Supporting Information

Kraynyukova and Tchumatchenko 10.1073/pnas.1700080115

In the following sections we present the details of the calculations related to the results derived in the main text.

Characteristic Function

The characteristic function $\mathcal{F}$ which we introduce in Eq. **15** is essential for the multiplicity analysis of the SSN steady states. In the main text we have derived $\mathcal{F}$ for the case $\det J > 0$. Here, we present the details of the derivation of the characteristic function for $\det J < 0$ and $\det J = 0$ whereby we follow steps similar to those for $\det J > 0$ in *Materials and Methods*. Before we do so, we mention why different definitions of $\mathcal{F}$ for $\det J \geq 0$ and $\det J < 0$ are necessary. Many of our multiplicity proofs are based on the Descartes' rule of signs and the second derivative argument (see *Multiplicity of Steady States*). Both arguments would lose their usefulness if we were to use the same definition of P and $\mathcal{F}$ for positive and negative values of $\det J$ as we point out in the corresponding sections. Therefore, we have opted for the particular definition of $\mathcal{F}$ as in Eq. **15**.

We start the derivation of the characteristic function $\mathcal{F}$ for $\det J < 0$ by considering P defined by Eq. **12** in the main text. We recognize that P is no longer a monotonically increasing function with respect to the variable z_E for $\det J < 0$. However, we note that the left side of Eq. **11** is monotonically increasing with respect to z_I. Therefore, here we change the definition of P to

$$P(z_I) = -\det J \cdot J_{IE}^{-1} \cdot (z_I)_+^{\alpha_I} + J_{IE}^{-1} J_{EE} z_I + C_-. \quad \textbf{[S1]}$$

Our goal now is to eliminate the variable z_E and reduce the problem to 1D. To this end, we substitute $z_E = P(z_I)$ into Eq **10**. Now the system of Eqs. **10** and **11** is equivalent to the equation $\mathcal{F}(z_I) = 0$ with respect to the variable z_I, where the function $\mathcal{F}$ is given by

$$\mathcal{F}(z_I) = J_{IE}(P(z_I))_+^{\alpha_E} - J_{II}(z_I)_+^{\alpha_I} - z_I + g_I \quad \textbf{[S2]}$$

with P defined by Eq. **S1**. In summary, for $\det J < 0$ we have shown that if r_E, r_I is a steady state, then z_I satisfies $\mathcal{F}(z_I) = 0$ with $\mathcal{F}$ defined by Eq. **S2**.

To establish a one-to-one bijective mapping between zero crossings and steady states we now show that the inverse is also true: All zero crossings of the characteristic function have a corresponding SSN steady state; i.e., there are no spurious zero crossings that have no corresponding steady states. To this end we consider an arbitrary real zero-crossing z^* of the function $\mathcal{F}$ defined by Eq. **S2**. To show that z^* corresponds to an SSN steady state we verify that the nonnegative numbers $r_E = (P(z^*))_+^{\alpha_E}$ and $r_I = (z^*)_+^{\alpha_I}$ represent a steady state of the SSN. To this end we substitute $r_E = (P(z^*))_+^{\alpha_E}$ and $r_I = (z^*)_+^{\alpha_I}$ into the zero-crossing equation

$$\mathcal{F}(z^*) = J_{IE}(P(z^*))_+^{\alpha_E} - J_{II}(z^*)_+^{\alpha_I} - z^* + g_I = 0$$

and obtain

$$J_{IE} r_E - J_{II} r_I + g_I = z^*. \quad \textbf{[S3]}$$

Next, we apply the function $(\cdot)_+^{\alpha_I}$ to both sides of Eq. **S3** to obtain

$$(J_{IE} r_E - J_{II} r_I + g_I)_+^{\alpha_I} = (z^*)_+^{\alpha_I} = r_I,$$

which is the second steady-state equation in Eq. **2**. Now, we insert z^* from Eq. **S3** and $r_I = (z^*)_+^{\alpha_I}$ into $r_E = (P(z^*))_+^{\alpha_E}$ with P defined by Eq. **S1** and obtain

$$r_E = (J_{EE} r_E - J_{EI} r_I + g_E)_+^{\alpha_E}.$$

This is the first steady-state equation in Eq. **2**. Taken together this proves that any zero crossing of $\mathcal{F}$ for $\det J < 0$ corresponds to a steady state as defined by Eq. **2**.

Finally, we derive the characteristic function for $\det J = 0$. We find that for $\det J = 0$, we cannot invert the system Eq. **7**. However, we note that $\det J = 0$ is equivalent to $J_{IE} = J_{II} J_{EE} J_{EI}^{-1}$. Using this formula we express the variable z_I as a function of z_E:

$$z_I = J_{IE} r_E - J_{II} r_I + g_I = J_{II} J_{EI}^{-1}(z_E - g_E) + g_I.$$

For $\det J = 0$ we define P as

$$P(z_E) = J_{II} J_{EI}^{-1}(z_E - g_E) + g_I = J_{EI}^{-1} J_{II} z_E + C_+. \quad \textbf{[S4]}$$

We note that this definition coincides with the definition of P in Eq. **12** for $\det J > 0$ if we set $\det J = 0$. The equilibrium equations in Eq. **2** are now equivalent to

$$r_E = (z_E)_+^{\alpha_E}, \quad r_I = (P(z_E))_+^{\alpha_I}. \quad \textbf{[S5]}$$

To obtain the equation with respect to the new variable z_E we substitute Eq. **S5** into the expression $z_E = J_{EE} r_E - J_{EI} r_I + g_E$. We obtain $\mathcal{F}(z_E) = 0$, where $\mathcal{F}$ is defined by Eq. **14**.

Next, we prove that there is a one-to-one mapping between zero crossings and steady states for $\det J = 0$ and show that all zero crossings of the characteristic function $\mathcal{F}$ correspond to the SSN steady states. To this end we consider a real number z^* such that $\mathcal{F}(z^*) = 0$ with $\mathcal{F}$ defined by Eq. **14**. Here, we consider the case $\det J = 0$ and use the corresponding definitions of $\mathcal{F}$ and P from Eq. **14** and Eq. **S4**. Next, we define $r_E = (z^*)_+^{\alpha_E}$, $r_I = (P(z^*))_+^{\alpha_I}$ and insert these numbers into $\mathcal{F}(z^*) = 0$. We obtain the first steady-state relation from Eq. **2** using similar algebraic steps as in the case $\det J > 0$ in *Materials and Methods*. Then, we use the relation $r_I = (P(z^*))_+^{\alpha_I}$ to obtain the second relation in Eq. **2**. This proves that r_E, r_I is an SSN steady state.

To summarize, we have established a one-to-one bijective mapping between the steady states of Eq. **1** and zero crossings of the characteristic function $\mathcal{F}$ defined by Eq. **15** in the following sense. If r_E, r_I is a steady state of Eq. **1**, then

$$z = \begin{cases} J_{EE} r_E - J_{EI} r_I + g_E, & \det J \geq 0 \\ J_{IE} r_E - J_{II} r_I + g_I, & \det J < 0 \end{cases}$$

satisfies $\mathcal{F}(z) = 0$. And vice versa, if z satisfies $\mathcal{F}(z) = 0$ whereby $\mathcal{F}$ is defined by Eq. **15**, then

$$r_E = \begin{cases} (z)_+^{\alpha_E}, & \det J \geq 0 \\ (P(z))_+^{\alpha_E}, & \det J < 0 \end{cases}, \quad r_I = \begin{cases} (P(z))_+^{\alpha_I}, & \det J \geq 0 \\ (z)_+^{\alpha_I}, & \det J < 0 \end{cases}$$

is a steady state of Eq. **1**.

Stability Conditions

Here we show that if $\det J \leq 0$ and $\tau_I \leq \tau_E$, then a steady state corresponding to a zero crossing of the characteristic function $\mathcal{F}$ with $\mathcal{F}'(z) < 0$ is always a stable node. To this end we prove that for $\det J \leq 0$ and $\tau_I \leq \tau_E$ the eigenvalues λ_1 and λ_2 of the Jacobian matrix at the steady state of interest are always real and both are negative. We denote the trace of the Jacobian matrix DG by $\mathrm{Tr}\, DG$ and use the following equation for the eigenvalues of DG:

$$\lambda_{1,2} = \frac{1}{2}\mathrm{Tr}\, DG \pm \frac{1}{2}\sqrt{(\mathrm{Tr}\, DG)^2 - 4\det DG}. \quad \textbf{[S6]}$$

Both eigenvalues are real, if $(\mathrm{Tr}\, DG)^2 - 4 \det\, DG \geq 0$. Using the representation for DG in *Materials and Methods* we obtain

$$
\begin{aligned}
&(\mathrm{Tr}\, DG)^2 - 4 \det\, DG \\
&= (\tau_I^{-1} - \tau_E^{-1})^2 + (\tau_E^{-1} J_{EE} \alpha_E (z_E)_+^{\alpha_E - 1} - \tau_I^{-1} J_{II} \alpha_I (z_I)_+^{\alpha_I - 1})^2 \\
&+ 2(\tau_I^{-1} - \tau_E^{-1})(\tau_E^{-1} J_{EE} \alpha_E (z_E)_+^{\alpha_E - 1} + \tau_I^{-1} J_{II} \alpha_I (z_I)_+^{\alpha_I - 1}) \\
&- 4 \det\, J \cdot \tau_E^{-1} \tau_I^{-1} \alpha_E \alpha_I (z_E)_+^{\alpha_E - 1} (z_I)_+^{\alpha_I - 1}. \qquad \mathbf{[S7]}
\end{aligned}
$$

Thus, if $\det\, J \leq 0$ and $\tau_I \leq \tau_E$, then the eigenvalues are real. Since for $\det\, J \leq 0$ and $\tau_I \leq \tau_E$

$$
\begin{aligned}
\mathrm{Tr}\, DG &< \tau_E^{-1} J_{EE} \alpha_E (z_E)_+^{\alpha_E - 1} - \tau_I^{-1} J_{II} \alpha_I (z_I)_+^{\alpha_I - 1} \\
&\leq \sqrt{(\mathrm{Tr}\, DG)^2 - 4 \det\, DG},
\end{aligned}
$$

then the eigenvalue corresponding to the "$-$" sign in Eq. **S6** is always negative. To obtain the second inequality in the above expression we used Eq. **S7**. Since we consider the case $\mathcal{F}'(z) < 0$, both eigenvalues have the same sign: If one is negative, then the second eigenvalue is also negative.

In summary, we have proved that if $\det\, J \leq 0$ and $\tau_I \leq \tau_E$, then the eigenvalues are always real and both eigenvalues are negative if $\mathcal{F}'(z) < 0$. Thus, Eq. **4** implies that the steady state corresponding to z is a stable node if $\mathcal{F}'(z) < 0$ and a saddle if $\mathcal{F}'(z) > 0$.

Multiplicity of Steady States

Here, we present the details of the multiplicity analysis of zero crossings in the nine classes outlined in Table 1. For completeness, we present below two statements A and B which we apply in our analysis. Statement A is Descartes' rule of signs which we use to determine the maximal number of zero crossings. Statement B is a corollary of the well-known mean-value theorem, which relates zero crossings of a function to zero crossings of its derivatives.

***A*) Descartes' Rule of Signs.** In a polynomial $\mathcal{P}$ which has order n and real coefficients a_k,

$$\mathcal{P}(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0,$$

the number of positive zero crossings z^* ($z^* \in \mathbb{R}$, $z^* > 0$) either is equal to the number of sign switches between subsequent nonzero coefficients or is an even number less than that number. Multiple roots are counted separately in this calculation.

***B*) Mean-Value Theorem.** If a smooth real function F is zero at two points $F(a) = 0$ and $F(b) = 0$, then there exists at least one point $a < z^* < b$ where $F'(z^*) = 0$. This statement implies that if there exists only one point z^* such that $F'(z^*) = 0$, then there can exist at most two zero crossings of F in $\mathbb{R}$. Otherwise, according to the mean-value theorem F' would have more than one zero crossing. Applied iteratively, this implies that if a continuous derivative F' has exactly N zero crossings, then the function F can have at most $N + 1$ zero crossings. Similarly, if the continuous kth derivative $F^{(k)}$ crosses the axis exactly N times, then the function F has at most $N + k$ zero crossings.

Now, we analyze the form of the characteristic function $\mathcal{F}$ in all nine parameter classes specified in Table 1. By z_0 we denote the point where P crosses the z axis. In each class we first address the case $n > 2$ and then the case $n = 2$.

i) Here, we consider $C_+ < 0$ and $\det\, J > 0$. Then $z_0 > 0$ and we obtain

$$
\mathcal{F}(z) = \begin{cases} J_{EE} z^n - J_{EI}(P(z))^n - z + g_E, & z_0 \leq z, \\ J_{EE} z^n - z + g_E, & 0 \leq z < z_0, \\ -z + g_E, & z < 0. \end{cases}
$$

First, we assume that $n > 2$. The function $\mathcal{F}$ has at most three zero crossings on $\mathbb{R}$. To show this, we consider $\mathcal{F}$ in the subintervals $(-\infty, 0)$, $(0, z_0)$, and $(z_0, +\infty)$. Using the *Proof* below we find that $\mathcal{F}''$ has at most one zero crossing in the interval $(z_0, +\infty)$ and is positive at the point z_0. Since $\mathcal{F}''$ is also positive in the interval $(0, z_0)$, we find that $\mathcal{F}''$ has only one zero crossing in the whole interval $(0, +\infty)$. Since $\mathcal{F}' = -1$ in $(-\infty, 0]$, we use mean-value theorem B and conclude that $\mathcal{F}'$ has at most two zero crossings on $\mathbb{R}$. Therefore, $\mathcal{F}$ has at most three zero crossings on $\mathbb{R}$. Hereby, the sign of $\mathcal{F}'$ at the zero crossings is alternating starting from $\mathcal{F}' < 0$ at the leftmost zero crossing. This leads to at most two zero crossings with the negative derivative and therefore to at most two stable steady states.

Proof:

Now we show that $\mathcal{F}''$ has at most one zero crossing in the interval (z_0, ∞). Specifically, we prove that there exists a unique point z_{cross} in the interval $(z_0, +\infty)$ such that $\mathcal{F}''(z_{\text{cross}}) = 0$, and $\mathcal{F}''$ is positive below z_{cross} and negative above z_{cross}. To start we calculate $\mathcal{F}''$:

$$
\begin{aligned}
\mathcal{F}''(z) = n(n-1)\Big(&- J_{EI} P^{n-2}(z)(P'(z))^2 \\
&+ z^{n-2}\Big(J_{EE} - n \cdot \det\, J \cdot P^{n-1}(z)\Big)\Big). \qquad \mathbf{[S8]}
\end{aligned}
$$

We recall that z_0 denotes the unique point where the monotonically increasing polynomial P crosses the axis; i.e., $P(z_0) = 0$. Then $P^{n-2}(z_0) = 0$ and $P^{n-1}(z_0) = 0$. We conclude that $\mathcal{F}''(z_0) = n(n-1) J_{EE} z_0^{n-2} > 0$. On the other hand we observe that $\lim_{z \to \infty} \mathcal{F}''(z) = -\infty$. This means that $\mathcal{F}''$ crosses the z axis in the interval $(z_0, +\infty)$ at least once. Our next goal is to show that a zero crossing of $\mathcal{F}''$ in $(z_0, +\infty)$ is unique.

To this end we substitute $P'(z) = n \cdot \det\, J \cdot J_{EI}^{-1} z^{n-1} + J_{EI}^{-1} J_{II}$ into Eq. **S8** and reformulate the condition $\mathcal{F}''(z) = 0$ via

$$
\begin{aligned}
&J_{EI}^{-1}(n \cdot \det\, J \cdot z^{n-1} + J_{II})^2 \\
&\quad = (J_{EE} - n \cdot \det\, J \cdot P^{n-1}(z)) z^{n-2} P^{2-n}(z). \qquad \mathbf{[S9]}
\end{aligned}
$$

Our next step is to show that there exists a unique point z_{cross}, which satisfies Eq. **S9**. To this end we prove that the functions on the right and left sides of Eq. **S9** intersect in the interval $(z_0, +\infty)$ at exactly one point z_{cross}. Specifically, we show that the function on the left side of Eq. **S9** is finite at z_0 and increases strictly monotonically to $+\infty$ whereas its counterpart on the right side is trending toward $+\infty$ at z_0 and decreases monotonically to $-\infty$.

Since the derivative of the function on the left side of Eq. **S9** is strictly positive, it increases strictly monotonically in the interval $(z_0, +\infty)$. Next, we address the function on the right side of Eq. **S9**. To simplify notation we name this function by H:

$$H(z) = (J_{EE} - n \cdot \det\, J \cdot P^{n-1}(z)) z^{n-2} P^{2-n}(z).$$

First, we find that $\lim_{z \to z_0^+} H(z) = +\infty$, which means that H is larger in the vicinity of the point z_0 than its counterpart on the left side of Eq. **S9**. Next, we prove that H decreases monotonically. To this end we show that $H' < 0$. We derive the derivative of H as

$$
\begin{aligned}
H'(z) = z^{n-3} P^{1-n}\Big(& J_{EE}(n-2)(P - zP') \\
&- n \cdot \det\, J \cdot (P^{n-1} zP' + (n-2)P^n)\Big).
\end{aligned}
$$

H' is negative in the interval $(z_0, +\infty)$ because the term in the parentheses is negative at z_0 and this term cannot grow because its derivative is negative. This implies that H decreases monotonically in $(z_0, +\infty)$ and intersects the monotonically increasing function on the left side of Eq. **S9** in exactly one point, z_{cross}.

Next, we address the case $n = 2$. Here, the mean-value theorem *B* cannot be applied to the function $\mathcal{F}''$ since it has jumps at 0 and z_0. However, $\mathcal{F}'$ is continuous. Therefore, we proceed with $\mathcal{F}'$ as

$$\mathcal{F}'(z) = \begin{cases} \sum_{k=0}^{3} a_k z^k, & z_0 \le z, \\ 2J_{EE} z - 1, & 0 \le z < z_0, \\ -1, & z < 0, \end{cases}$$

where the coefficients a_3 and a_2 are always negative whereas a_1 and a_0 are arbitrary real numbers. According to Descartes' rule of signs *A*, the polynomial $\sum_{k=0}^{3} a_k z^k$ can have at most two positive zero crossings provided a_1 is positive and a_0 is negative. In this case we obtain that $\sum_{k=0}^{3} a_k z^k$ first crosses the positive part of the z axis from below and then from above and finally decreases to $-\infty$. If $2J_{EE} z_0 - 1$ is nonnegative, then $\sum_{k=0}^{3} a_k z^k$ can cross the z axis for $z > z_0$ only once from above. On the contrary, if $2J_{EE} z_0 - 1$ is negative, then $\sum_{k=0}^{3} a_k z^k$ can have two zero crossings. Altogether we obtain that $\mathcal{F}'$ can have at most two zero crossings on $\mathbb{R}$. Then via the mean-value theorem *B* we conclude that $\mathcal{F}$ can cross the z axis at most three times with the negative derivative at the most left and right zero crossings. Therefore, the result for $n > 2$ remains valid for $n = 2$.

ii) Here, we consider $C_- < 0$ and $\det\, J < 0$. Then $z_0 > 0$ and we obtain

$$\mathcal{F}(z) = \begin{cases} J_{IE}(P(z))^n - J_{II} z^n - z + g_I, & z_0 \le z, \\ -J_{II} z^n - z + g_I, & 0 \le z < z_0, \\ -z + g_I, & z < 0. \end{cases}$$

We first analyze the case $n > 2$. The function $\mathcal{F}$ is monotonically decreasing in $(-\infty, z_0]$. Following the steps in class *i*, we find that there exists a unique point z_{cross} in the interval (z_0, ∞) such that $\mathcal{F}$ is strictly concave in (z_0, z_{cross}) and strictly convex in $(z_{\text{cross}}, +\infty)$. Since $\mathcal{F}$ decreases monotonically below z_{cross}, the first derivative $\mathcal{F}'$ has only one zero crossing, which is above z_{cross}. Therefore, the number of possible zero crossings of $\mathcal{F}$ is reduced to two, whereby we find $\mathcal{F}' < 0$ at the left and $\mathcal{F}' > 0$ at the right zero crossing. This means if $\mathcal{F}$ crosses the axis, then there is a stable and a saddle steady state. Finally, let us note that if we used the definition of $\mathcal{F}$ for $\det\, J \ge 0$ also for $\det\, J < 0$ in class *ii*, then we would no longer be able to provide a multiplicity bound. Our main argument is the observation that the monotonically decreasing function on the right side of Eq. **S9** intersects the monotonically increasing function on the left side of Eq. **S9** in exactly one point. This is no longer correct for $\det\, J < 0$, because both functions are not monotonic for $z > z_0$.

For $n = 2$ the function $\mathcal{F}'$ is continuous. We proceed as in class *i* and obtain

$$\mathcal{F}'(z) = \begin{cases} \sum_{k=0}^{3} a_k z^k, & z_0 \le z, \\ -2J_{II} z - 1, & 0 \le z < z_0, \\ -1, & z < 0, \end{cases}$$

whereby a_3 and a_2 are positive, a_0 is negative, and a_1 is either a positive or a negative real number. Descartes' rule of signs *A* informs us that $\sum_{k=0}^{3} a_k z^k$ crosses the positive part of the z axis exactly once from below and then increases to $+\infty$. Since this is the only intersection of $\mathcal{F}'$ with the z axis, $\mathcal{F}$ can have at most two zero crossings with the negative derivative at the left intersection. Thus, the results we derived for $n > 2$ remain valid for $n = 2$.

iii) Now we consider $C_+ < 0$ and $\det\, J = 0$. For $n > 2$ we rewrite $\mathcal{F}$ as in the case $\det\, J > 0$ in *i*. First, we consider $J_{EE} < J_{II}^n J_{EI}^{1-n}$ and find that for positive z the function $\mathcal{F}''$ is positive below the point z_{cross} and negative above it, where

$$z_{\text{cross}} = \frac{J_{II}^{-1} J_{EI} C_+}{-1 + (J_{EE} J_{II}^{-n} J_{EI}^{n-1})^{\frac{1}{n-2}}}. \qquad \textbf{[S10]}$$

We note that $\mathcal{F}'$ is negative for $z \le 0$. Since $\mathcal{F}''$ has only one zero crossing on the positive part of the z axis, then $\mathcal{F}'$ has at most two zero crossings and the function $\mathcal{F}$ can have up to three zero crossings on $\mathbb{R}$. $\mathcal{F}'$ starts off negative at the leftmost zero crossing and then switches signs two times. Therefore, the SSN model can be bistable in this class. For $J_{EE} \ge J_{II}^n J_{EI}^{1-n}$ we find that $\mathcal{F}''$ is positive for positive z. Since $\mathcal{F}'$ is negative for $z \le 0$, it has no zero crossings there. For $z > 0$ $\mathcal{F}'$ grows monotonically and converges to $+\infty$ as z increases. $\mathcal{F}'$ therefore has a unique positive zero crossing in the interval $z > 0$. Thus, $\mathcal{F}$ can have up to two zero crossings. The left crossing will have $\mathcal{F}' < 0$ and the right one $\mathcal{F}' > 0$.

For $n = 2$ we find that the maximal number of zero crossings is two. If $J_{EE} < J_{II}^2 J_{EI}^{-1}$, then $\mathcal{F}$ is a linearly decreasing function for negative z, a convex quadratic function between 0 and z_0, and a concave quadratic function to the right of z_0. However, we verified that $\mathcal{F}$ never can be negative between 0 and z_0. Thus, $\mathcal{F}$ has exactly one zero crossing with a negative derivative. If $J_{EE} > J_{II}^2 J_{EI}^{-1}$, then the form of $\mathcal{F}$ is the same as for $n > 2$, and $\mathcal{F}$ has at most two zero crossings. For $J_{EE} = J_{II}^2 J_{EI}^{-1}$ there is a slight difference to the case $n > 2$, because $\mathcal{F}(z) = (-2J_{II} C_+ - 1)z + g_E - J_{EI} C_+^2$ for $z \ge z_0$. Here, the behavior of $\mathcal{F}$ depends on the sign of the expression $-2J_{II} C_+ - 1$. If $-2J_{II} C_+ - 1 > 0$, then $\mathcal{F}$ has a unique minimum as in the case $J_{EE} > J_{II}^2 J_{EI}^{-1}$ and at most two zero crossings. If $-2J_{II} C_+ - 1 = 0$, then $\mathcal{F}$ is a positive constant to the right of the point z_0 which is simultaneously its minimum and does not have any zero crossings. Finally, $\mathcal{F}$ is monotonically decreasing if $-2J_{II} C_+ - 1 < 0$ and has exactly one zero crossing.

iv) Now, we address $\det\, J < 0$ and $C_- > 0$. We start with $n > 2$. In this case $z_0 < 0$ and we obtain

$$\mathcal{F}(z) = \begin{cases} J_{IE}(P(z))^n - J_{II} z^n - z + g_I, & 0 \le z, \\ J_{IE}(J_{IE}^{-1} J_{EE} z + C_-)^n - z + g_I, & z_0 \le z < 0, \\ -z + g_I, & z < z_0. \end{cases}$$

For $z > 0$ we rewrite P to have the form

$$(P(z))^n = \sum_{k=0}^{n^2} a_k z^k, \quad a_k \ge 0,$$

where the coefficients depend on the connectivity and input parameters. Then for $z > 0$

$$\mathcal{F}(z) = J_{IE} \sum_{k=n+1}^{n^2} a_k z^k + (a_n J_{IE} - J_{II}) z^n + J_{IE} \sum_{k=2}^{n-1} a_k z^k + (a_1 J_{IE} - 1)z + (a_0 J_{IE} + g_I)$$

and we apply Descartes' rule of signs to $\mathcal{F}$ for $z > 0$. The polynomial $\mathcal{F}$ above has at most four sign switches for $z > 0$. Next, we address the negative zero crossings of $\mathcal{F}$. $\mathcal{F}$ does not have zero crossings in the interval $[z_0, 0]$, because $\mathcal{F}(z) = 0$ would be equivalent to the relation $J_{IE}(P(z))^n + g_I = z$ which cannot be satisfied because its right side is negative and the left side is positive. For $z \le z_0$ the function $\mathcal{F}$ is positive. Therefore, $\mathcal{F}$ has at most four zero crossings on $\mathbb{R}$. Hereby, $\mathcal{F}'$ starts off negative at the leftmost point and then alternates signs. The SSN network can be bistable in this parameter class and have two saddle steady states. An example where $\mathcal{F}$ has four zero crossings is given in Fig. 2*D*. Finally, let us note that if we were to use the $\det J > 0$ definition of $\mathcal{F}$ for $\det J < 0$ in this case (as well as in class *vii*), then the number of sign switches in the polynomial $(P(z))^n$ would grow with n and we would not be able to derive an upper bound on the number of zero crossings.

Now we consider the case $n = 2$. Here, the characteristic function $\mathcal{F}$ can have at most two zero crossings for $z > 0$, because the term $J_{IE} \sum_{k=2}^{n-1} a_k z^k$ in the expression for $\mathcal{F}$ is equal to zero. Using the same argument as for $n > 2$ we conclude that $\mathcal{F}$ does not have zero crossings between z_0 and 0. Thus, the overall number of zero crossings does not exceed two for $n = 2$. In summary, the maximal number of steady states for $n = 2$ is less than that for $n > 2$.

v) Here, we consider $\det J > 0$ and $C_+ > 0$. We start with $n > 2$. In this case $z_0 < 0$ and

$$\mathcal{F}(z) = \begin{cases} J_{EE} z^n - J_{EI}(P(z))^n - z + g_E, & 0 \le z, \\ -J_{EI}(J_{EI}^{-1} J_{II} z + C_+)^n - z + g_E, & z_0 \le z < 0, \\ -z + g_E, & z < z_0. \end{cases}$$

We use the same representation for the polynomial P as in class *iv*. Then we rewrite $\mathcal{F}$ for $z > 0$ as

$$\mathcal{F}(z) = -J_{EI} \sum_{k=n+1}^{n^2} a_k z^k + (-a_n J_{EI} + J_{EE}) z^n - J_{EI} \sum_{k=2}^{n-1} a_k z^k + (-a_1 J_{EI} - 1) z + (-a_0 J_{EI} + g_E)$$

and apply Descartes' rule of signs to $\mathcal{F}'$ for $n > 2$. For positive z the number of sign switches between subsequent nonzero coefficients does not exceed two. Since $\mathcal{F}'$ is negative below 0, we obtain that $\mathcal{F}'$ has at most two zero crossings on $\mathbb{R}$. Therefore, $\mathcal{F}$ can have at most three zero crossings. $\mathcal{F}'$ starts off negative at the leftmost zero crossing and then alternates signs. There exist at most two stable steady states in this parameter class.

Now we address the case $n = 2$. Here, $J_{IE} \sum_{k=2}^{n-1} a_k z^k = 0$ in the expression for $\mathcal{F}$; however, the maximal number of steady states is the same as for $n > 2$. This is because the sign of coefficients in the expression for the continuous function $\mathcal{F}'$ can still switch two times.

vi) Here, we consider $\det J = 0$ and $C_+ > 0$. First, we consider $n > 2$. The functional form of $\mathcal{F}$ coincides with the expression we obtained for $\det J > 0$ and $C_+ > 0$ in class *v*. Following steps similar to those in class *iii*, we show that for $J_{EE} > J_{II}^n J_{EI}^{1-n}$ the function $\mathcal{F}''$ has a unique zero crossing z_{cross} for $z > 0$ given by Eq. **S10**. Moreover, $\mathcal{F}$ is concave and monotonically decreasing below and convex above the point z_{cross}. Thus, $\mathcal{F}$ has a unique minimum and can have at most two zero crossings. The characteristic function derivative $\mathcal{F}'$ starts off negative at the leftmost zero crossing and then switches sign. For $J_{EE} \le J_{II}^n J_{EI}^{1-n}$ $\mathcal{F}$ is concave and monotonically decreasing for all z, and therefore it can have exactly one stable steady state.

All of the conclusions hold for $n = 2$ with a small difference that z_{cross} in the case $J_{EE} > J_{II}^2 J_{EI}^{-1}$ coincides with 0.

vii) Now, we address $\det J < 0$ and $C_- = 0$. First, we consider $n > 2$. The polynomial P intersects the z axis in 0 and therefore, $z_0 = 0$. We obtain

$$\mathcal{F}(z) = \begin{cases} J_{IE}(P(z))^n - J_{II} z^n - z + g_I, & 0 \le z, \\ -z + g_I, & z < 0, \end{cases}$$

whereby we still use the representation $(P(z))^n = \sum_{k=n}^{n^2} a_k z^k$ with $a_k \ge 0$. We obtain for $z > 0$

$$\mathcal{F}(z) = J_{IE} \sum_{k=n+1}^{n^2} a_k z^k + (a_n J_{IE} - J_{II}) z^n - z + g_I$$

and apply Descartes' rule of signs to $\mathcal{F}'$. $\mathcal{F}'$ crosses the z axis at most once, because $\mathcal{F}'(z) = -1 < 0$ for $z \le 0$ and $\mathcal{F}'$ has at most one sign switch for $z > 0$. Thus, $\mathcal{F}$ can have at most two zero crossings on $\mathbb{R}$. The sign of $\mathcal{F}'$ starts off negative at the leftmost zero crossing and then switches sign, leaving only the possibility for one stable steady state. The results above remain valid for $n = 2$, because the same arguments can be applied to the continuous function $\mathcal{F}'$.

viii) Now, we consider $\det J > 0$ and $C_+ = 0$. We start with $n > 2$. As in class *vii* we obtain $z_0 = 0$ and

$$\mathcal{F}(z) = \begin{cases} J_{EE} z^n - J_{EI}(P(z))^n - z + g_E, & 0 \le z, \\ -z + g_E, & z < 0. \end{cases}$$

Here, we also apply Descartes' rule of signs to $\mathcal{F}'$ and use the form of P from class *vii*. We obtain for $z > 0$

$$\mathcal{F}(z) = -J_{EI} \sum_{k=n+1}^{n^2} a_k z^k + (-a_n J_{EI} + J_{EE}) z^n - z + g_E.$$

We note that for $n > 2$ the number of sign switches between subsequent nonzero coefficients of $\mathcal{F}'$ for $z > 0$ can be at most two. We also obtain that $\mathcal{F}'(z) = -1 < 0$ for $z \le 0$. Therefore, $\mathcal{F}'$ can have at most two zero crossings for all z and $\mathcal{F}$ has at most three zero crossings. $\mathcal{F}'$ starts off negative at the leftmost point and then alternates signs. Two stable states are possible in this parameter class. A parameter set with three zero crossings is shown in Fig. 3. Since $\mathcal{F}'$ for $n = 2$ is continuous, we apply similar arguments for $n = 2$ to derive the same conclusions.

ix) Finally, we consider $\det J = 0$, $C_+ = 0$, and $n \ge 2$. First, we note that the functional form of $\mathcal{F}$ is the same as in class *viii* except that for $z \ge 0$ the expression for $\mathcal{F}$ simplifies to $\mathcal{F}(z) = (J_{EE} - J_{EI}^{1-n} J_{II}^n) z^n - z + g_E$. For $J_{EE} > J_{II}^n J_{EI}^{1-n}$ the function $\mathcal{F}'$ has one zero crossing for $z > 0$ and is negative for $z < 0$. Thus, $\mathcal{F}$ can have at most two zero crossings. $\mathcal{F}'$ starts off negative at the leftmost point and then alternates signs. If $J_{EE} \le J_{II}^n J_{EI}^{1-n}$, then $\mathcal{F}$ is monotonically decreasing for all z such that $\mathcal{F}$ always has exactly one zero crossing with a negative derivative. Therefore, there exists at most one stable steady state in this class.

Here, we briefly summarize our multiplicity results. We have shown that for $\det J > 0$ the SSN has an odd number of steady states, either one or three. For $\det J < 0$ the number of steady states is even. In particular, this number is four or two or zero. The case $\det J = 0$ is an intermediate one: For a subclass of the SSN parameters indicated in classes *iii*, *vi*, and *ix* the characteristic function behaves similarly to the case $\det J > 0$ and for the rest of the parameters it is similar to $\det J < 0$. Based on our results in the section *Stability Conditions* in the main text we find a unique order of steady-state types. Since both r_E and r_I are strictly monotonically increasing functions of z, the steady states can be uniquely ordered according to their distance from the origin. Hereby, stability of steady states alternates as the sign of the characteristic function derivative switches: The steady states with an odd number are either stable or repelling and steady states with an even number are saddles.

Existence of Persistent States

Here, we present the steps for finding positive, stable steady states in the absence of inputs. Such steady states are often referred to as persistent states. Based on the results for the characteristic function, we know that a stable persistent state of the model in Eq. **1** exists if two conditions are fulfilled. First, for zero input the characteristic function $\mathcal{F}$ has a positive zero crossing with a negative derivative and second, the corresponding steady state satisfies the stability condition we derived in Eq. **5**. Below, we consider these two conditions in more detail and start with the first condition.

To find model parameters for a persistent state we start by noting that by setting the inputs $g_E = g_I = 0$ we automatically have $C_\pm = 0$. In the previous section we identified nine parameter classes of which only classes *vii*, *viii*, and *ix* comprise constants $C_\pm = 0$. Furthermore, we find that in these parameter classes the trivial point $z = 0$ is the first zero crossing of $\mathcal{F}$ from the left and that $\mathcal{F}'(0) = -1$. Thus, to obtain a positive zero crossing $z^* > 0$ with a negative derivative $\mathcal{F}'(z^*) < 0$ the characteristic function needs to have at least three zero crossings. This is possible only in the parameter class *viii* where $\det\ J > 0$, because in classes *vii* and *ix* the overall number of zero crossings does not exceed two. To obtain two positive zero crossings in addition to the trivial zero crossing at zero, the coefficients of the characteristic function have to switch signs at least two times in accordance with Descartes' rule of signs. Using the representation of the characteristic function for class *viii* in the previous section we recognize that all terms in the characteristic function except $(-J_{II}^n J_{EI}^{1-n} + J_{EE}) \cdot z^n$ are necessarily negative. Thus, for the coefficients of $\mathcal{F}$ to switch signs twice we need $-J_{II}^n J_{EI}^{1-n} + J_{EE} > 0$ to be fulfilled. This restricts the parameter set supporting a persistent state to $\det\ J > 0$ and $-J_{II}^n J_{EI}^{1-n} + J_{EE} > 0$. For consistency we rewrite the second inequality such that it represents an upper bound for $\det\ J$. To this end we multiply it by J_{II}, add to both sides the term $J_{EI} J_{IE}$, and move $J_{EE} J_{II}$ to the other side. The resulting two-sided estimate for $\det\ J$ is

$$0 < \det\ J < J_{EI} J_{IE} - J_{II}^{1+n} J_{EI}^{1-n}. \qquad \textbf{[S11]}$$

Outside of the parameter set specified by Eq. **S11** a persistent state does not exist. We now clarify whether it always exists when the SSN parameters belong to this class or the actual set of parameters is smaller. To this end we derive the exact condition when the characteristic function has a positive zero crossing with a negative derivative in the parameter class specified by Eq. **S11**. First, we recall that the zero crossings of $\mathcal{F}$ for $\det\ J > 0$ are solutions of the equation

$$\mathcal{F}(z) = z(J_{EE} z^{n-1} - J_{EI}^{1-n} z^{n-1} (\det\ J \cdot z^{n-1} + J_{II})^n - 1) = 0.$$

Since we are interested only in positive solutions of the above equation, we cancel the factor z and obtain

$$z^{n-1}(J_{EE} - J_{EI}^{1-n} (\det\ J \cdot z^{n-1} + J_{II})^n) = 1. \qquad \textbf{[S12]}$$

Next, we substitute

$$x = \det\ J \cdot z^{n-1} + J_{II} \qquad \textbf{[S13]}$$

into Eq. **S12** to simplify notation and obtain

$$(x - J_{II})(J_{EE} - J_{EI}^{1-n} x^n) = \det\ J. \qquad \textbf{[S14]}$$

Eq. **S13** implies that the new variable x satisfies $x > J_{II}$. Since $\det\ J > 0$, Eq. **S14** has solutions only on a subset of x for which the function on the left side is positive. Therefore, Eq. **S14** is constrained to the interval $(J_{II}, J_{EI}^{\frac{n-1}{n}} J_{EE}^{\frac{1}{n}})$. Let us note that this interval is not empty because we limited the parameters to the set $J_{EE} > J_{II}^n J_{EI}^{1-n}$. We recognize that the function on the left side of Eq. **S14** is strictly concave in the interval of interest and is equal to zero at the boundary points. Therefore, it has a unique maximum x_0 in this interval. To guarantee that this function crosses the value $\det\ J$ its maximum has to be higher than $\det\ J$. This is equivalent to the requirement that the inequality Eq. **17**

$$0 < \det\ J < (J_{EE} - J_{EI}^{1-n} x_0^n)(x_0 - J_{II})$$

is fulfilled, which we state in *Materials and Methods*. For completeness, let us note that the equation $\det\ J = (J_{EE} - J_{EI}^{1-n} x_0^n)(x_0 - J_{II})$ is fulfilled only by a zero crossing of the function $\mathcal{F}$ with $\mathcal{F}' = 0$; this is a local maximum which corresponds to a steady state with a zero eigenvalue. Since we have ignored throughout our article the nongeneric cases (destroyed by any perturbation of parameters) in which there is a steady state with a zero eigenvalue, we do not consider the case $\det\ J = (J_{EE} - J_{EI}^{1-n} x_0^n)(x_0 - J_{II})$ and use the $<$ sign in the equation above. We now proceed by recognizing that x_0 is the maximum of the function on the left side of Eq. **S14**. It is the unique solution of the vanishing derivative equation Eq. **18**

$$(n+1)x_0^n - J_{II} n x_0^{n-1} - J_{EI}^{n-1} J_{EE} = 0$$

in the interval $(J_{II}, J_{EI}^{\frac{n-1}{n}} J_{EE}^{\frac{1}{n}})$ presented in *Materials and Methods*. For $n = 2$ we can rewrite the upper estimate in the inequality Eq. **17** using only connectivity constants because the explicit solution of Eq. **18** can be easily determined. We obtain $x_0 = \frac{1}{3}(J_{II} + \sqrt{J_{II}^2 + 3J_{EI}J_{EE}})$. Then, the upper estimate in Eq. **17** contains only the connectivity constants

$$\det\ J < \frac{2}{27} J_{II}^3 J_{EI}^{-1} - \frac{2}{3} J_{II} J_{EE} + \frac{2}{27} J_{EI}^{-1} (J_{II}^2 + 3J_{EI}J_{EE})^{3/2}.$$

For $n \geq 2$ the estimate in Eq. **17** represents an exact upper bound for $\det\ J$. This means that outside of the parameter set specified by Eq. **17** a persistent state does not exist and moreover, if the connectivity constants and the power-law exponents fulfill Eq. **17**, then a persistent state always exists. Let us note that exact solutions to Eq. **18** can be found also for $n = 3, 4$; however, already here the corresponding expressions are long and difficult to interpret. Therefore, we decided to provide an easy-to-verify upper bound for $\det\ J$ in Eq. **6** of the main text which is valid for all $n \geq 2$. Using Eq. **6** our readers can quickly check whether a specific parameter set is a persistent-state candidate.

In the following, we prove that the upper bound we provide in Eq. **6** is tighter than Eq. **S11**, but in general larger than the exact upper bound in Eq. **17**. To this end, we first show that the set defined by Eq. **17** is contained in Eq. **6**. Second, we verify that Eq. **6** is tighter than Eq. **S11**. This would mean that the parameter set specified by Eq. **17** is contained in the set specified by Eq. **S11**. To verify that the set specified by Eq. **17** is contained in that of Eq. **6** we use Eq. **18** to derive a lower bound for x_0^n:

$$x_0^n > \frac{J_{EI}^{n-1} J_{EE}}{n+1}.$$

Next, we derive an upper bound on x_0. Since we originally defined x_0 to be in the interval $(J_{II}, J_{EI}^{(n-1)/n} J_{EE}^{1/n})$, we obtain that x_0 is constrained by $x_0 < J_{EI}^{(n-1)/n} J_{EE}^{1/n}$. Now we insert the lower bound for x_0^n into the first bracket and the upper bound on x_0 into the second bracket of Eq. **17** to obtain

$$0 < \det\ J < n/(n+1)(J_{EE}^{(n+1)/n} J_{EI}^{(n-1)/n} - J_{II} J_{EE}).$$

We have presented this inequality as a necessary condition for a persistent state in Eq. **6**.

Now, we move on to prove that the set defined by Eq. **6** is contained in Eq. **S11**. To this end, we show that if the inequality in Eq. **6** is fulfilled, then Eq. **S11** holds as well. Since $n/(n+1) < 1$, Eq. **6** implies the estimate

$$\det\ J < J_{EE}^{(n+1)/n} J_{EI}^{(n-1)/n} - J_{II} J_{EE}.$$

Next, we apply the definition $\det J = -J_{EE}J_{II} + J_{EI}J_{IE}$ to the left side of the inequality above, cancel $J_{II}J_{EE}$, raise the relation to the nth power, and obtain the following estimate:

$$J_{EI}J_{EE}^{-1} < J_{EE}^{n}J_{IE}^{-n}.$$

We have shown that the last estimate follows from Eq. **6** and now we prove that it implies the inequality in Eq. **S11**. To this end we reformulate the inequality in Eq. **S11**. We use the definition of $\det J$ again and cancel the term $J_{EI}J_{EI}$ on both sides of Eq. **S11**, rearrange the terms, and obtain an estimate equivalent to Eq. **S11**:

$$J_{EI}J_{EE}^{-1} < J_{II}^{-n}J_{EI}^{n}.$$

The condition $\det J > 0$ is equivalent to $J_{EE}^{n}J_{IE}^{-n} < J_{II}^{-n}J_{EI}^{n}$. Then the inequality $J_{EI}J_{EE}^{-1} < J_{EE}^{n}J_{IE}^{-n}$, which follows from Eq. **6**, implies the inequality $J_{EI}J_{EE}^{-1} < J_{II}^{-n}J_{EI}^{n}$ which is equivalent to Eq. **S11**, because the right side of the first inequality $J_{EE}^{n}J_{IE}^{-n}$ is smaller than $J_{II}^{-n}J_{EI}^{n}$. Thus, Eq. **6** is tighter than Eq. **S11**.

We have achieved our goal and have proved that the exact connectivity range corresponding to a persistent state as given by Eq. **17** belongs to the set given by Eq. **6** and the estimate Eq. **6** is tighter than Eq. **S11**. We have included Eq. **6** in the main text as a necessary persistent-state condition because it is easy to verify for any given J and n.

In summary, to guarantee the existence of a stable persistent state the following three conditions need to be fulfilled: First, the excitatory feedback needs to be sufficiently strong, $J_{EE} > J_{II}^{n}J_{EI}^{1-n}$. This requirement implies that the interval $(J_{II}, J_{EI}^{\frac{n-1}{n}}J_{EE}^{\frac{1}{n}})$ is not empty. Second, the inequality in Eq. **17** in *Materials and Methods* needs to be fulfilled. This condition implies a specific balance between the SSN parameters such that together they lead to a positive zero crossing of the characteristic function with the negative derivative. The final third condition is to verify that the persistent state is dynamically stable and thus fulfills the stability condition in Eq. **5**. This last condition can be achieved by choosing a sufficiently large excitatory time constant τ_E. Examples of parameters leading to a stable persistent state can be found in Table 2 and a corresponding steady state is presented in Fig. 3*B*.

Existence of Global Oscillations

Here, we present the details for how to prove the existence of stable limit cycles in the SSN network. To this end we determine model parameters leading to a Hopf bifurcation. A Hopf bifurcation takes place when the eigenvalues corresponding to a steady state cross the imaginary axis while one of the model parameters varies. Therefore, to obtain a Hopf bifurcation it is necessary to show existence of steady states with purely imaginary eigenvalues of the Jacobian.

We first consider the eigenvalues $\lambda_{1,2}$ of the Jacobian DG given by Eq. **S6** and recognize that they are purely imaginary for $\mathrm{Tr}\, DG = 0$ and $\det DG > 0$.

The condition $\mathrm{Tr}\, DG = 0$ is equivalent to

$$\frac{\tau_E}{\tau_I} = \frac{J_{EE}\alpha_E r_E^{\frac{\alpha_E - 1}{\alpha_E}} - 1}{J_{II}\alpha_I r_I^{\frac{\alpha_I - 1}{\alpha_I}} + 1}. \qquad \textbf{[S15]}$$

Eq. **S15** can be fulfilled only if the following holds:

$$J_{EE}\alpha_E r_E^{\frac{\alpha_E - 1}{\alpha_E}} > 1. \qquad \textbf{[S16]}$$

According to Eq. **16** the relation $\det DG > 0$ is equivalent to

$$\mathcal{F}'(z) < 0, \qquad \textbf{[S17]}$$

where z is a zero crossing of the characteristic function corresponding to the steady-state r_E, r_I. With $\mathrm{Tr}\, DG = 0$ and Eq. **16** the eigenvalues reduce to

$$\lambda_{1,2} = \pm\sqrt{-\det DG} = \pm i\sqrt{-\tau_E^{-1}\tau_I^{-1}\mathcal{F}'(z)}.$$

In Fig. 4 we present an example of a supercritical Hopf bifurcation, i.e., the bifurcation leading to a stable limit cycle. To prove that a stable limit cycle emerges after a stable point turns into a repeller during this Hopf bifurcation we perform the following steps. First, we determine a steady state which corresponds to a zero crossing of the characteristic function with $\mathcal{F}'(z) < 0$ and satisfies the condition in Eq. **S16**. We find that for $\det J > 0$ the inequality Eq. **S16** can always be satisfied by choosing a sufficiently large input g_E, since r_E increases strictly monotonically with g_E. Next, we ensure that the time constants τ_E, τ_I fulfill the zero trace condition in Eq. **S15**. We note that to control the frequency of the future limit cycle which is $f = (2\pi)^{-1}\sqrt{-\tau_E^{-1}\tau_I^{-1}\mathcal{F}'(z)}$, both τ_E and τ_I can be varied. This can be accomplished because Eq. **S15** places constraints only on the ratio τ_E/τ_I but not on their absolute values. While the input g_E increases and crosses the value in Eq. **S15**, the stable steady state bifurcates to a repelling one. We denote the inputs and the steady state corresponding to the bifurcation point by g_E^0 and g_I^0 and r_E^0 and r_I^0, respectively. By meeting Eq. **S15** we have achieved that at the bifurcation point the eigenvalues corresponding to the steady-state r_E^0 and r_I^0 are purely imaginary complex conjugate numbers. Next, we cross the bifurcation point by increasing the parameter g which we add to the excitatory input: $g_E^0 \to g_E^0 + g$. As g crosses the origin the real part of the eigenvalues turns positive, the stable steady-state r_E^0 and r_I^0 becomes repelling, and a limit cycle emerges (Fig. 4). The next task is to verify that the emerging limit cycle is stable.

A planar dynamical system which undergoes a Hopf bifurcation can be transformed to a normal form by an appropriate substitution of variables (ref. 1, pp. 91–99). In the normal form, the coefficient in front of the cubic term evaluated at the bifurcation point is referred to as the Lyapunov coefficient. Its sign determines whether the resulting limit cycle is stable (supercritical Hopf bifurcation) or unstable (subcritical Hopf bifurcation). To show that the bifurcation presented in Fig. 4 is supercritical we compute here the Lyapunov coefficient using the equation on p. 99 in ref. 1 and prove that it is negative in the vicinity of the bifurcation point. For this calculation we have assumed, similar to that in *Multiplicity* of *Steady States*, that the power-law exponents are equal integers $\alpha_E = \alpha_I = n$. To apply the theory from ref. 1 we first move the steady state to the origin via a g-dependent coordinate transformation $\tilde{r}_E = r_E - r_E^0$, $\tilde{r}_I = r_I - r_I^0$. For the new variables $\tilde{r}_E$, $\tilde{r}_I$ the SSN dynamics read

$$\begin{aligned}\tau_E\dot{\tilde{r}}_E &= -\tilde{r}_E - r_E^0 + (J_{EE}\tilde{r}_E - J_{EI}\tilde{r}_I + z_E^0)_+^n \\ \tau_I\dot{\tilde{r}}_I &= -\tilde{r}_I - r_I^0 + (J_{IE}\tilde{r}_E - J_{II}\tilde{r}_I + z_I^0)_+^n,\end{aligned} \qquad \textbf{[S18]}$$

whereby the rescaled inputs are given by $(z_E^0(g), z_I^0(g)) = J(r_E^0(g), r_I^0(g)) + (g_E^0 + g, g_I^0)$ and the connectivity matrix is as before, $J = (J_{EE}, -J_{EI}; J_{IE}, -J_{II})$. The Jacobian matrix DG of the system in Eq. **S18** and its eigenvalues and eigenvectors depend on g. Since the dynamical system undergoes a Hopf bifurcation at $g = 0$, the eigenvalues of DG are complex conjugates of each other in the vicinity of the point $g = 0$. By $q(g)$ we denote the eigenvector of the matrix $DG(g)$ corresponding to the eigenvalue $\lambda(g) = 1/2\ TrDG(g) + i/2\sqrt{-(\mathrm{Tr}DG(g))^2 + 4\det DG(g)}$. By $p(g)$ we denote the eigenvector of the transposed matrix $DG^T(g)$ corresponding to the complex conjugate eigenvalue $\bar{\lambda}(g)$. We assume that p and q are chosen such that they fulfill $\langle p, q\rangle = \bar{p}_1 q_1 + \bar{p}_2 q_2 = 1$. By p_1, q_1 we denote the first and by p_2, q_2

the second entries of the respective vectors and by $\bar{p}_1, \bar{q}_1$ and $\bar{p}_2, \bar{q}_2$ their respective complex conjugates. For the corresponding theory we refer to ref. 1. For $g = 0$ we have $\mathrm{Tr}\ DG(0) = 0$ and $\lambda(0)$ is a purely imaginary complex number. Now, we follow the steps outlined in ref. 1 and make a coordinate transformation. We substitute the 2D real variable $\tilde{r} = (\tilde{r}_E, \tilde{r}_I)$ with a 1D complex variable z which we define as $z = \langle p, \tilde{r} \rangle$. The inverse transformation is given via the equation $\tilde{r} = zq + \bar{z}\bar{q}$. For the variable z the system in Eq. **S18** can be expressed as a Taylor series with respect to the variables z and $\bar{z}$,

$$\dot{z} = \lambda z + \sum_{2 \le k+l \le 3} \frac{g_{kl}}{k!l!} z^k \bar{z}^l + O(|z|^4), \qquad \textbf{[S19]}$$

with the complex numbers $g_{kl}(g)$. Then the Lyapunov coefficient l_1 takes the form specified on p. 99 in ref. 1,

$$l_1(0) = \frac{1}{2 \det\ DG(0)} \mathbf{Re}(i g_{20}(0) g_{11}(0) + g_{21}(0)\sqrt{\det\ DG(0)}),$$

whereby $\mathbf{Re}(\cdot)$ denotes the real part. The complex numbers $g_{20}(0)$, $g_{11}(0)$, and $g_{21}(0)$ are the coefficients in front of the terms z^2, $z\bar{z}$, and $z^2\bar{z}$ on the right side of the Taylor expansion in Eq. **S19** corresponding to $g = 0$, respectively. Using the form in Eq. **S19** we have computed these coefficients for the SSN model at the input value $g = 0$ and obtained

$$\begin{aligned} g_{20} &= n(n-1)(\bar{p}_1 (z_E^0)^{n-2} \beta_1^2 \tau_E^{-1} + \bar{p}_2 (z_I^0)^{n-2} \beta_2^2 \tau_I^{-1}), \\ g_{11} &= n(n-1)(\bar{p}_1 (z_E^0)^{n-2} |\beta_1|^2 \tau_E^{-1} + \bar{p}_2 (z_I^0)^{n-2} |\beta_2|^2 \tau_I^{-1}), \\ g_{21} &= n(n-1)(n-2)(\bar{p}_1 (z_E^0)^{n-3} \beta_1 |\beta_1|^2 \tau_E^{-1} + \\ &\quad \bar{p}_2 (z_I^0)^{n-3} \beta_2 |\beta_2|^2 \tau_I^{-1}). \end{aligned}$$

Here, $(z_E^0, z_I^0) = (z_E^0(0), z_I^0(0))$, and the eigenvectors $p(0)$ and $q(0)$ at $g = 0$ are chosen as

$$p(0) = (p_1, p_2) = \frac{1}{2b\sqrt{\det\ DG(0)}} \left(\sqrt{\det\ DG(0)} + ia, ib\right),$$
$$q(0) = (q_1, q_2) = \left(b, -a + i\sqrt{\det\ DG(0)}\right),$$

whereby

$$\begin{aligned} a &= -\tau_E^{-1} + \tau_E^{-1} J_{EE} n (z_E^0(0))^{n-1}, \\ b &= -\tau_E^{-1} J_{EI} n (z_E^0(0))^{n-1}, \\ \beta &= (\beta_1, \beta_2) = Jq(0). \end{aligned}$$

For convenience, we provide Mathematica code for our readers (*Code Availability* in the main text) which implements the above Lyapunov coefficient and calculates its value for the example we considered in Fig. 4. We find that the Lyapunov coefficient in Fig. 4 is approximately $-1{,}244.41$ in the vicinity of the bifurcation point and the emerging limit cycle is therefore stable. This proves that the SSN model can support stable limit cycles.

Rescaling the Connectivity and Input Constants to Match a Desired Firing-Rate Range

We assume that the firing rates $r_E > 0$ and $r_I > 0$ satisfy the SSN steady-state equations

$$r_E = (J_{EE} r_E - J_{EI} r_I + g_E)_+^{\alpha_E} \quad r_I = (J_{IE} r_E - J_{II} r_I + g_I)_+^{\alpha_I}.$$

If the values of r_E and r_I are not within the desired range for a given application, then it is possible to rescale them to any arbitrary pair of values $\bar{r}_E > 0$, $\bar{r}_I > 0$. To map $\{r_E, r_I\} \to \{\bar{r}_E, \bar{r}_I\}$ such that the new values remain a steady state of the SSN we rescale the connectivity and input constants as follows:

$$\bar{J}_{EE} = J_{EE} \left(\frac{r_E}{\bar{r}_E}\right)^{1-\frac{1}{\alpha_E}}, \quad \bar{J}_{EI} = \frac{J_{EI} r_I}{\bar{r}_I} \left(\frac{r_E}{\bar{r}_E}\right)^{-\frac{1}{\alpha_E}},$$
$$\bar{J}_{IE} = \frac{J_{IE} r_E}{\bar{r}_E} \left(\frac{r_I}{\bar{r}_I}\right)^{-\frac{1}{\alpha_I}}, \quad \bar{J}_{II} = J_{II} \left(\frac{r_I}{\bar{r}_I}\right)^{1-\frac{1}{\alpha_I}},$$
$$\bar{g}_E = g_E \left(\frac{r_E}{\bar{r}_E}\right)^{-\frac{1}{\alpha_E}}, \quad \bar{g}_I = g_I \left(\frac{r_I}{\bar{r}_I}\right)^{-\frac{1}{\alpha_I}}.$$

The new, rescaled steady state now fulfills the following equations:

$$\bar{r}_E = (\bar{J}_{EE} \bar{r}_E - \bar{J}_{EI} \bar{r}_I + \bar{g}_E)_+^{\alpha_E}, \ \bar{r}_I = (\bar{J}_{IE} \bar{r}_E - \bar{J}_{II} \bar{r}_I + \bar{g}_I)_+^{\alpha_I}.$$

Location of SSN Steady States Is Preserved in the Presence of Firing-Rate Saturation

The power-law activation function in the SSN model belongs to the class of unbounded activation functions. Other prominent members of this class are generalized linear models (GLMs) with exponential nonlinearities or rate models with unbounded threshold linear activation functions. In general, this class of models does not provide mechanisms to keep firing rates below a biologically plausible maximal value, which exists due to physiological constraints of neurons. Instead, these models are designed to describe the activity in a regime that reflects the operational state in vivo. This state is located far below the maximal firing rate and is often fluctuation driven (2, 3). One possibility to control growth of firing rates in a rate model is enforcing saturation in the activation function. However, it is an open question how the presence of a maximal firing rate in the activation function impacts the dynamics and the location of steady states. There are two conceivable scenarios. First, the introduction of saturation fundamentally alters the dynamics and the steady states. Second, it is possible to identify some steady states of the saturating system by considering a corresponding unbounded model. Below, we show that the second scenario applies for the SSN model provided the saturation threshold is far above the operational state of neurons. Specifically, we prove that the SSN steady states calculated for the unbounded power-law activation function (see Eq. **2**) can also be found in the corresponding bounded network (Fig. S2). At the same time, saturation in the activation function imposes global boundedness for the firing-rate trajectories. For convenience, we present the corresponding upper bounds for the firing rates at the end of this section.

To show that the steady states of the unbounded SSN model can be found in a bounded model we consider the following saturating activation functions for the excitatory and inhibitory neurons:

$$(g)_{+,B}^{\alpha_E} = \begin{cases} 0, & g < 0 \\ g^{\alpha_E}, & 0 \le g < B, \\ B^{\alpha_E}, & B \le g, \end{cases} \qquad \textbf{[S20]}$$
$$(g)_{+,B}^{\alpha_I} = \begin{cases} 0, & g < 0 \\ g^{\alpha_I}, & 0 \le g < B, \\ B^{\alpha_I}, & B \le g. \end{cases}$$

We assume B to be large but finite to keep the model tractable. Here, we show that all steady states of the unbounded SSN model in Eq. **1** are also steady states in the model with a saturating activation function as in Eq. **S20**, provided that B is chosen sufficiently large. In addition to the steady states which are “inherited” from the original unbounded SSN model the bounded model can exhibit additional steady states whose location we also discuss below. We first introduce the characteristic function for the bounded model

$$\mathcal{F}_B(z) = \begin{cases} J_{EE}(z)_{+,B}^{\alpha_E} - J_{EI}(P_B(z))_{+,B}^{\alpha_I} - z + g_E, & \det\ J \geq 0, \\ J_{IE}(P_B(z))_{+,B}^{\alpha_E} - J_{II}(z)_{+,B}^{\alpha_I} - z + g_I, & \det\ J < 0, \end{cases}$$

where the function P_B is defined by

$$P_B(z) = \begin{cases} \det\ J \cdot J_{EI}^{-1}(z)_{+,B}^{\alpha_E} + J_{EI}^{-1}J_{II}z + C_+, & \det\ J \geq 0 \\ -\det\ J \cdot J_{IE}^{-1}(z)_{+,B}^{\alpha_I} + J_{IE}^{-1}J_{EE}z + C_-, & \det\ J < 0. \end{cases}$$

Now, we discuss how to find the threshold B and a point $z_B \leq B$ such that the interval $(-\infty, z_B)$ contains all zero crossings of the function $\mathcal{F}$ and $\mathcal{F}$ coincides with $\mathcal{F}_B$ on $(-\infty, z_B)$. This would imply that steady states of the SSN can also be found in the bounded model.

First, we choose z_B to be larger than the rightmost zero crossing of $\mathcal{F}$ if zero crossings exist; otherwise z_B is arbitrary. This will satisfy the requirement that $(-\infty, z_B)$ contains all zero crossings of $\mathcal{F}$.

The goal now is to choose B and if necessary adapt z_B such that $(P_B(z))_{+,B}$ coincides with $(P(z))_+$ and $(z)_{+,B}$ with $(z)_+$ on the interval $(-\infty, z_B)$. This would imply that $\mathcal{F}$ coincides with $\mathcal{F}_B$ on this interval.

If we choose $B \geq z_B$, then naturally $(z)_{+,B} = (z)_+$ and $P_B(z) = P(z)$ on $(-\infty, z_B)$. It remains to show that the saturated version of the function P, $(P)_{+,B}$, coincides with $(P)_+$ on $(-\infty, z_B)$. To fulfill this relation we need to carefully choose z_B and eventually shift it to the right, because P can theoretically exceed B before its argument z reaches the point z_B. To analyze the behavior of P and z systematically we consider four parameter classes and explain how to choose z_B and B to satisfy $(P)_{+,B} = (P)_+$ on $(-\infty, z_B)$ in each of them:

- For $\det\ J \neq 0$ or $\det\ J = 0$ with $J_{EI}^{-1}J_{II} > 1$ the polynomial P is either located above the line $y = z$ or crosses this line from below at some point on the z axis and remains above this line for all larger z. In the case where P lies above the line $y = z$, the point z_B can remain at the same location. If P crosses $y = z$ from below, we shift the point z_B rightward such that it is located to the right side from the intersection point. Then, we set $B = P(z_B)$. This ensures that B satisfies $B > z_B$, because $P(z_B) > z_B$. This class is the most prevalent in the SSN model.
- For $\det\ J = 0$ with $J_{EI}^{-1}J_{II} < 1$ the line $y = z$ intersects the straight line P from below at one point. Here, we shift z_B rightward to ensure that z_B is located to the right side from the intersection point and set $B = z_B$.
- For $\det\ J = 0$ with $J_{EI}^{-1}J_{II} = 1$ and $C_\pm \leq 0$ the line $y = z$ is above and parallel or coincides with the straight line P on the whole axis. Here, we do not change the location of z_B and set $B = z_B$.
- For $\det\ J = 0$ with $J_{EI}^{-1}J_{II} = 1$ and $C_\pm > 0$ the straight line P lies above and parallel with respect to the line $y = z$ on the whole z axis. Here, we do not change the location of z_B and set $B = P(z_B)$.

Herewith we have shown how to choose a sufficiently large threshold current B such that $\mathcal{F}$ and $\mathcal{F}_B$ coincide on the interval $(-\infty, z_B)$ with an appropriate $z_B \leq B$. Hereby, the interval $(-\infty, z_B)$ contains all zero crossings of $\mathcal{F}$. For such B, all steady states of the SSN model are automatically also steady states of the saturated model. In other words, introducing saturation to the excitatory and inhibitory activation functions in Eq. **1** preserves the location of the SSN steady states if B is sufficiently large. We illustrate this finding in Fig. S2. If the threshold B is smaller than we specified above, then $\mathcal{F}$ and $\mathcal{F}_B$ will coincide on a smaller interval and not all but only a subset of the steady states will carry over. How many steady states will carry over will depend on the specific magnitude of B.

In addition to the steady states which are inherited from the unbounded activation function, new steady states can appear in the saturating SSN model. Below we show that saturation of the form in Eq. **S20** can introduce zero, one, or even two additional steady states. For concreteness, we consider here the case $\det\ J \neq 0$. Since the zero crossings of $\mathcal{F}_B$ and $\mathcal{F}$ coincide in the interval $(-\infty, z_B)$, additional zero crossings can appear only for $z > z_B$ and lead to the new steady states. These steady states are unique to $\mathcal{F}_B$, they do not appear in the original characteristic function $\mathcal{F}$, and their location depends on B. For $z_B \leq z < B$ we obtain

$$\mathcal{F}_B(z) = \begin{cases} J_{EE}z^{\alpha_E} - J_{EI}B^{\alpha_I} - z + g_E, & \det\ J > 0, \\ J_{IE}B^{\alpha_E} - J_{II}z^{\alpha_I} - z + g_I, & \det\ J < 0 \end{cases}$$

and for $z \geq B$

$$\mathcal{F}_B(z) = \begin{cases} J_{EE}B^{\alpha_E} - J_{EI}B^{\alpha_I} - z + g_E, & \det\ J > 0, \\ J_{IE}B^{\alpha_E} - J_{II}B^{\alpha_I} - z + g_I, & \det\ J < 0. \end{cases}$$

For $\det\ J > 0$ the characteristic function $\mathcal{F}$ has an odd number of zero crossings and is negative in the point z_B. The function $\mathcal{F}_B$ can monotonically increase between z_B and B and it monotonically decreases for $z \geq B$. Then the function $\mathcal{F}_B$ can have either two (example in Fig. S2*B*) or no additional zero crossings (example in Fig. S2*C*) for $z > z_B$. Thus, in cases where two additional zero crossings appear the smaller zero crossing belongs to the interval (z_B, B) and has a positive derivative and the larger one has a negative derivative and is larger than B. Hereby, the steady state corresponding to the right zero crossing is always stable because the corresponding Jacobian is the diagonal matrix with the diagonal elements $-\tau_E^{-1}$ and $-\tau_I^{-1}$.

If $\det\ J < 0$, then $\mathcal{F}$ has an even number of zero crossings or does not have any zero crossings. In all cases $\mathcal{F}_B$ is positive at the point z_B. Since $\mathcal{F}_B$ is monotonically decreasing for $z \geq z_B$, it has an additional right zero crossing with a negative derivative (Fig. S2*A*). This zero crossing always corresponds to a stable steady state because the first row of the corresponding Jacobian is the vector with the elements $-\tau_E^{-1}$ and 0 and the second element of the second row is always negative.

Finally, we show that firing-rate saturation will guarantee that all firing-rate trajectories will remain bounded. To show this, we assume that the activation function for the excitatory unit is bounded by a constant $B_E^{\alpha_E}$. Then the first equation in Eq. **1** implies the estimate

$$\tau_E \dot{r}_E(t) + r_E(t) \leq B_E^{\alpha_E}. \quad \textbf{[S21]}$$

Now we multiply Eq. **S21** by e^{t/τ_E} to obtain

$$\frac{d}{dt}\left(r_E(t) \cdot e^{t/\tau_E}\right) \leq \frac{B_E^{\alpha_E} \cdot e^{t/\tau_E}}{\tau_E}. \quad \textbf{[S22]}$$

Finally, we integrate Eq. **S22** with respect to t and obtain an upper bound for the function r_E:

$$r_E(t) \leq e^{-t/\tau_E}(r_E(0) - B_E^{\alpha_E}) + B_E^{\alpha_E} \leq \max\{r_E(0), B_E^{\alpha_E}\}.$$

An analogous estimate holds for the function r_I under the assumption that the inhibitory activation function is bounded by a constant $B_I^{\alpha_I}$. Thus, both r_E and r_I are globally bounded:

$$r_E(t) \leq \max\{r_E(0), B_E^{\alpha_E}\},\ r_I(t) \leq \max\{r_I(0), B_I^{\alpha_I}\}. \quad \textbf{[S23]}$$

1. Kuznetsov Y (2013) *Elements of Applied Bifurcation Theory* (Springer, New York), Vol 112.
2. Priebe NJ, Ferster D (2008) Inhibition, spike threshold, and stimulus selectivity in primary visual cortex. *Neuron* 57:482–497.
3. Margrie TW, Brecht MSB (2002) In vivo, low-resistance, whole-cell recordings from neurons in the anaesthetized and awake mammalian brain. *Pflugers Arch* 444: 491–498.

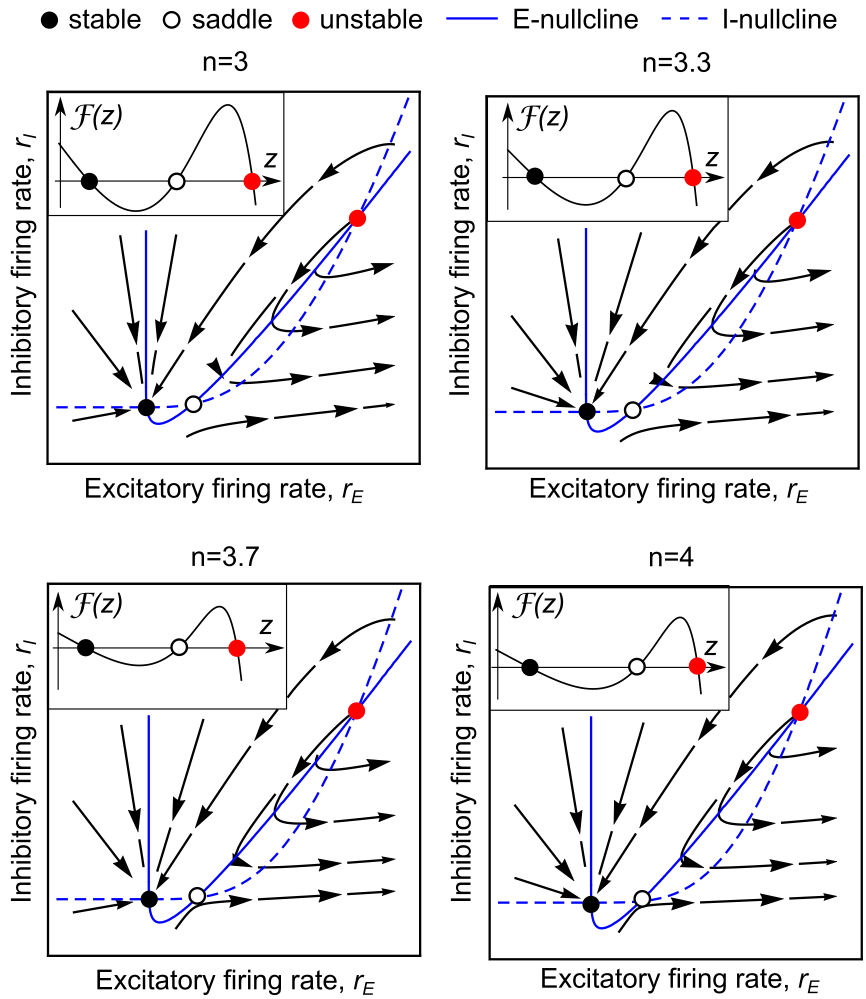

Fig. S1. Phase space comparison between real and integer exponents. Shown is the SSN phase space as a function of the power-law exponent n. The exponent varies from $n = 3$ to $n = 4$, and all other parameters are as in Fig. 2*C* and Table 2.

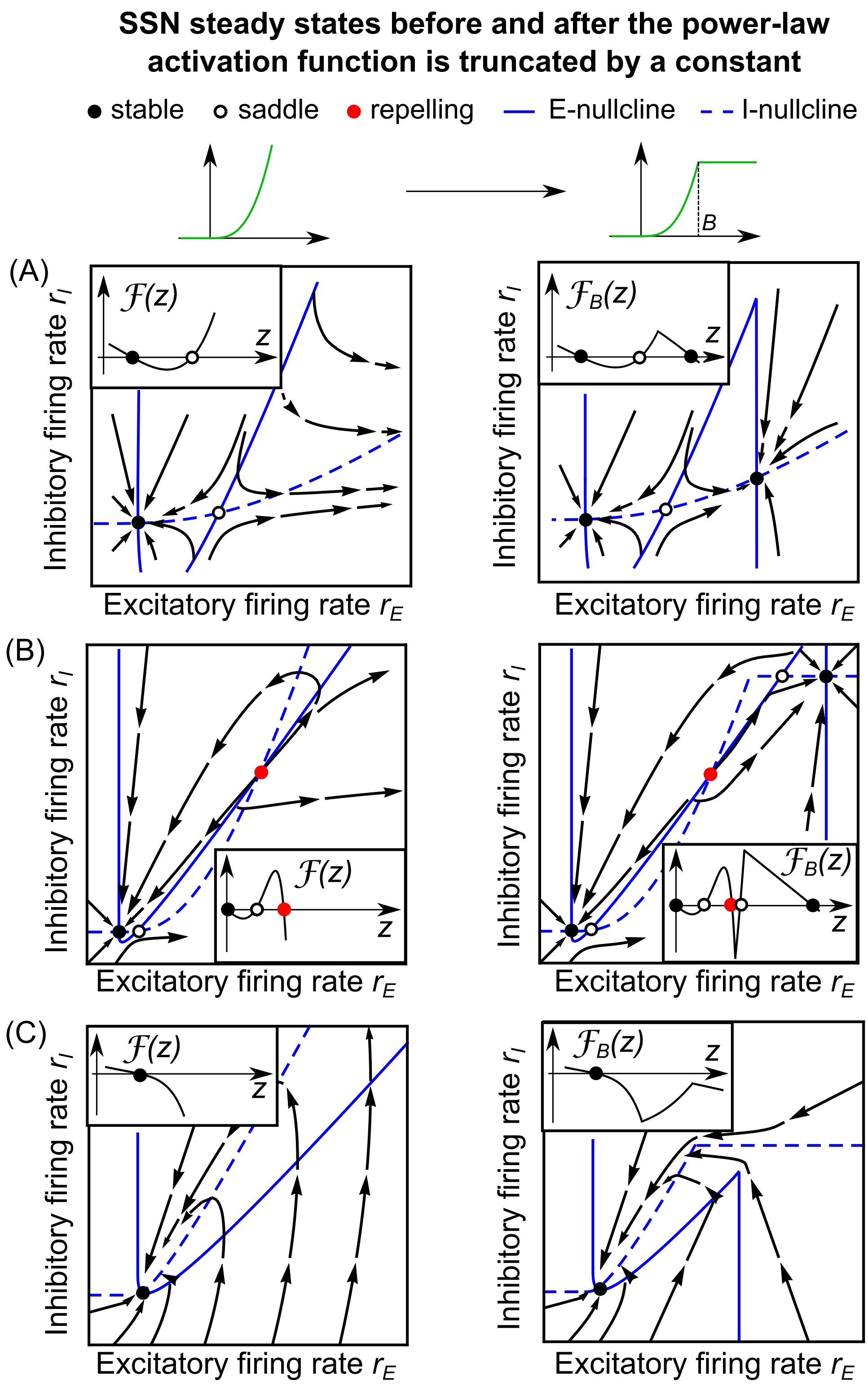

Fig. S2. SSN phase space before and after firing-rate saturation is introduced. Here, we consider SSN dynamics from Fig. 2 *A*–*C* in the main text. (*A*) The two steady states (a stable point and a saddle) which are present in the unbounded model are preserved after saturation is introduced. Saturation contributes an additional stable steady state (*A*, *Right*). Here, det $J < 0$, $B = 2$, and other parameters are adapted from Fig. 2*B*. (*B*) In addition to the three steady states which are inherited from the unbounded model, saturation introduces two steady states which are a saddle and a stable point. Here, $B = 2$, and other parameters are adapted from Fig. 2*C*. (*C*) Saturation does not always introduce additional steady states. Here, we show the dynamics from Fig. 2*A* ($B = 1.4$) where the single steady state prevails after saturation. Parameters are as in Fig. 2*A*.