 (90)	73 (69)	97 (97)	78 (72)	95 (93)
CLZ	9 (4)	71 (57)	92 (85)	85 (74)	99 (97)	57 (41)	75 (64)
CLX	7 (5)	34 (29)	55 (48)	79 (74)	98 (97)	86 (84)	97 (97)
BZ	5 (5)	25 (23)	40 (38)	49 (46)	80 (77)	8 (7)	11 (9)
CQ	5 (5)	25 (23)	40 (38)	49 (46)	80 (77)	8 (7)	11 (9)
SD	4 (4)	64 (66)	88 (90)	55 (56)	86 (87)	21 (22)	34 (35)

The size (type I error) or power outside and inside parentheses were calculated from asymptotics- and permutation-based p-values, respectively. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

Table 9. *Empirical test sizes and powers (%) based on 1,000 replicates of multivariate normal data with sample sizes $n_1 = n_2 = 50$, dimension $p = 200$, and the first-order autoregressive covariance matrix. The covariance matrix was estimated by the sample covariance matrix.*

Equal variances ($\mathbf{D} = \mathbf{I}_p$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.04$	$r = 0.06$	$r = 0.6$	$r = 0.9$	$r = 1.5$	$r = 2.5$
SPU(1)	4 (5)	77 (76)	91 (91)	12 (12)	14 (14)	6 (5)	6 (5)
SPU(2)	2 (5)	29 (47)	54 (69)	31 (49)	58 (75)	6 (14)	9 (22)
SPU(3)	4 (5)	70 (70)	88 (87)	47 (50)	77 (76)	14 (18)	37 (40)
SPU(4)	3 (5)	27 (39)	47 (61)	61 (71)	91 (95)	29 (41)	71 (78)
SPU(5)	4 (6)	47 (51)	70 (72)	67 (70)	92 (93)	46 (48)	82 (83)
SPU(6)	3 (5)	22 (31)	35 (48)	66 (76)	93 (96)	50 (58)	87 (89)
SPU(∞)	6 (5)	18 (16)	25 (23)	67 (59)	90 (87)	63 (63)	92 (90)
aSPU	5 (6)	62 (66)	82 (84)	67 (68)	92 (93)	56 (54)	90 (87)
CLZ	12 (6)	56 (37)	77 (58)	82 (67)	98 (93)	40 (23)	65 (42)
CLX	6 (5)	18 (16)	25 (22)	67 (60)	90 (88)	63 (63)	92 (90)
BZ	6 (6)	48 (47)	70 (70)	52 (49)	77 (75)	13 (15)	21 (23)
CQ	6 (6)	48 (47)	70 (70)	52 (49)	77 (75)	13 (15)	21 (23)
SD	4 (6)	43 (46)	67 (69)	46 (49)	73 (74)	11 (14)	18 (22)

Unequal variances ($\mathbf{D} = \mathrm{diag}(d_{ii})$ with $d_{ii} \sim U(0.1, 10)$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.15$	$r = 0.25$	$r = 2.5$	$r = 3.5$	$r = 3.0$	$r = 4.5$
SPU(1)	5 (5)	69 (68)	90 (90)	10 (10)	12 (12)	5 (5)	5 (5)
SPU(2)	2 (5)	18 (28)	40 (53)	23 (35)	40 (55)	3 (6)	3 (7)
SPU(3)	4 (4)	43 (42)	66 (66)	32 (33)	52 (52)	5 (5)	5 (5)
SPU(4)	3 (4)	13 (17)	23 (29)	37 (41)	57 (63)	3 (5)	4 (6)
SPU(5)	4 (4)	17 (20)	29 (32)	35 (37)	56 (59)	4 (4)	5 (6)
SPU(6)	2 (4)	8 (12)	13 (18)	32 (37)	50 (59)	2 (5)	4 (6)
SPU(∞)	6 (5)	54 (48)	89 (86)	78 (74)	95 (93)	76 (74)	94 (93)
aSPU	4 (5)	62 (63)	90 (89)	71 (66)	91 (88)	66 (60)	91 (88)
CLZ	12 (5)	88 (72)	99 (97)	81 (63)	94 (85)	48 (26)	68 (45)
CLX	6 (5)	54 (50)	89 (86)	78 (74)	95 (93)	76 (74)	94 (93)
BZ	6 (4)	32 (28)	57 (54)	39 (35)	58 (55)	7 (6)	8 (7)
CQ	6 (4)	32 (28)	57 (54)	39 (35)	58 (55)	7 (6)	8 (7)
SD	4 (5)	71 (73)	97 (97)	40 (42)	61 (63)	12 (13)	20 (21)

The size (type I error) or power outside and inside parentheses were calculated from asymptotics- and permutation-based p-values, respectively. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

Table 10. *Empirical test sizes and powers (%) based on 1,000 replicates of multivariate normal data with sample sizes $n_1 = n_2 = 50$, dimension $p = 200$, and overlapping block diagonal covariance matrix. The covariance matrix was estimated by the sample covariance matrix.*

Equal variances ($\mathbf{D} = \mathbf{I}_p$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.04$	$r = 0.06$	$r = 0.6$	$r = 0.9$	$r = 1.8$	$r = 2.7$
SPU(1)	5 (5)	64 (63)	81 (80)	10 (9)	12 (12)	5 (5)	5 (5)
SPU(2)	3 (5)	28 (38)	49 (59)	28 (41)	53 (66)	7 (12)	10 (18)
SPU(3)	5 (5)	61 (60)	79 (79)	42 (42)	70 (71)	17 (18)	37 (38)
SPU(4)	4 (6)	27 (34)	44 (53)	60 (67)	90 (93)	40 (48)	74 (80)
SPU(5)	4 (5)	40 (42)	63 (64)	64 (66)	91 (92)	56 (58)	84 (85)
SPU(6)	4 (5)	22 (28)	34 (42)	67 (74)	93 (95)	64 (69)	89 (92)
SPU(∞)	7 (6)	19 (16)	26 (21)	67 (60)	91 (90)	77 (73)	95 (94)
aSPU	6 (6)	49 (52)	71 (73)	65 (65)	92 (91)	69 (64)	91 (90)
CLZ	16 (6)	56 (34)	75 (53)	82 (65)	98 (92)	48 (24)	66 (40)
CLX	7 (6)	19 (16)	26 (21)	67 (61)	91 (89)	77 (74)	95 (93)
BZ	6 (5)	42 (38)	64 (59)	45 (41)	71 (66)	15 (12)	22 (19)
CQ	6 (5)	42 (38)	64 (59)	45 (41)	71 (66)	15 (12)	22 (19)
SD	5 (5)	36 (37)	58 (59)	38 (41)	64 (66)	11 (12)	17 (18)

Unequal variances ($\mathbf{D} = \mathrm{diag}(d_{ii})$ with $d_{ii} \sim U(0.1, 10)$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.15$	$r = 0.25$	$r = 2.5$	$r = 3.5$	$r = 3.0$	$r = 4.5$
SPU(1)	5 (5)	56 (56)	78 (77)	10 (9)	11 (11)	5 (5)	5 (5)
SPU(2)	3 (5)	19 (25)	37 (46)	21 (29)	36 (47)	4 (7)	5 (8)
SPU(3)	6 (6)	34 (34)	57 (56)	28 (28)	45 (44)	6 (6)	7 (7)
SPU(4)	5 (5)	14 (16)	23 (27)	34 (39)	56 (61)	6 (6)	7 (8)
SPU(5)	5 (6)	17 (18)	27 (29)	31 (34)	55 (57)	5 (6)	6 (7)
SPU(6)	4 (6)	9 (12)	14 (18)	32 (37)	50 (58)	5 (7)	6 (8)
SPU(∞)	7 (6)	56 (50)	86 (83)	80 (74)	94 (93)	76 (73)	95 (93)
aSPU	6 (6)	56 (53)	85 (84)	70 (64)	92 (88)	66 (60)	90 (87)
CLZ	16 (6)	86 (67)	99 (95)	80 (58)	96 (83)	48 (24)	66 (40)
CLX	7 (6)	56 (51)	86 (84)	80 (74)	94 (93)	76 (73)	95 (93)
BZ	7 (5)	29 (26)	50 (46)	34 (30)	50 (47)	9 (7)	10 (8)
CQ	7 (5)	29 (26)	50 (46)	34 (30)	50 (47)	9 (7)	10 (8)
SD	5 (5)	64 (65)	93 (93)	33 (35)	51 (53)	11 (12)	17 (18)

The size (type I error) or power outside and inside parentheses were calculated from asymptotics- and permutation-based p-values, respectively. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

Table 11. *Empirical test sizes and powers (%) based on 1,000 replicates of multivariate normal data with sample sizes $n_1 = n_2 = 50$, dimension $p = 200$, and compound symmetric covariance matrix. The covariance matrix was estimated by the sample covariance matrix.*

Equal variances ($\mathbf{D} = \mathbf{I}_p$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.02$	$r = 0.04$	$r = 0.6$	$r = 0.9$	$r = 1.8$	$r = 2.7$
SPU(1)	6 (6)	10 (9)	11 (10)	7 (7)	7 (7)	7 (6)	7 (6)
SPU(2)	8 (6)	11 (10)	14 (12)	12 (9)	14 (11)	9 (7)	10 (8)
SPU(3)	7 (6)	10 (10)	13 (12)	13 (11)	18 (16)	9 (8)	11 (10)
SPU(4)	6 (6)	10 (9)	12 (12)	18 (18)	34 (35)	15 (14)	30 (30)
SPU(5)	5 (6)	9 (10)	11 (12)	23 (26)	42 (47)	24 (26)	48 (52)
SPU(6)	5 (6)	8 (10)	10 (13)	29 (36)	55 (64)	36 (45)	74 (81)
SPU(∞)	6 (6)	8 (8)	11 (11)	53 (55)	77 (79)	75 (77)	95 (96)
aSPU	6 (7)	8 (9)	11 (12)	41 (49)	65 (72)	68 (72)	92 (95)
CLZ	23 (6)	28 (10)	34 (12)	67 (14)	93 (30)	34 (7)	44 (8)
CLX	6 (7)	8 (9)	11 (11)	53 (56)	77 (79)	75 (77)	95 (96)
BZ	8 (6)	12 (10)	14 (12)	13 (9)	15 (11)	10 (7)	10 (8)
CQ	8 (6)	12 (10)	14 (12)	13 (9)	15 (11)	10 (7)	10 (8)
SD	3 (6)	5 (9)	7 (12)	5 (9)	6 (11)	4 (7)	4 (8)

Unequal variances ($\mathbf{D} = \mathrm{diag}(d_{ii})$ with $d_{ii} \sim U(0.1, 10)$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.2$	$r = 0.3$	$r = 2.5$	$r = 4.0$	$r = 3$	$r = 5$
SPU(1)	6 (6)	11 (11)	13 (13)	7 (7)	8 (7)	6 (6)	6 (6)
SPU(2)	8 (6)	14 (11)	17 (14)	11 (9)	13 (11)	8 (6)	8 (6)
SPU(3)	7 (6)	12 (11)	14 (13)	12 (10)	15 (14)	7 (6)	7 (6)
SPU(4)	6 (6)	11 (11)	13 (13)	13 (13)	24 (25)	6 (6)	7 (6)
SPU(5)	6 (6)	10 (10)	11 (13)	14 (16)	28 (32)	6 (6)	6 (6)
SPU(6)	5 (6)	8 (10)	10 (12)	14 (20)	32 (40)	5 (6)	5 (7)
SPU(∞)	6 (6)	55 (57)	81 (82)	69 (71)	94 (95)	75 (76)	97 (97)
sSPU	6 (7)	43 (50)	70 (76)	57 (64)	89 (92)	68 (72)	95 (96)
CLZ	22 (6)	64 (19)	86 (34)	64 (12)	91 (24)	34 (7)	47 (8)
CLX	6 (7)	55 (58)	81 (82)	69 (71)	94 (95)	75 (77)	97 (97)
BZ	8 (6)	14 (11)	17 (13)	12 (9)	14 (11)	8 (6)	9 (6)
CQ	8 (6)	14 (11)	17 (13)	12 (9)	14 (11)	8 (6)	9 (6)
SD	3 (6)	10 (17)	15 (25)	5 (9)	6 (11)	4 (7)	4 (8)

The size (type I error) or power outside and inside parentheses were calculated from asymptotics- and permutation-based p-values, respectively. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

Table 12. *Empirical test sizes and powers (%) based on 1,000 replicates of multivariate normal data with sample sizes $n_1 = n_2 = 50$, dimension $p = 200$, and 1-band covariance matrix. The covariance matrix was estimated by the L_1-penalized covariance matrix estimator of Xue et al. (2012).*

Equal variances ($\mathbf{D} = \mathbf{I}_p$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.02$	$r = 0.04$	$r = 0.6$	$r = 0.9$	$r = 1.8$	$r = 2.7$
SPU(1)	9 (4)	87 (79)	99 (98)	25 (18)	33 (24)	10 (6)	10 (6)
SPU(2)	7 (6)	36 (28)	70 (63)	74 (65)	93 (89)	29 (20)	41 (33)
SPU(3)	7 (5)	74 (67)	96 (94)	74 (69)	95 (92)	37 (34)	65 (61)
SPU(4)	8 (6)	26 (23)	57 (53)	86 (83)	99 (98)	66 (63)	90 (86)
SPU(5)	6 (6)	41 (41)	72 (72)	83 (80)	98 (97)	67 (67)	92 (90)
SPU(6)	5 (5)	16 (18)	35 (38)	82 (82)	98 (98)	74 (74)	95 (93)
SPU(∞)	7 (5)	12 (10)	18 (16)	69 (61)	92 (88)	77 (72)	96 (92)
aSPU	7 (6)	76 (69)	97 (96)	84 (80)	98 (97)	75 (67)	94 (90)
CLZ	9 (6)	32 (21)	62 (48)	87 (78)	99 (97)	49 (35)	71 (57)
CLX	7 (5)	12 (10)	18 (16)	69 (61)	92 (88)	77 (72)	96 (92)
BZ	5 (6)	29 (28)	61 (63)	67 (65)	90 (90)	21 (20)	34 (34)
CQ	5 (6)	29 (28)	61 (63)	67 (65)	90 (90)	21 (20)	34 (34)
SD	4 (6)	25 (28)	57 (63)	64 (64)	88 (88)	17 (20)	30 (34)

The size (type I error) or power outside and inside parentheses were calculated from asymptotics- and permutation-based p-values, respectively. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

2·4. *Results for various sample sizes*

Tables 13–15 present results when the sample sizes n_1 and n_2 vary from 25 to 200 and the sparsity parameter is $\beta = 0.1$, 0.2, and 0.5. The signal strength r was fixed at certain values. These results show that sample size has little impact on the selection of parameter γ in the sum-of-powers tests for fixed signal sparsity β, signal strength r, and dimension p.

To study the impact of p, Table 16 further presents the results when $\beta = 0.2$, $n_1 = n_2 = 50$, and the dimension p varies from 200 to 1000. We can see that when $p = 200$, the sum-of-powers test with $\gamma = 1$ is not as powerful as those with $\gamma = 2$ and 3; however, as p becomes larger, the sum-of-powers test with $\gamma = 1$ becomes more powerful. These observations are consistent with the theoretical result that the sum-of-powers test with $\gamma = 1$ is asymptotically most powerful when signals are dense.

Table 13. *Empirical test sizes and powers (%) based on 1,000 replicates of multivariate normal data with sample sizes* $n_1 = n_2 = 25$*, 50, 100, and 200, dimension* $p = 200$*, and the first-order autoregressive covariance matrix* $\Sigma_1 = \Sigma_2 = (0.6^{|i-j|})$*. The sparsity parameter is* $\beta = 0.1$*.*

Test	$n_1 = n_2 = 25$			$n_1 = n_2 = 50$			$n_1 = n_2 = 100$			$n_1 = n_2 = 200$		
	Size	$r = 0.04$	$r = 0.06$	Size	$r = 0.04$	$r = 0.06$	Size	$r = 0.04$	$r = 0.06$	Size	$r = 0.04$	$r = 0.06$
SPU(1)	6 (6)	78 (76)	92 (89)	5 (5)	78 (76)	92 (91)	5 (5)	77 (76)	91 (90)	4 (5)	77 (75)	91 (91)
SPU(2)	5 (5)	45 (44)	71 (69)	5 (5)	47 (47)	69 (69)	6 (5)	46 (44)	72 (68)	7 (6)	46 (44)	71 (69)
SPU(3)	5 (5)	69 (68)	87 (86)	4 (5)	71 (70)	88 (87)	5 (4)	72 (70)	87 (87)	5 (5)	73 (71)	89 (89)
SPU(4)	4 (5)	34 (39)	58 (60)	5 (5)	38 (39)	61 (61)	6 (6)	40 (37)	63 (61)	7 (5)	43 (39)	65 (61)
SPU(5)	3 (5)	44 (49)	63 (68)	4 (6)	47 (51)	70 (72)	5 (6)	50 (50)	72 (72)	5 (5)	53 (52)	73 (73)
SPU(6)	3 (6)	20 (29)	37 (48)	4 (5)	26 (31)	42 (48)	6 (6)	29 (29)	49 (48)	6 (4)	31 (30)	51 (49)
SPU(∞)	10 (6)	27 (14)	35 (22)	6 (5)	18 (16)	25 (23)	6 (6)	18 (17)	22 (23)	4 (5)	15 (16)	21 (23)
aSPU	8 (5)	65 (65)	84 (84)	6 (6)	66 (66)	85 (84)	5 (5)	65 (66)	83 (85)	6 (5)	63 (66)	83 (85)
CLZ	18 (6)	65 (35)	83 (56)	12 (6)	56 (37)	77 (58)	11 (5)	55 (36)	76 (59)	12 (6)	53 (37)	74 (59)
CLX	10 (5)	27 (15)	35 (22)	6 (5)	18 (16)	25 (22)	6 (6)	18 (18)	22 (22)	4 (5)	15 (16)	21 (24)
BZ	7 (6)	47 (45)	73 (70)	6 (6)	48 (47)	70 (70)	6 (5)	46 (44)	72 (69)	7 (6)	47 (44)	72 (69)
CQ	7 (6)	47 (45)	73 (70)	6 (6)	48 (47)	70 (70)	6 (5)	46 (44)	72 (69)	7 (6)	47 (44)	72 (69)
SD	4 (5)	42 (44)	66 (69)	4 (6)	43 (46)	67 (69)	5 (5)	43 (44)	68 (69)	6 (6)	44 (44)	68 (69)

The size (type I error) or power outside and inside parentheses were calculated from asymptotics- and permutation-based p-values, respectively. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

Table 14. *Empirical test sizes and powers (%) based on 1,000 replicates of multivariate normal data with sample sizes* $n_1 = n_2 = 25$*, 50, 100, and 200, dimension* $p = 200$*, and the first-order autoregressive covariance matrix* $\Sigma_1 = \Sigma_2 = (0.6^{|i-j|})$*. The sparsity parameter is* $\beta = 0.2$*.*

Test	$n_1 = n_2 = 25$			$n_1 = n_2 = 50$			$n_1 = n_2 = 100$			$n_1 = n_2 = 200$		
	Size	$r = 0.10$	$r = 0.15$	Size	$r = 0.10$	$r = 0.15$	Size	$r = 0.10$	$r = 0.15$	Size	$r = 0.10$	$r = 0.15$
SPU(1)	6 (6)	72 (70)	88 (85)	5 (5)	74 (71)	88 (87)	5 (5)	71 (70)	86 (85)	4 (5)	69 (68)	87 (86)
SPU(2)	5 (5)	71 (70)	94 (94)	5 (5)	70 (70)	92 (92)	6 (5)	70 (68)	93 (91)	7 (6)	73 (70)	94 (93)
SPU(3)	5 (5)	75 (75)	93 (93)	4 (5)	78 (77)	94 (94)	5 (4)	77 (76)	93 (93)	5 (5)	77 (76)	95 (94)
SPU(4)	4 (5)	59 (62)	88 (90)	5 (5)	63 (63)	89 (88)	6 (6)	67 (64)	90 (88)	7 (5)	69 (66)	92 (89)
SPU(5)	3 (5)	58 (63)	84 (87)	4 (6)	65 (67)	87 (88)	5 (6)	68 (69)	88 (88)	5 (5)	68 (66)	89 (89)
SPU(6)	3 (6)	40 (51)	67 (78)	4 (5)	48 (53)	76 (79)	6 (6)	54 (54)	80 (80)	6 (4)	58 (55)	82 (81)
SPU(∞)	10 (6)	38 (25)	54 (37)	6 (5)	31 (27)	47 (41)	6 (6)	28 (27)	41 (42)	4 (5)	25 (27)	39 (42)
aSPU	8 (5)	70 (70)	93 (93)	6 (6)	72 (73)	92 (92)	5 (5)	74 (74)	92 (91)	6 (5)	73 (72)	93 (92)
CLZ	18 (6)	85 (57)	97 (87)	12 (6)	79 (60)	95 (86)	11 (5)	77 (61)	94 (87)	12 (6)	77 (63)	96 (88)
CLX	10 (5)	38 (25)	54 (37)	6 (5)	31 (27)	47 (41)	6 (6)	28 (27)	41 (42)	4 (5)	25 (26)	39 (42)
BZ	7 (6)	73 (71)	94 (94)	6 (6)	71 (70)	93 (92)	6 (5)	71 (68)	93 (92)	7 (6)	73 (71)	94 (93)
CQ	7 (6)	73 (71)	94 (94)	6 (6)	71 (70)	93 (92)	6 (5)	71 (68)	93 (92)	7 (6)	73 (71)	94 (93)
SD	4 (5)	67 (69)	92 (93)	4 (6)	67 (70)	91 (91)	5 (5)	67 (69)	91 (92)	6 (6)	69 (71)	92 (93)

The size (type I error) or power outside and inside parentheses were calculated from asymptotics- and permutation-based p-values, respectively. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

Table 15. *Empirical test sizes and powers (%) based on 1,000 replicates of multivariate normal data with sample sizes $n_1 = n_2 = 25$, 50, 100, and 200, dimension $p = 200$, and autoregressive covariance matrix $\Sigma_1 = \Sigma_2 = (0.6^{|i-j|})$. The sparsity parameter is $\beta = 0.5$.*

Test	$n_1 = n_2 = 25$			$n_1 = n_2 = 50$			$n_1 = n_2 = 100$			$n_1 = n_2 = 200$		
	Size	$r = 0.4$	$r = 0.6$	Size	$r = 0.4$	$r = 0.6$	Size	$r = 0.4$	$r = 0.6$	Size	$r = 0.4$	$r = 0.6$
SPU(1)	6 (6)	19 (17)	26 (23)	5 (5)	19 (17)	24 (23)	5 (5)	17 (16)	23 (22)	4 (5)	17 (16)	22 (22)
SPU(2)	5 (5)	56 (55)	84 (84)	5 (5)	58 (57)	85 (84)	6 (5)	59 (56)	86 (84)	7 (6)	62 (60)	87 (86)
SPU(3)	5 (5)	52 (52)	80 (81)	4 (5)	56 (54)	83 (82)	5 (4)	55 (55)	82 (81)	5 (5)	53 (52)	82 (82)
SPU(4)	4 (5)	65 (69)	94 (95)	5 (5)	72 (71)	95 (95)	6 (6)	73 (71)	96 (95)	7 (5)	78 (75)	96 (96)
SPU(5)	3 (5)	59 (65)	90 (93)	4 (6)	68 (68)	92 (93)	5 (6)	68 (68)	92 (92)	5 (5)	70 (70)	94 (94)
SPU(6)	3 (6)	60 (69)	91 (95)	4 (5)	69 (71)	93 (94)	6 (6)	72 (72)	95 (95)	6 (4)	75 (74)	96 (96)
SPU(∞)	10 (6)	59 (46)	84 (72)	6 (5)	57 (52)	86 (81)	6 (6)	54 (54)	82 (82)	4 (5)	55 (56)	82 (83)
aSPU	8 (5)	66 (62)	92 (93)	6 (6)	69 (66)	94 (93)	5 (5)	70 (66)	95 (92)	6 (5)	74 (68)	95 (93)
CLZ	18 (6)	86 (65)	99 (94)	12 (6)	84 (68)	98 (94)	11 (5)	82 (69)	97 (93)	12 (6)	82 (71)	98 (95)
CLX	10 (5)	59 (46)	84 (72)	6 (5)	57 (52)	86 (81)	6 (6)	54 (54)	82 (82)	4 (5)	55 (57)	82 (83)
BZ	7 (6)	58 (55)	86 (84)	6 (6)	59 (57)	86 (84)	6 (5)	60 (56)	87 (84)	7 (6)	63 (60)	87 (86)
CQ	7 (6)	58 (55)	86 (84)	6 (6)	59 (57)	86 (84)	6 (5)	60 (56)	87 (84)	7 (6)	63 (60)	87 (86)
SD	4 (5)	51 (54)	80 (82)	4 (6)	55 (58)	84 (85)	5 (5)	55 (57)	84 (84)	6 (6)	59 (59)	85 (86)

The size (type I error) or power outside and inside parentheses were calculated from asymptotics- and permutation-based p-values, respectively. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

Table 16. *Empirical test sizes and powers (%) based on 1,000 replicates of multivariate normal data with sample sizes $n_1 = n_2 = 50$, sparsity parameter $\beta = 0.2$, dimension $p = 200$, 500, and 1000, and autoregressive covariance matrix $\Sigma_1 = \Sigma_2 = (0.6^{|i-j|})$.*

Test	$p = 200$			$p = 500$			$p = 1000$		
	Size	$r = 0.1$	$r = 0.15$	Size	$r = 0.04$	$r = 0.06$	Size	$r = 0.04$	$r = 0.06$
SPU(1)	5 (5)	74 (71)	88 (87)	7 (6)	63 (60)	80 (78)	6 (5)	85 (81)	95 (94)
SPU(2)	5 (5)	70 (70)	92 (92)	6 (6)	46 (45)	70 (69)	6 (6)	62 (61)	88 (88)
SPU(3)	4 (5)	78 (77)	94 (94)	5 (5)	60 (60)	82 (81)	5 (5)	84 (84)	96 (95)
SPU(4)	5 (5)	63 (63)	89 (88)	5 (6)	36 (38)	61 (63)	4 (5)	52 (57)	81 (84)
SPU(5)	4 (6)	65 (67)	87 (88)	4 (5)	44 (46)	65 (68)	3 (4)	65 (69)	88 (89)
SPU(6)	4 (5)	48 (53)	76 (79)	4 (6)	24 (28)	42 (48)	3 (5)	35 (42)	63 (70)
SPU(∞)	6 (5)	31 (27)	47 (41)	9 (6)	17 (13)	24 (18)	8 (4)	21 (13)	29 (20)
aSPU	6 (6)	72 (73)	92 (92)	6 (5)	56 (56)	77 (78)	6 (5)	78 (79)	95 (96)
CLZ	12 (6)	79 (60)	95 (86)	18 (6)	62 (32)	81 (57)	19 (6)	78 (45)	93 (76)
CLX	6 (5)	31 (27)	47 (41)	9 (6)	17 (14)	24 (18)	8 (4)	21 (14)	29 (21)
BZ	6 (6)	71 (70)	93 (92)	7 (6)	47 (45)	72 (69)	6 (6)	64 (62)	89 (88)
CQ	6 (6)	71 (70)	93 (92)	7 (6)	47 (45)	72 (69)	6 (6)	64 (62)	89 (88)
SD	4 (6)	67 (70)	91 (91)	6 (6)	42 (45)	65 (68)	4 (6)	56 (62)	85 (88)

The size (type I error) or power outside and inside parentheses were calculated from asymptotics- and permutation-based p-values, respectively. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

2·5. *Results for* $\Sigma_1 \neq \Sigma_2$

This section assumes that $\Sigma_1 \neq \Sigma_2$. Since Bai & Saranadasa (1996) and Srivastava & Du (2008) did not discuss the case of unequal covariance matrices for their test statistics, this section will not consider their methods.

The procedure introduced in the main text similarly applies to calculate the asymptotics-based p-values of the class of sum-of-powers tests and the tests proposed by Chen & Qin (2010), Cai et al. (2014), and Chen et al. (2014); however, the permutation method is invalid to calculate p-values when $\Sigma_1 \neq \Sigma_2$, because the two groups no longer share a common distribution under the null hypothesis. Alternatively, we used the parametric bootstrap resampling method to calculate p-values, and compared these with the asymptotics-based p-values. The parametric resampling has also been used by Chen et al. (2014) to deal with inflated type I error of their thresholding test statistic. Specifically, the bootstrap samples under the null hypothesis were drawn from $N(0, \hat{\Sigma}_1)$ and $N(0, \hat{\Sigma}_2)$, where $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$ are consistent estimates of Σ_1 and Σ_2, respectively. As before, we used the banding approach (Bickel & Levina, 2008) to obtain the consistent estimates. Since the banding estimate is not necessarily positive-definite, we followed the suggesting in Remark 3 of Cai et al. (2010) to diagonalize the banding estimate and then replace negative eigenvalues by a small positive constant, say 0.001.

Table 17 presents the results. The adaptive sum-of-powers test still controls type I errors well and has high powers under most simulation settings for $\Sigma_1 \neq \Sigma_2$.

Table 17. *Empirical test sizes and powers (%) based on 1,000 replicates of multivariate normal data with sample sizes $n_1 = n_2 = 50$, dimension $p = 200$, and autoregressive covariance matrix with order 1. The covariance matrices of the two groups are $\Sigma_1 = (0.2^{|i-j|})$ and $\Sigma_2 = (0.8^{|i-j|})$.*

Equal variances ($\mathbf{D} = \mathbf{I}_p$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.04$	$r = 0.06$	$r = 0.6$	$r = 0.9$	$r = 1.5$	$r = 2.5$
SPU(1)	5 (5)	70 (71)	86 (86)	11 (11)	12 (13)	6 (6)	6 (6)
SPU(2)	6 (3)	45 (40)	69 (63)	49 (42)	76 (69)	14 (10)	22 (16)
SPU(3)	4 (3)	67 (64)	82 (82)	46 (44)	73 (71)	14 (13)	34 (33)
SPU(4)	5 (3)	38 (32)	60 (52)	72 (66)	96 (93)	41 (33)	78 (72)
SPU(5)	4 (3)	47 (42)	68 (62)	68 (65)	92 (91)	44 (42)	81 (80)
SPU(6)	4 (3)	28 (23)	43 (38)	74 (69)	96 (95)	56 (53)	89 (88)
SPU(∞)	7 (5)	20 (15)	26 (21)	66 (60)	90 (87)	65 (60)	92 (90)
aSPU	6 (4)	55 (58)	77 (79)	70 (61)	94 (89)	57 (50)	90 (86)
CLZ	12 (4)	56 (33)	77 (54)	84 (65)	98 (93)	43 (21)	66 (43)
CLX	7 (5)	20 (15)	26 (21)	66 (60)	90 (87)	65 (60)	92 (90)
CQ	6 (4)	46 (41)	71 (64)	51 (43)	78 (71)	15 (10)	23 (18)

Unequal variances ($\mathbf{D} = \mathrm{diag}(d_{ii})$ with $d_{ii} \sim U(0.1, 10)$):

Test	Size	$\beta = 0.1$		$\beta = 0.6$		$\beta = 0.9$	
		$r = 0.15$	$r = 0.25$	$r = 2.5$	$r = 3.5$	$r = 3$	$r = 4.5$
SPU(1)	4 (5)	62 (63)	83 (83)	10 (10)	11 (11)	5 (5)	5 (5)
SPU(2)	5 (4)	32 (28)	55 (48)	37 (32)	56 (50)	8 (6)	9 (7)
SPU(3)	4 (3)	39 (37)	63 (61)	31 (29)	49 (46)	4 (3)	4 (4)
SPU(4)	5 (3)	19 (15)	31 (27)	42 (36)	66 (58)	6 (4)	8 (6)
SPU(5)	4 (4)	18 (16)	29 (28)	35 (33)	56 (55)	4 (4)	5 (5)
SPU(6)	4 (4)	10 (9)	17 (16)	33 (31)	55 (53)	4 (4)	6 (5)
SPU(∞)	7 (5)	54 (47)	88 (83)	78 (74)	95 (93)	76 (72)	95 (93)
aSPU	6 (4)	58 (58)	88 (87)	73 (62)	92 (87)	67 (59)	91 (88)
CLZ	12 (4)	87 (70)	99 (96)	81 (64)	95 (85)	51 (25)	70 (47)
CLX	7 (5)	54 (47)	88 (83)	78 (74)	95 (93)	76 (72)	95 (93)
CQ	6 (4)	33 (29)	56 (49)	38 (33)	57 (50)	8 (6)	10 (7)

The size (type I error) or power outside and inside parentheses were calculated from asymptotics- and resampling-based p-values, respectively. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

3. Simulations: single nucleotide polymorphisms data

To illustrate the performance of the proposed method and the asymptotic approximations for non-Gaussian data, we next applied the various methods to test association between a binary trait, such as disease status, and a set of single nucleotide polymorphisms in genome-wide association study. Suppose there are n subjects in total. For the ith subject ($i = 1, \ldots, n$), we assigned it to two groups according to its binary trait $Y_i = 0$ or 1. Also, a p-dimensional predictor $X_i = (X_{i1}, \ldots, X_{ip})^{\mathrm{T}}$ is available, where the genotype score $X_{ij} = 0$, 1, or 2 is the count of the minor allele at the jth single nucleotide polymorphism. We simulated genotypes for each subject following Wang & Elston (2007). First, a latent p-dimensional vector $Z = (Z_1, \ldots, Z_p)^{\mathrm{T}}$ was generated from a multivariate normal distribution $N(0, R)$, where $R = (r_{ij})$ is an autoregressive correlation matrix with order 1 and $r_{ij} = 0.2^{|i-j|}$. Second, we dichotomized the latent vector Z to indicate a haplotype with some specified minor allele frequencies, which were uniformly sampled between 0.05 to 0.5. Third, we used the above two steps to independently generate two haplotypes, which were combined to form the genotype X_i for subject i.

For each set-up, we simulated 1,000 independent replicates. Each replicate consisted of 200 cases and 200 controls; for each subject, there were $p = 1,000$ single nucleotide polymorphisms, 5% of which were causal with odds ratios generated from $U(1, a)$. Note that $a = 1$ indicates that none of the single nucleotide polymorphisms was causal, which was used to estimate type I errors for a null case. For non-null cases, we considered the values of a at 1.1, 1.2, 1.3, 1.4, and 1.5 respectively. Note that it is realistic to assume 5% associated single nucleotide polymorphisms in light of the current knowledge on the genetic architecture for complex diseases (The International Schizophrenia Consortium, 2009).

Table 18 presents empirical type I errors and power calculated by both asymptotics-based and permutation-based methods.The type I errors of the sum-of-squares tests based on asymptotics were very close to those based on permutations.The asymptotic approximations gave a little inflated type I errors for the sum-of-powers test with $\gamma = 1$. This shows the good performance of the developed theoretical results.

Since permutation-based p-values could control the type I error satisfactorily for all tests, we used them to compare the power of the tests. Again the sum-of-squares type tests, including the sum-of-powers test with $\gamma = 2$, performed similarly well, but not the best, while the supremum type test was low powered, given the non-sparse alternatives. However, since the non-zero signals were not dense enough, if not counting the adaptive sum-of-powers test, it is notable that the sum-of-powers tests with $\gamma = 3$, 4, and 5 were often more powerful than the sum-of-squares type tests. For example, at $a = 1.4$, the power of the sum-of-powers test with $\gamma = 4$ was 0.619, much higher than about 0.57 of the sum-of-squares type tests, 0.555 of the thresholding test and 0.275 of the supremum type test. Note that, presumably due to weak signals, thresholding adopted in Chen et al. (2014) was not as effective as weighting used in the sum-of-powers test with $\gamma = 4$. Most importantly, the adaptive sum-of-powers test could retain the highest power across all the settings. We have also used simple covariance matrix to estimate Σ and similar performance as in Section 2·3 can be observed; Table 19 presents the results.

Table 18. *Empirical test sizes and powers (%) based on 1,000 replicates of simulated single nucleotide polymorphisms data. There were $n_1 = n_2 = 200$ cases and controls with $p = 1,000$ single nucleotide polymorphisms; 5% of the single nucleotide polymorphisms were causal. The association odds ratios were randomly drawn from $U(1, a)$, where a varied from 1 to 1.5.*

Test	$a = 1$ (Size)	$a = 1.2$	$a = 1.3$	$a = 1.4$	$a = 1.5$
SPU(1)	9 (5)	23 (13)	31 (26)	45 (36)	58 (51)
SPU(2)	6 (5)	16 (15)	34 (32)	59 (57)	82 (83)
SPU(3)	5 (4)	20 (14)	37 (37)	64 (60)	88 (86)
SPU(4)	5 (5)	15 (16)	33 (35)	65 (62)	91 (88)
SPU(5)	4 (4)	14 (12)	31 (32)	61 (60)	88 (88)
SPU(6)	5 (4)	13 (12)	28 (28)	57 (54)	87 (84)
SPU(∞)	5 (5)	8 (8)	14 (14)	29 (28)	55 (57)
aSPU	7 (6)	21 (15)	41 (39)	71 (66)	94 (90)
CLZ	7 (5)	15 (13)	30 (26)	62 (56)	90 (84)
CLX	5 (5)	8 (8)	14 (14)	29 (28)	55 (57)
BZ	6 (5)	15 (16)	33 (32)	58 (57)	82 (82)
CQ	6 (5)	15 (16)	33 (32)	58 (57)	82 (82)
SD	5 (5)	12 (14)	29 (32)	54 (56)	80 (82)

The size (type I error) or power outside and inside parentheses were calculated from asymptotics- and permutation-based p-values, respectively. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

Table 19. *Empirical test sizes and powers (%) based on 1,000 replicates of simulated single nucleotide polymorphisms data. There were $n_1 = n_2 = 200$ cases and controls with $p = 1,000$ single nucleotide polymorphisms; 5% of the single nucleotide polymorphisms were causal. The association odds ratios were randomly drawn from $U(1, a)$, where a varied from 1 to 1.5. The covariance matrix was estimated by the sample covariance matrix.*

Test	$a = 1$ (Size)	$a = 1.2$	$a = 1.3$	$a = 1.4$	$a = 1.5$
SPU(1)	4 (5)	16 (13)	24 (26)	38 (36)	50 (51)
SPU(2)	0 (5)	1 (15)	7 (32)	18 (57)	41 (83)
SPU(3)	4 (4)	17 (14)	32 (37)	61 (60)	86 (86)
SPU(4)	1 (5)	5 (16)	16 (35)	39 (62)	75 (88)
SPU(5)	4 (4)	13 (12)	30 (32)	60 (60)	88 (88)
SPU(6)	3 (4)	8 (12)	19 (28)	45 (54)	81 (84)
SPU(∞)	5 (5)	8 (8)	14 (14)	29 (28)	55 (57)
aSPU	4 (6)	12 (15)	27 (39)	55 (66)	87 (90)
CLZ	7 (5)	15 (13)	30 (26)	62 (56)	90 (84)
CLX	5 (5)	8 (8)	14 (14)	29 (28)	55 (57)
BS	6 (5)	15 (16)	33 (32)	58 (57)	82 (82)
CQ	6 (5)	15 (16)	33 (32)	58 (57)	82 (82)
SD	5 (5)	12 (14)	29 (32)	54 (56)	80 (82)

The size (type I error) or power outside and inside parentheses were calculated from asymptotics- and permutation-based p-values, respectively. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

4. REAL DATA ANALYSIS

Table 20 presents the complete results of all 22 autosomes in the data collected by The Wellcome Trust Case Control Consortium (2007) for bipolar disorder.

Table 20. *The 100 × p-values of various tests for the Wellcome Trust Case Control Consortium bipolar disease data.*

Test	Chromosome (number of single nucleotide polymorphisms)										
	1 (3340)	2 (3194)	3 (2710)	4 (2617)	5 (2590)	6 (2424)	7 (2270)	8 (2050)	9 (1986)	10 (2224)	11 (2039)
SPU(1)	63.6 (64.3)	17.0 (17.8)	0.7 (0.7)	0.2 (0.2)	38.6 (38.4)	3.1 (3.5)	6.9 (7.3)	67.5 (68.1)	5.7 (6.1)	87.5 (87.6)	77.3 (77.5)
SPU(2)	<0.1 (<0.1)	<0.1 (<0.1)	<0.1 (<0.1)	1.5 (1.7)	0.2 (0.2)	<0.1 (<0.1)	<0.1 (<0.1)	12.9 (13.3)	<0.1 (<0.1)	16.2 (16.6)	0.7 (0.8)
SPU(3)	73.8 (74.5)	0.6 (0.7)	<0.1 (<0.1)	3.1 (3.1)	96.3 (96.3)	4.2 (4.7)	0.2 (0.3)	55.2 (55.9)	28.1 (28.4)	33.1 (32.6)	78.8 (79.0)
SPU(4)	<0.1 (<0.1)	<0.1 (<0.1)	<0.1 (<0.1)	2.0 (2.7)	0.1 (0.4)	<0.1 (<0.1)	<0.1 (<0.1)	15.3 (15.0)	<0.1 (<0.1)	10.9 (11.3)	8.5 (9.3)
SPU(5)	74.2 (73.2)	0.2 (0.3)	0.4 (0.5)	37.5 (36.1)	57.3 (55.9)	32.1 (30.8)	<0.1 (0.2)	52.1 (49.7)	66.2 (64.5)	8.0 (7.5)	81.1 (80.6)
SPU(6)	<0.1 (<0.1)	<0.1 (0.1)	<0.1 (0.3)	2.7 (4.1)	1.0 (2.5)	0.5 (1.5)	<0.1 (0.4)	19.4 (16.7)	<0.1 (0.2)	10.4 (10.0)	38.9 (34.4)
SPU(∞)	13.1 (11.8)	4.5 (4.3)	70.4 (72.9)	12.1 (11.9)	4.8 (4.5)	31.7 (31.9)	19.3 (18.8)	45.6 (47.1)	4.8 (4.1)	10.0 (9.1)	24.4 (23.9)
aSPU	<0.1 (<0.1)	<0.1 (<0.1)	<0.1 (<0.1)	1.0 (1.2)	0.6 (1.0)	<0.1 (<0.1)	<0.1 (<0.1)	50.2 (43.3)	<0.1 (<0.1)	27.1 (28.3)	3.9 (4.0)
CLZ	<0.1 (<0.1)	<0.1 (0.3)	0.3 (0.7)	9.6 (10.2)	0.8 (1.5)	<0.1 (<0.1)	<0.1 (0.2)	23.3 (19.2)	<0.1 (0.3)	12.9 (12.7)	12.6 (12.9)
CLX	13.1 (11.8)	4.5 (4.3)	70.4 (72.9)	12.1 (11.9)	4.8 (4.5)	31.7 (31.9)	19.3 (18.8)	45.6 (47.1)	4.8 (4.1)	10.0 (9.1)	24.4 (23.9)
BZ	<0.1 (<0.1)	<0.1 (<0.1)	<0.1 (<0.1)	1.5 (1.7)	0.2 (0.2)	<0.1 (<0.1)	<0.1 (<0.1)	12.8 (13.3)	<0.1 (<0.1)	16.2 (16.6)	0.7 (0.8)
CQ	<0.1 (<0.1)	<0.1 (<0.1)	<0.1 (<0.1)	1.5 (1.7)	0.2 (0.2)	<0.1 (<0.1)	<0.1 (<0.1)	12.8 (13.3)	<0.1 (<0.1)	16.0 (16.6)	0.7 (0.8)
SD	<0.1 (<0.1)	<0.1 (<0.1)	<0.1 (<0.1)	<0.1 (1.0)	<0.1 (0.4)	<0.1 (<0.1)	<0.1 (<0.1)	<0.1 (10.0)	<0.1 (<0.1)	<0.1 (11.6)	<0.1 (2.5)

Test	Chromosome (number of single nucleotide polymorphisms)										
	12 (2135)	13 (1592)	14 (1434)	15 (1428)	16 (1476)	17 (1405)	18 (1421)	19 (1026)	20 (1257)	21 (698)	22 (776)
SPU(1)	3.3 (3.3)	3.7 (3.7)	61.1 (62.2)	82.5 (82.5)	60.1 (60.2)	80.6 (81.0)	33.0 (32.3)	17.9 (18.1)	89.2 (89.2)	19.5 (19.9)	32.2 (32.0)
SPU(2)	<0.1 (<0.1)	2.7 (2.9)	<0.1 (<0.1)	<0.1 (<0.1)	<0.1 (<0.1)	15.0 (14.5)	28.9 (28.7)	<0.1 (0.1)	<0.1 (<0.1)	34.1 (33.0)	0.6 (0.8)
SPU(3)	3.6 (3.6)	12.9 (12.6)	2.1 (2.1)	100.0 (100.0)	34.9 (34.4)	98.2 (98.2)	18.7 (17.4)	3.4 (3.6)	94.6 (94.7)	41.5 (41.0)	67.9 (67.5)
SPU(4)	1.0 (1.5)	<0.1 (0.2)	<0.1 (<0.1)	<0.1 (<0.1)	<0.1 (<0.1)	23.2 (21.7)	35.3 (33.1)	0.1 (0.4)	1.2 (2.0)	16.5 (15.6)	2.8 (3.6)
SPU(5)	15.6 (14.7)	39.4 (37.1)	0.2 (0.7)	32.9 (30.5)	2.5 (3.3)	97.6 (97.6)	25.9 (23.4)	0.4 (1.0)	80.6 (79.6)	72.4 (69.1)	75.5 (73.0)
SPU(6)	10.0 (10.3)	<0.1 (0.4)	<0.1 (0.2)	<0.1 (0.2)	<0.1 (<0.1)	21.9 (17.9)	44.8 (38.6)	0.3 (1.5)	18.1 (15.8)	9.7 (9.2)	13.0 (11.4)
SPU(∞)	61.4 (63.2)	8.8 (8.0)	11.2 (10.3)	14.3 (13.8)	6.4 (5.8)	18.7 (18.0)	0.5 (0.4)	9.9 (9.4)	78.3 (81.0)	9.8 (9.0)	66.2 (68.1)
aSPU	0.2 (0.4)	<0.1 (1.3)	<0.1 (<0.1)	<0.1 (<0.1)	<0.1 (<0.1)	46.3 (45.9)	1.4 (1.9)	0.4 (0.7)	0.2 (0.5)	26.6 (31.1)	3.4 (3.7)
CLZ	2.5 (3.4)	0.2 (0.5)	<0.1 (<0.1)	<0.1 (0.2)	<0.1 (<0.1)	47.0 (30.1)	5.6 (6.6)	0.3 (0.7)	2.1 (2.9)	0.9 (1.4)	3.2 (4.2)
CLX	61.4 (63.2)	8.8 (8.0)	11.2 (10.3)	14.3 (13.8)	6.4 (5.8)	18.7 (18.0)	0.5 (0.4)	9.9 (9.4)	78.3 (81.0)	9.8 (9.0)	66.2 (68.1)
BZ	<0.1 (<0.1)	2.6 (2.9)	<0.1 (<0.1)	<0.1 (<0.1)	<0.1 (<0.1)	14.9 (14.5)	28.8 (28.7)	<0.1 (0.1)	<0.1 (<0.1)	34.1 (33.0)	0.6 (0.8)
CQ	<0.1 (<0.1)	2.7 (2.9)	<0.1 (<0.1)	<0.1 (<0.1)	<0.1 (<0.1)	14.9 (14.5)	29.0 (28.7)	<0.1 (0.1)	<0.1 (<0.1)	34.1 (33.0)	0.6 (0.8)
SD	<0.1 (0.2)	<0.1 (11.4)	<0.1 (<0.1)	<0.1 (<0.1)	<0.1 (<0.1)	<0.1 (8.2)	<0.1 (9.7)	<0.1 (<0.1)	<0.1 (0.1)	<0.1 (33.1)	<0.1 (0.5)

The p-values outside parentheses were calculated from asymptotic distributions; those inside parentheses were based on permutations. SPU and aSPU: the sum-of-powers tests and the adaptive test; CLZ: Chen et al. (2014); CLX: Cai et al. (2014); BS: Bai & Saranadasa (1996); CQ: Chen & Qin (2010); SD: Srivastava & Du (2008).

5. R PACKAGE AND CODE

An R package `highmean` implementing the tests studied in this paper is available on CRAN. Below is R code for Table 1 of the main file.

```
library(snowfall)
# the snowfall package is used for parallel computing on a multi-CPU computer
sfInit(parallel = TRUE, cpus = 10, type = "SOCK")

begin <- proc.time()

#install.packages("highmean")
sfLibrary(highmean) # package highmean version 2.0
sfLibrary(mnormt)
sfLibrary(mvtnorm)
sfLibrary(CVTuningCov)

pow <- c(1:6, Inf) # the candidate parameters (gamma) for sum-of-powers tests

iter <- 1000 # number of Monte Carlo iterations
n1 <- 50 # sample size of group 1
n2 <- 50 # sample size of group 2
p <- 200 # dimension
sigma <- 0.6^abs(outer(1:p, 1:p, "-")) # first-order autoregressive covariance

beta <- 0.1 # the signal sparsity parameter ranging between 0 and 1
r <- 0.06 # the signal strength parameter (0, 0.02, ..., 0.08 in Table 1);
          # we take r = 0.06 as an example

m <- floor(p^(1 - beta)) # number of non-zero signals in mu2
mu1 <- rep(0, p) # mean vector of group 1
mu2 <- rep(0, p)
if(m != 0){
diff.mean <- rep(sqrt(2*r*log(p)*(1/n1 + 1/n2)), m)
set.seed(1234)
m.entries <- sample(p)[1:m]
mu2[m.entries] <- diff.mean # mean vector of group 2
}

set.seed(12345)
iter.seeds <- sample(1000000, iter)

# asymptotics-based
sim.asym <- function(n1, n2, p, mu1, mu2, sigma, pow, seed){
set.seed(seed)
sam1 <- rmnorm(n = n1, mean = mu1, sigma)
sam2 <- rmnorm(n = n2, mean = mu2, sigma)
set.seed(seed)
n <- n1 + n2 - 2
sam.cov <- ((n1 - 1)*cov(sam1) + (n2 - 1)*cov(sam2))/n
sam1.ctr <- sam1 - matrix(rep(colMeans(sam1), n1), n1, p, byrow = TRUE)
sam2.ctr <- sam2 - matrix(rep(colMeans(sam2), n2), n2, p, byrow = TRUE)
out <- regular.CV(rbind(sam1.ctr, sam2.ctr),
k.grid = seq(from = 0, to = 50, by = 1), method = "banding",
fold = 5, norm = "F")
cov.est <- sam.cov*banding(p, out$CV.k[1])
```

```
pval.spu <- apval_aSPU(sam1, sam2, pow = pow, cov.est = cov.est)$pval
pval.bai1996 <- apval_Bai1996(sam1, sam2)$pval
pval.cai2014 <- apval_Cai2014(sam1, sam2)$pval
pval.chen2010 <- apval_Chen2010(sam1, sam2)$pval
pval.chen2014 <- apval_Chen2014(sam1, sam2)$pval
pval.sri2008 <- apval_Sri2008(sam1, sam2)$pval
pval <- c(pval.spu, pval.chen2014, pval.cai2014, pval.bai1996,
pval.chen2010, pval.sri2008)
return(pval)
}

sfExport("n1", "n2", "p", "mu1", "mu2", "sigma", "pow", "iter.seeds")
sfExport("sim.asym")

pval.asym <- sfLapply(1:iter, function(i){
sim.asym(n1, n2, p, mu1, mu2, sigma, pow, seed = iter.seeds[i])
})
pval.asym <- unlist(pval.asym)
test.names <- unique(names(pval.asym))
powers.asym <- numeric(length(test.names))
for(i in 1:length(test.names)){
pval.temp <- pval.asym[names(pval.asym) == test.names[i]]
powers.asym[i] <- sum(pval.temp <= 0.05)/iter
}
powers.asym <- as.matrix(powers.asym)
rownames(powers.asym) <- test.names
round(100*powers.asym) # powers outside the parentheses when r = 0.06 in Table 1


# permutation-based
perm.iter <- 1000

sim.perm <- function(n1, n2, p, mu1, mu2, sigma, pow, perm.iter, seed){
set.seed(seed)
sam1 <- rmnorm(n = n1, mean = mu1, sigma)
sam2 <- rmnorm(n = n2, mean = mu2, sigma)
set.seed(seed)
seeds <- sample(1000000, perm.iter)
pval.spu <- epval_aSPU(sam1, sam2, pow = pow, n.iter = perm.iter, seeds = seeds)$pval
pval.bai1996 <- epval_Bai1996(sam1, sam2, n.iter = perm.iter, seeds = seeds)$pval
pval.cai2014 <- epval_Cai2014(sam1, sam2, n.iter = perm.iter, seeds = seeds)$pval
pval.chen2010 <- epval_Chen2010(sam1, sam2, n.iter = perm.iter, seeds = seeds)$pval
pval.chen2014 <- epval_Chen2014(sam1, sam2, n.iter = perm.iter, seeds = seeds)$pval
pval.sri2008 <- epval_Sri2008(sam1, sam2, n.iter = perm.iter, seeds = seeds)$pval
pval <- c(pval.spu, pval.chen2014, pval.cai2014, pval.bai1996,
pval.chen2010, pval.sri2008)
return(pval)
}

sfExport("n1", "n2", "p", "mu1", "mu2", "sigma", "pow", "perm.iter", "iter.seeds")
sfExport("sim.perm")

pval.perm <- sfLapply(1:iter, function(i){
sim.perm(n1, n2, p, mu1, mu2, sigma, pow, perm.iter, seed = iter.seeds[i])
})
pval.perm <- unlist(pval.perm)
```

```
test.names <- unique(names(pval.perm))
powers.perm <- numeric(length(test.names))
for(i in 1:length(test.names)){
pval.temp <- pval.perm[names(pval.perm) == test.names[i]]
powers.perm[i] <- sum(pval.temp <= 0.05)/iter
}
powers.perm <- as.matrix(powers.perm)
rownames(powers.perm) <- test.names
round(100*powers.perm) # powers inside the parentheses when r = 0.06 in Table 1

end <- proc.time()
end - begin

sfStop()
```

REFERENCES

BAI, Z. D. & SARANADASA, H. (1996). Effect of high dimension: by an example of a two sample problem. *Statist. Sinica* **6**, 311–329.

BICKEL, P. J. & LEVINA, E. (2008). Regularized estimation of large covariance matrices. *Ann. Statist.* **36**, 199–227.

CAI, T. T., LIU, W. & XIA, Y. (2014). Two-sample test of high dimensional means under dependence. *J. R. Statist. Soc. B* **76**, 349–372.

CAI, T. T., ZHANG, C.-H. & ZHOU, H. H. (2010). Optimal rates of convergence for covariance matrix estimation. *Ann. Statist.* **38**, 2118–2144.

CHEN, S. X., LI, J. & ZHONG, P.-S. (2014). Two-sample tests for high dimensional means with thresholding and data transformation .

CHEN, S. X. & QIN, Y.-L. (2010). A two-sample test for high-dimensional data with applications to gene-set testing. *Ann. Statist.* **38**, 808–835.

GUYON, X. (1995). *Random Fields on a Network: Modeling, Statistics, and Applications*. New York, NY: Springer-Verlag.

HSING, T. (1995). A note on the asymptotic independence of the sum and maximum of strongly mixing stationary random variables. *Ann. Prob.* **23**, 938–947.

IBRAGIMOV, I. A. & LINNIK, Y. V. (1971). *Independent and Stationary Sequences of Random Variables*. The Netherlands: Wolters-Noordhoff.

KIM, T. Y. (1994). Moment bounds for non-stationary dependent sequences. *J. Appl. Probab.* **31**, 731–742.

ROTHMAN, A. J. (2012). Positive definite estimators of large covariance matrices. *Biometrika* **99**, 733–740.

SRIVASTAVA, M. S. & DU, M. (2008). A test for the mean vector with fewer observations than the dimension. *J. Multivar. Anal.* **99**, 386–402.

THE INTERNATIONAL SCHIZOPHRENIA CONSORTIUM (2009). Common polygenic variation contributes to risk of schizophrenia and bipolar disorder. *Nature* **460**, 748–752.

THE WELLCOME TRUST CASE CONTROL CONSORTIUM (2007). Genome-wide association study of 14,000 cases of seven common diseases and 3,000 shared controls. *Nature* **447**, 661–678.

WANG, T. & ELSTON, R. C. (2007). Improved power by use of a weighted score test for linkage disequilibrium mapping. *Am. J. Hum. Genet.* **80**, 353–360.

XUE, L., MA, S. & ZOU, H. (2012). Positive-definite ℓ_1-penalized estimation of large covariance matrices. *J. Am. Statist. Assoc.* **107**, 1480–1491.

ZHONG, P.-S., CHEN, S. X. & XU, M. (2013). Tests alternative to higher criticism for high-dimensional means under sparsity and column-wise dependence. *Ann. Statist.* **41**, 2820–2851.