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Supplementary Material for "Identification and Estimation of Causal Effects with Outcomes Truncated by Death"

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SUMMARY

In this Supplementary Material we provide proofs of theorems and propositions in the paper, and alternative causal diagrams that are compatible with the structural equations (4) in the paper, but allow for certain dependencies between error terms. Some additional equations and figures are created within this document, which we label as equation (S1), Figure S1 and so forth, to distinguish them from those in the main text.

1. PROOF OF THEOREMS 1–3

By definition, $\Delta_{\mathrm{LL}} = E\{Y(1) \mid G = \mathrm{LL}\} - E\{Y(0) \mid G = \mathrm{LL}\}$, where $E\{Y(z) \mid G = \mathrm{LL}\}$ is given in (1). It then suffices to identify $\pi_{\mathrm{LL}|W}$ and $\mu_{z,\mathrm{LL},W}$ for identification of Δ_{LL}.

Identification of $\pi_{\mathrm{LL}|W}$: under the assumptions of Theorem 1 or 3, $\pi_{\mathrm{LL}|W}$ can be identified from (2), whereas under the assumptions of Theorem 2, $\pi_{\mathrm{LL}|W}$ can be identified from the following equations:

$$\mathrm{pr}\{S(1) = 1 \mid W\} = \mathrm{pr}(S = 1 \mid Z = 1, W) = \pi_{\mathrm{LL}|W} + \pi_{\mathrm{LD}|W};$$
$$\mathrm{pr}\{S(0) = 1 \mid W\} = \mathrm{pr}(S = 1 \mid Z = 0, W) = \pi_{\mathrm{LL}|W} + \pi_{\mathrm{DL}|W};$$
$$\pi_{\mathrm{LL}|W} = \mathrm{pr}\{S(0) = 1 \mid W\}\mathrm{pr}\{S(1) = 1 \mid S(0) = 1, W\};$$

$$\mathrm{pr}\{S(1) = 1 \mid S(0) = 1, W\} = \mathrm{pr}\{S(1) = 1 \mid W\} + \rho(W)\left(\min\left[1, \frac{\mathrm{pr}\{S(1) = 1 \mid W\}}{\mathrm{pr}\{S(0) = 1 \mid W\}}\right] - \mathrm{pr}\{S(1) = 1 \mid W\}\right),$$

where due to Assumption 2, $\mathrm{pr}\{S(z) = 1 \mid W\} = \mathrm{pr}(S = 1 \mid Z = z, W)$ and thus is identifiable for $z = 0, 1$. Note that under these assumptions, $\pi_{g|W}$ is also identifiable with $g \in \{\mathrm{LD}, \mathrm{DL}, \mathrm{DD}\}$.

Identification of $\mu_{z,\mathrm{LL},W}$: under the assumptions of Theorem 1, $\mu_{0,\mathrm{LL},W}$ can be identified from (3). First recall that $W = (A, X)$. As described above, $p_{g|z,w,s=1} = \mathrm{pr}(G = g \mid X =$

$x, A = a)/\mathrm{pr}(S = 1 \mid Z = z, X = x, A = a)$ is identifiable from the observed data. Note that by consistency, $0 = p_{\mathrm{DL}|1,x,a,s=1} = p_{\mathrm{LD}|0,x,a,s=1} = p_{\mathrm{DD}|z,x,a,s=1}$, $z \in \{0,1\}$. Furthermore, Assumption 5 implies that for all x, there exist $a_0 \neq a_1$ such that $p_{\mathrm{LL}|1,x,a_1,s=1} \neq p_{\mathrm{LL}|1,x,a_0,s=1}$. Due to Assumption 4, for $g = \mathrm{LL}, \mathrm{LD}$,

$$\mu_{1,g,w} = E(Y \mid Z = 1, G = g, X = x, A = a) = E(Y \mid Z = 1, G = g, X = x) \equiv \mu_{1,g,x}.$$

Furthermore, $\mu_{1,\mathrm{LL},x}$ can be identified from

$$E(Y \mid Z = 1, S = 1, X = x, A = a_1) = p_{\mathrm{LL}|1,x,a_1,s=1}\mu_{1,\mathrm{LL},x} + (1 - p_{\mathrm{LL}|1,x,a_1,s=1})\mu_{1,\mathrm{LD},x},$$
$$E(Y \mid Z = 1, S = 1, X = x, A = a_0) = p_{\mathrm{LL}|1,x,a_0,s=1}\mu_{1,\mathrm{LL},x} + (1 - p_{\mathrm{LL}|1,x,a_0,s=1})\mu_{1,\mathrm{LD},x}. \quad \text{(S1)}$$

Under the assumptions of Theorem 2, we can obtain similarly that for $z \in \{0,1\}$, $\mu_{z,\mathrm{LL},w} = \mu_{z,\mathrm{LL},x}$, where $\mu_{1,\mathrm{LL},x}$ is identified by (S1), and $\mu_{0,\mathrm{LL},x}$ can be identified from

$$E(Y \mid Z = 0, S = 1, X = x, A = a_1') = p_{\mathrm{LL}|0,x,a_1',s=1}\mu_{0,\mathrm{LL},x} + (1 - p_{\mathrm{LL}|0,x,a_1',s=1})\mu_{0,\mathrm{DL},x},$$
$$E(Y \mid Z = 0, S = 1, X = x, A = a_0') = p_{\mathrm{LL}|0,x,a_0',s=1}\mu_{0,\mathrm{LL},x} + (1 - p_{\mathrm{LL}|0,x,a_0',s=1})\mu_{0,\mathrm{DL},x},$$

where a_1' and a_0' are two distinct values of A such that $p_{\mathrm{LL}|0,x,a_1',s=1} \neq p_{\mathrm{LL}|0,x,a_0',s=1}$.

Similarly, under the assumptions of Theorem 3, $\mu_{0,\mathrm{LL},W}$ is identified by (3) and $\mu_{1,\mathrm{LL},x,a_1}$ can be identified from the following equations:

$$E(Y \mid Z = 1, S = 1, X = x, A = a_1) = p_{\mathrm{LL}|1,x,a_1,s=1}\mu_{1,\mathrm{LL},x,a_1} + (1 - p_{\mathrm{LL}|1,x,a_1,s=1})\mu_{1,\mathrm{LD},x,a_1};$$
$$E(Y \mid Z = 1, S = 1, X = x, A = a_0) = p_{\mathrm{LL}|1,x,a_0,s=1}\mu_{1,\mathrm{LL},x,a_0} + (1 - p_{\mathrm{LL}|1,x,a_0,s=1})\mu_{1,\mathrm{LD},x,a_0};$$
$$\mu_{1,\mathrm{LL},x,a_1} - \mu_{1,\mathrm{LL},x,a_0} = \mu_{1,\mathrm{LD},x,a_1} - \mu_{1,\mathrm{LD},x,a_0};$$
$$\mu_{1,\mathrm{LL},x,a_1} - \mu_{1,\mathrm{LL},x,a_0} = \mu_{0,\mathrm{LL},x,a_1} - \mu_{0,\mathrm{LL},x,a_0};$$
$$E(Y \mid Z = 0, S = 1, X = x, A = a_1) = \mu_{0,\mathrm{LL},x,a_1};$$
$$E(Y \mid Z = 0, S = 1, X = x, A = a_0) = \mu_{0,\mathrm{LL},x,a_0},$$

where $\mu_{z,g,x,a} \equiv E(Y \mid Z = z, G = g, X = x, A = a)$.

For any $a \in \mathcal{A} \setminus \{a_1\}$, $\mu_{1,\mathrm{LL},x,a}$ can be identified via the following equations:

$$\mu_{1,\mathrm{LL},x,a} - \mu_{1,\mathrm{LL},x,a_1} = \mu_{0,\mathrm{LL},x,a} - \mu_{0,\mathrm{LL},x,a_1};$$
$$E(Y \mid Z = 0, S = 1, X = x, A = a) = \mu_{0,\mathrm{LL},x,a};$$
$$E(Y \mid Z = 0, S = 1, X = x, A = a_1) = \mu_{0,\mathrm{LL},x,a_1}.$$

2. PROOF THAT THE NONPARAMETRIC STRUCTURAL EQUATION MODEL WITH INDEPENDENT ERRORS IN SECTION 3.3 IMPLIES ASSUMPTION 2–4

Under (4), $\epsilon_Z \perp\!\!\!\perp (\epsilon_X, \epsilon_A, \epsilon_S, \epsilon_Y)$ implies that $Z \perp\!\!\!\perp (\epsilon_S, \epsilon_Y) \mid X, A$, which then implies Assumptions 2 and 3; recall here that the error terms can be interpreted as the set of one-step-ahead counterfactuals. To see that Assumption 4 holds under the structural equations (4), note the following:

$$\begin{aligned}\mathrm{pr}\{Y \mid A = a_1, Z = 1, X = x, S(1) = 1, S(0)\} &= \mathrm{pr}\{Y \mid A = a_1, Z = 1, X = x, S = 1, S(0)\}\\ &= \mathrm{pr}(Y \mid A = a_1, Z = 1, X = x, S = 1)\\ &= \mathrm{pr}(Y \mid A = a_0, Z = 1, X = x, S = 1)\\ &= \mathrm{pr}\{Y \mid A = a_0, Z = 1, X = x, S = 1, S(0)\}\\ &= \mathrm{pr}\{Y \mid A = a_0, Z = 1, X = x, S(1) = 1, S(0)\},\end{aligned}$$

where the first and last lines follow from consistency; the second and fourth follow from $Y \perp\!\!\!\perp S(0) \mid A, Z = 1, X, S = 1$, which holds as Y is d-separated from ϵ_S given X, A, Z, S; the third line follows since Y is d-separated from A conditioning on X, Z, S.

As Assumption 4 holds trivially for the subgroup with $S(1) = 0$, we complete the proof.

3. PROOF OF PROPOSITION 1

Constraint (8) is standard. To prove (9), note that under Assumption 4, (S1) holds for all a, i.e.

$$E(Y \mid Z = 1, S = 1, X = x, A = a) = p_{\mathrm{LL}|1,x,a,s=1}\mu_{1,\mathrm{LL},x} + (1 - p_{\mathrm{LL}|1,x,a,s=1})\mu_{1,\mathrm{LD},x}. \quad \text{(S2)}$$

It follows that

$$|E(Y \mid Z = 1, S = 1, X = x, A = a)| \le \max(|\mu_{1,\mathrm{LL},x}|, |\mu_{1,\mathrm{LD},x}|).$$

Hence for all x, $E(Y \mid Z = 1, S = 1, X = x, A = a)$ is bounded as a function of a. On the other hand, suppose that for all x, $E(Y \mid Z = 1, S = 1, X = x, A = a)$ is bounded as a function of a. Let

$$\bar{f}(x) = \sup_a E(Y \mid Z = 1, S = 1, X = x, A = a), \text{ and } \underline{f}(x) = \inf_a E(Y \mid Z = 1, S = 1, X = x, A = a).$$

Then (S2) holds with

$$p_{\mathrm{LL}|1,x,a,s=1} = \frac{\bar{f}(x) - E(Y \mid Z = 1, S = 1, X = x, A = a)}{\bar{f}(x) - \underline{f}(x)}, \ \mu_{1,\mathrm{LL},x} = \underline{f}(x) \text{ and } \mu_{1,\mathrm{LD},x} = \bar{f}(x).$$

Hence (9) summarizes all the constraints on the observed data law derived from (S2). The proof of (11) is similar and hence omitted.

Constraint (10) follows immediately from Assumption 5 by noting that

$$\begin{aligned}\frac{\mathrm{pr}(S = 1 \mid Z = 0, X = x, A = a)}{\mathrm{pr}(S = 1 \mid Z = 1, X = x, A = a)} &= \frac{\mathrm{pr}(G = \mathrm{LL} \mid X = x, A = a)}{\mathrm{pr}(S = 1 \mid Z = 1, X = x, A = a)} \\ &= \mathrm{pr}(G = \mathrm{LL} \mid Z = 1, S = 1, X = x, A = a).\end{aligned}$$

To show (12), note that Assumption 7 implies that for all a_1, a_0,

$$\mu_{1,\mathrm{LL},x,a_1} - \mu_{0,\mathrm{LL},x,a_1} = \mu_{1,\mathrm{LL},x,a_0} - \mu_{0,\mathrm{LL},x,a_0},$$
$$\mu_{1,\mathrm{LD},x,a_1} - \mu_{0,\mathrm{LL},x,a_1} = \mu_{1,\mathrm{LD},x,a_0} - \mu_{0,\mathrm{LL},x,a_0}.$$

It follows that $\mu_{1,\mathrm{LL},w} - \mu_{0,\mathrm{LL},w} = \mu_{1,\mathrm{LL},x} - \mu_{0,\mathrm{LL},x}$ and $\mu_{1,\mathrm{LD},w} - \mu_{0,\mathrm{LL},w} = \mu_{1,\mathrm{LD},x} - \mu_{0,\mathrm{LL},x}$. We then have

$$\begin{aligned}&E(Y \mid Z = 1, S = 1, X = x, A = a) - E(Y \mid Z = 0, S = 1, X = x, A = a) \\ &= p_{\mathrm{LL}|1,x,a,s=1}\mu_{1,\mathrm{LL},x,a} + p_{\mathrm{LD}|1,x,a,s=1}\mu_{1,\mathrm{LD},x,a} - \mu_{0,\mathrm{LL},x,a} \\ &= p_{\mathrm{LL}|1,x,a,s=1}(\mu_{1,\mathrm{LL},x,a} - \mu_{0,\mathrm{LL},x,a}) + p_{\mathrm{LD}|1,x,a,s=1}(\mu_{1,\mathrm{LD},x,a} - \mu_{0,\mathrm{LL},x,a}) \\ &= p_{\mathrm{LL}|1,x,a,s=1}(\mu_{1,\mathrm{LL},x} - \mu_{0,\mathrm{LL},x}) + p_{\mathrm{LD}|1,x,a,s=1}(\mu_{1,\mathrm{LD},x} - \mu_{0,\mathrm{LL},x})\end{aligned}$$

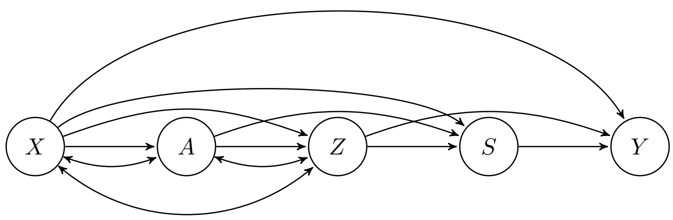
so that

$$\begin{aligned}&|E(Y \mid Z = 1, S = 1, X = x, A = a) - E(Y \mid Z = 0, S = 1, X = x, A = a)| \\ &\le \max(|\mu_{1,\mathrm{LL},x} - \mu_{0,\mathrm{LL},x}|, |\mu_{1,\mathrm{LD},x} - \mu_{0,\mathrm{LL},x}|).\end{aligned}$$

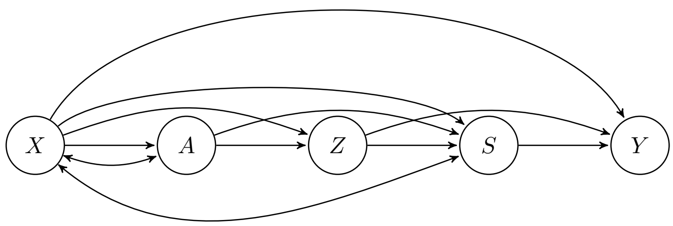
The rest of the proof is similar to that for (9) and is hence omitted.

4. ALTERNATIVE CAUSAL DIAGRAMS

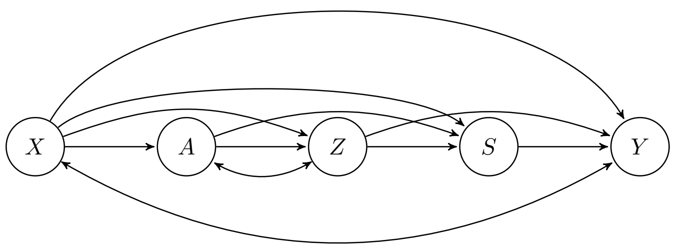
Figure S1 shows some more complicated causal diagrams that imply assumptions Assumptions 2–4 but allow for dependence of errors in the structural equations (4) representing unmeasured confounding between nodes in Fig. 1. Here unmeasured confounding is denoted by a bi-directed edge between observed nodes. In general, using graphical terminology (Richardson, 2003), the mixed graphs, such as those in Fig. S1, imply Assumptions 2–4 so long as the pairs (Z, S), (Z, Y), (A, Y) and (S, Y) are not in the same district, where a district is a connected component of the graph obtained by removing all edges that are not bi-directed. Specifically, Assumption 2 requires that Z and S cannot be in the same district, Assumption 3 requires that neither Z and Y nor S and Y can be in the same district, and Assumption 4 requires additionally that A and Y cannot be in the same district.



(i)



(ii)



(iii)

Fig. S1. More complicated causal diagrams that imply Assumptions 2–4. The districts in these graphs are: (i) $\{X, A, Z\}, \{S\}$ and $\{Y\}$; (ii) $\{X, A, S\}, \{Z\}$ and $\{Y\}$; (iii) $\{X, Y\}, \{A, Z\}$ and $\{S\}$.

REFERENCES

RICHARDSON, T. S. (2003). Markov properties for acyclic directed mixed graphs. *Scandinavian Journal of Statistics* **30**, 145–157.