Appendix Q factor

1 Harmonic oscillator, definition of Q factor and damping

The simplest system that exhibits damped oscillations is the harmonic oscillator. We here study the harmonic oscillator driven by white noise to show that an intrinsic relationship exists between the damping of the oscillation and the width of the resonance peak. The width of the resonance peak can be quantified by the Q-factor, defined as the ratio of the resonance frequency and the half-width of the power spectral density. We start with a short review of the properties of a noise-driven harmonic oscillator

$$\frac{d^2y}{dt^2} + 2\rho\omega_0 \frac{dy}{dt} + \omega_0^2 y = \xi(t). \tag{1}$$

Driven by white noise with the autocorrelation function $\langle \xi(t)\xi(t+\tau)\rangle = \sigma^2\delta(\tau)$, the solution in Fourier domain is

$$Y(\omega) = \frac{X(\omega)}{\omega_0^2 - \omega^2 + 2i\rho\omega_0\omega}. \tag{2}$$

Hence the power spectrum $S(\omega) = |Y(\omega)|^2$ takes the form

$$\begin{aligned} S(\omega) &= \frac{\sigma^2}{|-\omega^2 + 2i\rho\omega_0\omega + \omega_0^2|^2} \qquad (3) \\ &= \frac{\sigma^2}{(\omega^2 - \omega_0^2)^2 + (2\rho\omega_0\omega)^2}. \end{aligned}$$

The peak of the power spectrum is reached at the minimum of the denominator, which is given by $0 = 4(\omega^2 - \omega_0^2)\omega + 8(\rho\omega_0)^2\omega = 0$ which has the roots $\omega_{\max} = 0$ and $\omega_{\max} = \omega_0\sqrt{1-2\rho^2}$, so there is only a resonance peak, if $\rho < 1/\sqrt{2}$. The peak height at the maximum is

$$\begin{aligned} S_{\max} &= \frac{\sigma^2}{\omega_0^4(1-2\rho^2-1)^2 + 4\rho^2\omega_0^4(1-2\rho^2)} \\ &= \frac{\sigma^2}{4\rho^2\omega_0^4(1-\rho^2)}. \qquad (4) \end{aligned}$$

The power spectrum $S(\omega)$ decays to the value $\beta S_{\max}$ at the frequencies ω satisfying

$$\begin{aligned}4\rho^2\omega_0^4(1-\rho^2)/\beta &= (\omega^2-\omega_0^2)^2+(2\rho\omega_0\omega)^2\\ 0 &= (\omega^2)^2-2\omega^2\omega_0^2(1-2\rho^2)+\omega_0^4-4\rho^2\omega_0^4(1-\rho^2)/\beta,\end{aligned}$$

so that the solution of this quadratic equation in ω^2 yields

$$\begin{aligned}\omega_{1,2}^2 &= \omega_0^2(1-2\rho^2)\pm\sqrt{\omega_0^4(1-2\rho^2)^2-\omega_0^4+4\rho^2\omega_0^4(1-\rho^2)/\beta}\\ &= \omega_0^2\left((1-2\rho^2)\pm 2\rho\sqrt{(\rho^2-1)(1-\beta^{-1})}\right).\end{aligned}$$

The width $\Delta\omega_\beta$ at which the power has decayed to $\beta S_{\max}$ is hence

$$\begin{aligned}\Delta\omega_\beta &= \omega_0\left(\sqrt{(1-2\rho^2)+2\rho\sqrt{(\rho^2-1)(1-\beta^{-1})}}-\sqrt{(1-2\rho^2)-2\rho\sqrt{(\rho^2-1)(1-\beta^{-1})}}\right)\\ &= \left(\sqrt{\omega_0^2(1-2\rho^2)+\omega_0^2 2\rho\sqrt{(\rho^2-1)(1-\beta^{-1})}}-\sqrt{\omega_0^2(1-2\rho^2)-\omega_0^2 2\rho\sqrt{(\rho^2-1)(1-\beta^{-1})}}\right)\\ &= \left(\sqrt{\omega_{\max}^2+\frac{\omega_{\max}^2}{1-2\rho^2}2\rho\sqrt{(\rho^2-1)(1-\beta^{-1})}}-\sqrt{\omega_{\max}^2-\frac{\omega_{\max}^2}{1-2\rho^2}2\rho\sqrt{(\rho^2-1)(1-\beta^{-1})}}\right)\\ &= \omega_{\max}\left(\sqrt{1+\frac{2\rho\sqrt{(\rho^2-1)(1-\beta^{-1})}}{1-2\rho^2}}-\sqrt{1-\frac{2\rho\sqrt{(\rho^2-1)(1-\beta^{-1})}}{1-2\rho^2}}\right)\end{aligned} \tag{5}$$

The quality factor or Q-factor is the ratio of the free oscillation frequency to the half-width (i.e. for $\beta=\frac{1}{2}$)

$$\begin{aligned}Q &= \frac{\omega_0}{\Delta\omega_{1/2}} \qquad (6)\\ &= \frac{1}{\sqrt{1-2\rho^2}\left(\sqrt{1+\frac{2\rho\sqrt{1-\rho^2}}{1-2\rho^2}}-\sqrt{1-\frac{2\rho\sqrt{1-\rho^2}}{1-2\rho^2}}\right)}\\ &= \frac{1}{\sqrt{1-2\rho^2+2\rho\sqrt{1-\rho^2}}-\sqrt{1-2\rho^2-2\rho\sqrt{1-\rho^2}}}\end{aligned}$$

The highest damping ρ for which the power spectrum still decays below β^{-1} at low frequencies is given by

$$(1-2\rho^2)-2\rho\sqrt{(\rho^2-1)(1-\beta^{-1})}<0,$$

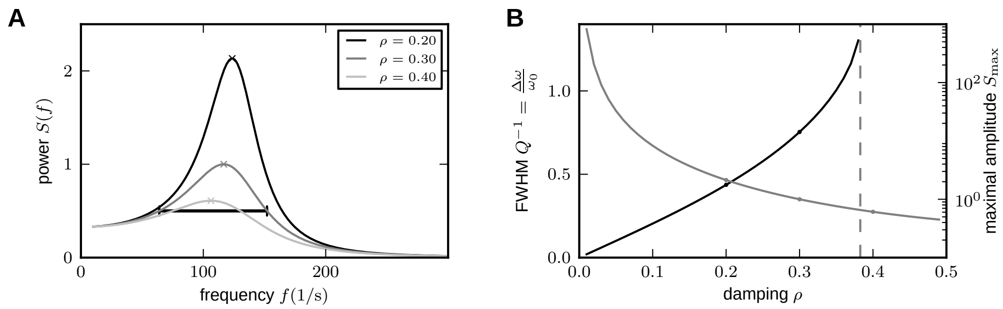

Figure 1: Power spectra of a harmonic oscillator driven by white noise. **A** Power spectrum (3) for different values of the damping normalized to the peak value for $\rho = 0.3$. The full width half mean is indicated for the curve of $\rho = 0.3$. **B** Inverse Q-factor (black, (6)) as a function of the damping constant ρ. Peak value (4) of the power spectrum as a function of the damping (gray, (4)). The dashed vertical line indicates the limit for which a half mean can be defined. Beyond this line, the power does not decay below half the peak value at small frequencies.

so at

$$\begin{aligned}(1-2\rho^2)^2 &= 4\rho^2(\rho^2-1)(1-\beta^{-1})\\ 1 &= -\beta^{-1}4\rho^2(\rho^2-1)\\ \rho^4-\rho^2+\frac{\beta}{4} &= 0\\ \rho^2 &= \frac{1}{2} \pm \sqrt{\frac{1-\beta}{4}}.\end{aligned}$$

The negative root yields the lower bound at which the solution vanishes first, i.e. above this value the quality factor cannot be defined. The analytical results are displayed in figure 1.