# SUPPLEMENTAL MATERIAL:

# Synthetic wavelength interferometry of an optical frequency comb for absolute distance measurement

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## Calculation of the effective center wavelength of the interference signal

The electric field of an optical frequency comb in vacuum can be expressed by:

$$E(t,x)=\sum_n A_n e^{i[2\pi f_n(t-x/c)+\varphi_n]} \tag{1}$$

where $f_n = nf_{rep}+f_{ceo}$ is optical frequency of $n^{th}$ mode, $A_n$ is amplitude of $n^{th}$ mode, $\varphi_n$ is initial phase of $n^{th}$ mode, $t$ is time, $x$ is the path length, and $c$ is speed of light in vacuum.

We denote $E_1$ and $E_2$ as the electric fields of measurement arm and reference arm respectively. Note that $E_2$ has a frequency shift $\Delta f$ by AOM. They can be expressed by:

$$E_1=\sum_n A_n e^{i[2\pi f_n(t-x_1/c)+\varphi_n]}\,, \tag{2}$$

$$E_2=\sum_n A_n e^{i[2\pi (f_n+\Delta f)(t-x_2/c)+\varphi_n]}\,, \tag{3}$$

where $A_n$ and the total modes number $n$ are determined by the spectrum of the band-pass filter used before the detector.

The interference signal is

$$I=\left|E_1+E_2\right|^2=(E_1+E_2)\cdot(E_1+E_2)^*\,. \tag{4}$$

The heterodyne interference signal (after low pass filter) is

$$I_{\text{Het}}=\sum_n A_n^2\cos[2\pi\Delta f(t-\frac{x_2}{c})+2\pi f_n\frac{x_1-x_2}{c}]\,. \tag{5}$$

Please note the heterodyne beat frequency can be $\Delta f$ or ($\Delta f$–$f_{rep}$). In Eq. (5), we use $\Delta f$ for simplicity. $I_{\text{Het}}$ is the sum of a series of cosine signals with a same frequency $\Delta f$, thus itself can be expressed as a simple cosine function with the same frequency $\Delta f$:

$$I_{\text{Het}}=a\cos[2\pi\Delta f(t-\frac{x_2}{c})+\alpha]\,, \tag{6}$$

where $a$ is the amplitude of $I_{Het}$ and $\alpha$ is the phase of $I_{Het}$. According to Eqs. (5) and (6), we can deduce that

$$a\cos\alpha = \sum_n A_n^2 \cos 2\pi f_n \frac{x_1 - x_2}{c}, \tag{7}$$

$$a\sin\alpha = \sum_n A_n^2 \sin 2\pi f_n \frac{x_1 - x_2}{c}. \tag{8}$$

So that

$$\tan\alpha = \frac{\sum_n A_n^2 \sin 2\pi f_n \frac{x_1 - x_2}{c}}{\sum_n A_n^2 \cos 2\pi f_n \frac{x_1 - x_2}{c}}. \tag{9}$$

Using a numerical method, we can set a series of ($x_1$−$x_2$) and calculate a serices of $\alpha$ according to Eq. (9). Then we can calculate the effective center frequency (wavelength) of the interference signal.

**In particular**, if the optical spectrum of the band-pass filter is symmetric with the center frequency $f_c$, the calculation can simplified. We consider two mode lines symmetric with $f_c$, and modes orders are $n_c$-$n_i$ and $n_c$+$n_i$, where $n_c$ is the order of $f_c$. We have $A_{n_c-n_i} = A_{n_c+n_i}$, so the sum signal of the heterodyne interference from the two spectral lines is

$$\begin{aligned}
&\cos[2\pi\Delta f(t - \frac{x_2}{c}) + 2\pi f_{n_c-n_i}\frac{x_1 - x_2}{c}] + \cos[2\pi\Delta f(t - \frac{x_2}{c}) + 2\pi f_{n_c+n_i}\frac{x_1 - x_2}{c}] \\
&= \cos[2\pi\Delta f(t - \frac{x_2}{c}) + 2\pi(f_{n_c} - n_i f_{\text{rep}})\frac{x_1 - x_2}{c}] \\
&\quad + \cos[2\pi\Delta f(t - \frac{x_2}{c}) + 2\pi(f_{n_c} + n_i f_{\text{rep}})\frac{x_1 - x_2}{c}] \\
&= 2\cos(2\pi n_i f_{\text{rep}}\frac{x_1 - x_2}{c})\cos[2\pi\Delta f(t - \frac{x_2}{c}) + 2\pi f_{n_c}\frac{x_1 - x_2}{c}]
\end{aligned} \tag{10}$$

Thus we have

$$I_{\text{Het}} = \cos[2\pi\Delta f(t - \frac{x_2}{c}) + 2\pi f_{n_c}\frac{x_1 - x_2}{c}]\sum_n A_n^2 \cos[2\pi(n - n_c)f_{\text{rep}}\frac{x_1 - x_2}{c}]. \tag{11}$$

We denote that

$$I_{\text{Het}} = b\cos[2\pi\Delta f(t - \frac{x_2}{c}) + 2\pi f_{n_c}\frac{x_1 - x_2}{c}], \tag{12}$$

where

$$b = \sum_n A_n^2 \cos[2\pi(n - n_c)f_{\text{rep}}\frac{x_1 - x_2}{c}]. \tag{13}$$

Comparing Eqs. (6) and (12), we can obtain that

$$\alpha = 2\pi f_{n_c} \frac{x_1 - x_2}{c} \tag{14}$$

Therefore, the integration of phases from all comb modes is equal to the phase of the center mode when the optical spectrum of the band-pass filter is symmetric with the center frequency $f_c$.

For the present experimental system, optical spectrum of the two band-pass filters are both nearly symmetric with their center frequencies. Thus, we use the center wavelengths of the filters as the effective center wavelength of the interference signal. They are measured by an OSA (Optical Spectrum Analyzer, AQ6370C, YOKOGAWA). We also use the numerical method according to Eq. (9) to verify the effective center wavelengths, and find the differences are both less than 0.03 nm, the corresponding difference in synthetic wavelength is about 87 nm. The results are shown as follows:

| | $\lambda_1$ (nm) | $\lambda_2$ (nm) | $\lambda_s$ (μm) |
|---|---|---|---|
| OSA | 1558.919 | 1566.945 | 304.440 |
| Numerical method | 1558.947 | 1566.971 | 304.353 |

As discussed in the manuscript, no matter how long distance we measure, the fringe order ($N_s$) of the synthetic wavelength is always zero. Thus, the accuracy of the synthetic wavelength itself is not a severe factor in the measurement since its error will not be accumulated by the fringe order. In our experiment (Figure 5), the phase change of synthetic wavelength ($\phi_s$) is less than 90 deg., thus the error caused by the $\lambda_s$ is less than 87/2×(90/360) = 11 nm, which is negligible.