Supplementary Materials for "Bayesian Multiresolution Variable Selection for Ultra-High Dimensional Neuroimaging Data"
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Appendix A: Standard Posterior Computation Algorithm

We provide the details of the standard posterior computation algorithm in Section 2.3 which is implemented via a Gibbs sampler. The joint posterior distribution of all the parameters given the data is

$$\begin{aligned}
&\pi(\mathbf{z}, \boldsymbol{\beta}, \boldsymbol{\alpha}, \mathbf{c}, \boldsymbol{\gamma}, \sigma_\beta^2, \eta_1, \eta_2, \xi_1, \xi_2 \mid \mathbf{S}, \mathbf{X}, \mathbf{y}) \\
\propto\ & \pi(\mathbf{y} \mid \mathbf{z})\pi(\mathbf{z} \mid \boldsymbol{\beta}, \boldsymbol{\alpha}, \mathbf{c}, \boldsymbol{\gamma}, \mathbf{S}, \mathbf{X})\pi(\boldsymbol{\beta} \mid \sigma_\beta^2)\pi(\boldsymbol{\alpha}) \\
& \pi(\mathbf{c} \mid \eta_1, \xi_1)\left[\prod_{r=1}^{R} \pi(\boldsymbol{\gamma}_r \mid \eta_2, \xi_2)\right]\pi(\sigma_\beta^2)\pi(\boldsymbol{\eta})\pi(\boldsymbol{\xi})
\end{aligned} \tag{1}$$

where $\boldsymbol{\eta} = (\eta_1, \eta_2)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2)$.

In the Gibbs sampler, the sampling schemes are as follows.

Sampling scheme for z: for $i = 1, \dots, n$, draw

$$\begin{aligned}
& z_i \mid \boldsymbol{\beta}, \boldsymbol{\alpha}, \mathbf{c}, \boldsymbol{\gamma}, y_i, \mathbf{S}, \mathbf{X} \\
& \qquad \sim y_i \mathrm{N}_{[0,+\infty)}(\widetilde{\mu}_i, 1) + (1 - y_i)\mathrm{N}_{(-\infty,0)}(\widetilde{\mu}_i, 1),
\end{aligned} \tag{2}$$

where $\mathrm{N}_{\mathcal{A}}(\mu, \Sigma)$ denote a normal distribution with mean μ and covariance Σ truncated on region $\mathcal{A}$, and $\widetilde{\mu}_i = \mathbf{S}_i\boldsymbol{\alpha} - \mathbf{X}_i\{\boldsymbol{\lambda} \circ \boldsymbol{\beta}\}$, where $\mathbf{S}_i, \mathbf{X}_i$ are row i for $\mathbf{S}, \mathbf{X}$.

Sampling scheme for $\boldsymbol{\alpha}$: draw

$$[\boldsymbol{\alpha} \mid \boldsymbol{\beta}, \mathbf{z}, \mathbf{c}, \boldsymbol{\gamma}, \mathbf{S}, \mathbf{X}] \sim \mathrm{N}(\widetilde{\boldsymbol{\mu}}_\alpha, \widetilde{\boldsymbol{\Sigma}}_\alpha), \tag{3}$$

where $\widetilde{\boldsymbol{\Sigma}}_\alpha = (\mathbf{S}'\mathbf{S} + \sigma_\alpha^{-2}\mathbf{I}_p)^{-1}$ and $\widetilde{\boldsymbol{\mu}}_\alpha = \widetilde{\boldsymbol{\Sigma}}_\alpha(\mathbf{z} - \mathbf{X}\{\boldsymbol{\lambda} \circ \boldsymbol{\beta}\})\mathbf{S}$.

Sampling scheme for σ_α^2: draw

$$[\sigma_\alpha^2 \mid \boldsymbol{\alpha}] \sim \mathrm{IG}(a_\alpha + P/2, b_\alpha + (1/2)\sum_{p=1}^{P} \alpha_p^2). \tag{4}$$

Sampling scheme for σ_β^2: draw

$$[\sigma_\beta^2 \mid \boldsymbol{\beta}] \sim \mathrm{IG}(a_\beta + V/2, b_\beta + (1/2)\sum_{r=1}^{R}\sum_{v=1}^{V_r} \beta_{rv}^2). \tag{5}$$

Sampling scheme for c: for $r = 1, \ldots, R$, the full conditional of c_r is given by

$$\begin{aligned} &\pi(c_r \mid \boldsymbol{\beta}, \mathbf{z}, \boldsymbol{\alpha}, \mathbf{c}_{-r}, \boldsymbol{\gamma}, \mathbf{S}, \mathbf{X}) \\ \propto \quad & \exp\left(\eta_1 \sum_{r=1}^{R} c_r + \xi_1 \sum_{r'=1}^{R} f_{r'r} I[c_{r'} = c_r]\right) \\ & \prod_{i=1}^{n} \phi\left(z_i - \mathbf{S}_i \boldsymbol{\alpha} - \mathbf{X}_i \left\{\boldsymbol{\lambda} \circ \boldsymbol{\beta}\right\}\right), \end{aligned} \tag{6}$$

where $\mathbf{c}_{-r} = (c_{r'}, r' \neq r)$.

Sampling scheme for $\boldsymbol{\gamma}$: for $r = 1, \ldots, R$ and $v = 1, \ldots, V_r$, the full conditional of γ_{rv} is given by

$$\begin{aligned} &\pi(\gamma_{rv} \mid \boldsymbol{\beta}, \mathbf{z}, \boldsymbol{\alpha}, \mathbf{c}, \boldsymbol{\gamma}_{-rv}, \mathbf{S}, \mathbf{X}) \\ \propto \quad & \exp\left(\eta_2 \sum_{v=1}^{V_r} \gamma_{rv} + \xi_2 \sum_{v'=1}^{V_r} l_{rv'v} I[\gamma_{rv'} = \gamma_{rv}]\right) \\ & \prod_{i=1}^{n} \phi\left(z_i - \mathbf{S}_i \boldsymbol{\alpha} - \mathbf{X}_i \left\{\boldsymbol{\lambda} \circ \boldsymbol{\beta}\right\}\right), \end{aligned} \tag{7}$$

where $\boldsymbol{\gamma}_{-rv} = (\gamma_{st}, s \neq r \text{ or } t \neq v)$.

Sampling scheme for $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$: the parameters in Ising priors are updated using the auxiliary variable method by Møller et al. [1].

Sampling scheme for β: As discussed in Section 2.3, based on the full conditional, update $\boldsymbol{\beta}$ via a block update.

Appendix B: SRS-MCMC Algorithm

The updating scheme for $\mathbf{z}^{(k)}, \boldsymbol{\alpha}^{(k)}, \boldsymbol{\beta}^{(k)}, \sigma_\beta^{2(k)}$, $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ follows the standard posterior computation algorithm in Appendix .

Sampling scheme for $\mathbf{c}^{(k)}$ and $\boldsymbol{\gamma}^{(k)}$:

- when $k = 1$, for $g = 1, \ldots, G^{(k)}; r = 1, \ldots, R; h = 1, \ldots, H_r^{(k)}$, the full conditionals of $c_g^{(k)}$ and $\gamma_{rh}^{(k)}$ are

$$\begin{aligned} &\pi(c_g^{(k)} \mid \boldsymbol{\beta}^{(k)}, \mathbf{z}^{(k)}, \boldsymbol{\alpha}^{(k)}, \mathbf{c}_{-g}^{(k)}, \boldsymbol{\gamma}^{(k)}, \mathbf{S}, \mathbf{X}) \\ \propto \quad & \prod_{i=1}^{n} \phi\left(z_i^{(k)} - \mathbf{S}_i \boldsymbol{\alpha}^{(k)} - \mathbf{X}_i \left\{\boldsymbol{\lambda}^{(k)} \circ \boldsymbol{\beta}^{(k)}\right\}\right); \end{aligned} \tag{8}$$

$$\begin{aligned} &\pi(\gamma_{rh}^{(k)} \mid \boldsymbol{\beta}^{(k)}, \mathbf{z}^{(k)}, \boldsymbol{\alpha}^{(k)}, \mathbf{c}^{(k)}, \boldsymbol{\gamma}_{r[-h]}^{(k)}, \boldsymbol{\gamma}_{[-r]}^{(k)}, \mathbf{S}, \mathbf{X}) \\ \propto \quad & \prod_{i=1}^{n} \phi\left(z_i^{(k)} - \mathbf{S}_i \boldsymbol{\alpha}^{(k)} - \mathbf{X}_i \left\{\boldsymbol{\lambda}^{(k)} \circ \boldsymbol{\beta}^{(k)}\right\}\right); \end{aligned} \tag{9}$$

- when $1 < k < K$, the sampling scheme is stated in Section 3.2 with the full conditional updates of $c_g^{(k)}$ and $\gamma_{rh}^{(k)}$ in the moving step.

- when $k = K$, the sampling scheme is stated in Section 3.2 with the full conditional updates of $c_g^{(k)}$ and $\gamma_{rh}^{(k)}$ in the moving step.

Appendix C: fastSRS-MCMC Algorithm

The updating scheme for $\mathbf{z}^{(k)}$, $\boldsymbol{\alpha}^{(k)}$, $\sigma_\beta^{2(k)}$, $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ follows the standard posterior computation algorithm in Appendix .

Sampling scheme for $\mathbf{c}^{(k)}, \boldsymbol{\gamma}^{(k)}, \widetilde{\boldsymbol{\beta}}^{(k)}$:

- when $k = 1$, the full conditional of $\mathbf{c}^{(k)}$ follows (8). For the full conditional of $(\boldsymbol{\gamma}^{(k)}, \widetilde{\boldsymbol{\beta}}^{(k)})$,

$$\pi(\gamma_{rh'}^{(k)} \mid \bullet) = \int \pi(\widetilde{\boldsymbol{\beta}}_{rh'}^{(k)}, \boldsymbol{\gamma}_{rh'}^{(k)} \mid \bullet)\mathrm{d}\widetilde{\boldsymbol{\beta}}_{rh'}^{(k)} \propto \phi(\mathbf{0}; \boldsymbol{\mu}_{rh'}^{(k)}, \boldsymbol{\Sigma}_{rh'}^{(k)}),$$

$$\pi(\widetilde{\boldsymbol{\beta}}_{rh'1}^{(k)} \mid \boldsymbol{\gamma}_{rh'}^{(k)}, \bullet) = \phi(\widetilde{\boldsymbol{\beta}}_{rh'1}^{(k)}; \boldsymbol{\mu}_{rh'}^{(k)}, \boldsymbol{\Sigma}_{rh'}^{(k)}),$$
$$\pi(\widetilde{\boldsymbol{\beta}}_{rh'0}^{(k)} \mid \boldsymbol{\gamma}_{rh'}^{(k)}, \bullet) = \delta_{\mathbf{0}}(\widetilde{\boldsymbol{\beta}}_{rh'0}^{(k)}),$$

  with

$$\begin{aligned} \boldsymbol{\Sigma}_{rh'}^{(k)} &= \left(\sigma_\beta^{-2(k)}\mathbf{I} + \mathbf{X}_{rh'}^\top \mathbf{X}_{rh'}\right)^{-1}; \\ \boldsymbol{\mu}_{rh'}^{(k)} &= \boldsymbol{\Sigma}_{rh'}^{(k)} \mathbf{X}_{rh'}^\top \left(\mathbf{z}^{(k)} - \mathbf{S}\boldsymbol{\alpha}^{(k)}\right), \end{aligned} \tag{10}$$

  where $\mathbf{X}_{rh'} = (\mathbf{x}_{rv}, a_{rvh}^{(k)} = 1; \gamma_{rh}^{(k)} = 1; \widetilde{a}_{rhh'}^{(k)} = 1)$.

- when $1 < k < K$, the sampling scheme is stated in Section 3.3.

- when $k = K$, the sampling scheme is stated in Section 3.3 with the full conditional updates of $c_g^{(k)}$ and $(\boldsymbol{\gamma}^{(k)}, \widetilde{\boldsymbol{\beta}}^{(k)})$ in the moving step.

References

[1] J. Møller, A. N. Pettitt, R. Reeves, and K. K. Berthelsen, "An efficient markov chain monte carlo method for distributions with intractable normalising constants," Biometrika, vol. 93, no. 2, pp. 451–458, 2006.