

Web-based Supplementary Materials for “Quantifying Publication Bias in Meta-Analysis” by

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Web Appendix: Proofs

Let X_1, \dots, X_n be iid random variables and denote the k th central moment $\beta_k = E(X_1 - \beta)^k$, where $\beta = E(X_1)$. Also, denote the sample k th central moment $m_k = n^{-1} \sum_{i=1}^n (X_i - \bar{X})^k$, where $\bar{X} = n^{-1} \sum_{i=1}^n X_i$. We have the following lemma regarding the asymptotic distribution of the sample k th central moment.

Lemma 1. *If X_1, \dots, X_n are iid with mean β and $\beta_{2k} < \infty$ for $k \geq 1$, then*

$$m_k - \beta_k = \frac{1}{n} \sum_{i=1}^n [(X_i - \beta)^k - \beta_k - k\beta_{k-1}(X_i - \beta)] + o_p(n^{-1/2}).$$

as $n \rightarrow \infty$.

Proof of Lemma 1. See page 72 in Serfling (1980). □

Now, let us back to the notation in the main text. Specifically, let $x_i = (s_i^2 + \tau^2)^{-1/2}$ and $z_i = y_i(s_i^2 + \tau^2)^{-1/2}$. The regression test is $z_i = \alpha + \mu x_i + \epsilon_i$, where ϵ_i 's are iid following a distribution with mean zero; $\hat{\alpha}$ and $\hat{\mu}$ are the least squares estimates of α and μ respectively, and the residuals $\hat{\epsilon}_i = y_i - \hat{\mu}x_i - \hat{\alpha}$. Also, $\beta_k = E(\epsilon_1 - \beta)^k$ is the k th central moment of ϵ_i 's, where $\beta = E(\epsilon_1) = 0$, and $m_k = n^{-1} \sum_{i=1}^n (\epsilon_i - \bar{\epsilon})^k$. The true skewness of ϵ_i 's is $\gamma = \beta_3/\beta_2^{3/2}$. Let $\hat{m}_k = n^{-1} \sum_{i=1}^n (\hat{\epsilon}_i - \bar{\hat{\epsilon}})^k$ be the sample k th central moment by plugging in the residuals $\hat{\epsilon}_i$, where $\bar{\hat{\epsilon}} = n^{-1} \sum_{i=1}^n \hat{\epsilon}_i = 0$. The sample skewness of $\epsilon = (\epsilon_1, \dots, \epsilon_n)^T$ is $\text{Skew}(\epsilon) = m_3/s^3$, where $s = \sqrt{nm_2/(n-1)}$, and $T_S = \text{Skew}(\hat{\epsilon})$ is obtained by plugging $\hat{\epsilon} = (\hat{\epsilon}_1, \dots, \hat{\epsilon}_n)^T$ in $\text{Skew}(\epsilon)$.

Proof of Proposition 1. First, we show that $\sqrt{n}(\text{Skew}(\epsilon) - \gamma) \xrightarrow{D} N(0, v)$ as $n \rightarrow \infty$, where

$$v = 9 + \frac{35}{4}\beta_2^{-3}\beta_3^2 - 6\beta_2^{-2}\beta_4 + \beta_2^{-3}\beta_6 + \frac{9}{4}\beta_2^{-5}\beta_3^2\beta_4 - 3\beta_2^{-4}\beta_3\beta_5.$$

Because $\text{Skew}(\epsilon) = [(n-1)/n]^{3/2}m_3/m_2^{3/2}$, $\text{Skew}(\epsilon)$ have the same asymptotic distribution as $m_3/m_2^{3/2}$. By Lemma 1, we have

$$\begin{bmatrix} m_2 \\ m_3 \end{bmatrix} - \begin{bmatrix} \beta_2 \\ \beta_3 \end{bmatrix} = \frac{1}{n} \sum_{i=1}^n \begin{bmatrix} \epsilon_i^2 - \beta_2 \\ \epsilon_i^3 - \beta_3 - 3\beta_2\epsilon_i \end{bmatrix} + o_p(n^{-1/2}).$$

Therefore,

$$\sqrt{n} \left(\begin{bmatrix} m_2 \\ m_3 \end{bmatrix} - \begin{bmatrix} \beta_2 \\ \beta_3 \end{bmatrix} \right) \xrightarrow{D} N \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \beta_4 - \beta_2^2 & \beta_5 - 4\beta_2\beta_3 \\ \beta_5 - 4\beta_2\beta_3 & \beta_6 - \beta_3^2 - 6\beta_2\beta_4 + 9\beta_2^3 \end{bmatrix} \right).$$

Denote the asymptotic covariance matrix above as Σ . Let $g(r, s) = s/r^{3/2}$, then $g'(r, s) = (-\frac{3}{2}sr^{-5/2}, r^{-3/2})^T$. By the delta method,

$$\sqrt{n}(g(m_2, m_3) - g(\beta_2, \beta_3)) \xrightarrow{D} N(0, [g'(\beta_2, \beta_3)]^T \Sigma [g'(\beta_2, \beta_3)]);$$

that is,

$$\sqrt{n}(\text{Skew}(\epsilon) - \gamma) \xrightarrow{D} N(0, v).$$

Second, we show that $\sqrt{n}(T_S - \text{Skew}(\epsilon)) \xrightarrow{D} 0$ as $n \rightarrow \infty$. We write $\text{Skew}(\epsilon) = [(n-1)/n]^{3/2} f(\delta)$, where $f(\delta) = m_3/m_2^{3/2}$ is a continuous and differentiable function of $\delta = (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5)^T = (\bar{\epsilon}^2, \bar{\epsilon}^3, \bar{\epsilon} \cdot \bar{\epsilon}^2, \bar{\epsilon}^2, \bar{\epsilon}^3)^T$; here, $\bar{\epsilon}^k = n^{-1} \sum_{i=1}^n \epsilon_i^k$. Specifically, $f(\delta) = (\delta_5 - 3\delta_3 + 2\delta_2)(\delta_4 - \delta_1)^{-3/2}$; it is free of n . Also, $T_S = \text{Skew}(\hat{\epsilon}) = [(n-1)/n]^{3/2} f(\hat{\delta})$, where $\hat{\delta} = \left((\hat{\bar{\epsilon}})^2, (\hat{\bar{\epsilon}})^3, \hat{\bar{\epsilon}} \cdot \hat{\bar{\epsilon}}^2, \hat{\bar{\epsilon}}^2, \hat{\bar{\epsilon}}^3 \right)^T$, and $\hat{\bar{\epsilon}}^k = n^{-1} \sum_{i=1}^n \hat{\epsilon}_i^k$. Because the average of the residuals is $\bar{\epsilon} = 0$, we have $\hat{\delta} = (0, 0, 0, \bar{\epsilon}^2, \bar{\epsilon}^3)^T$. By multivariate Taylor expansion,

$$f(\hat{\delta}) = f(\delta) + [\mathbf{h}(\delta)]^T (\hat{\delta} - \delta) + O_p(\|\hat{\delta} - \delta\|^2),$$

where $\mathbf{h}(\delta) = \nabla f(\delta)$ is the gradient of $f(\delta)$ and $\|\cdot\|$ is the Euclidean norm. Specifically,

$$\mathbf{h}(\delta) = \begin{bmatrix} h_1(\delta) \\ h_2(\delta) \\ h_3(\delta) \\ h_4(\delta) \\ h_5(\delta) \end{bmatrix} = \begin{bmatrix} \frac{3}{2}(\delta_5 - 3\delta_3 + 2\delta_2)(\delta_4 - \delta_1)^{-5/2} \\ 2(\delta_4 - \delta_1)^{3/2} \\ -3(\delta_4 - \delta_1)^{3/2} \\ -\frac{3}{2}(\delta_5 - 3\delta_3 + 2\delta_2)(\delta_4 - \delta_1)^{-5/2} \\ (\delta_4 - \delta_1)^{3/2} \end{bmatrix}.$$

Since $\delta_1, \delta_2, \delta_3 \xrightarrow{P} 0$, $\delta_4 \xrightarrow{P} \beta_2 > 0$, and $\delta_5 \xrightarrow{P} \beta_3$, we have $h_j(\delta) = O_p(1)$ for $j = 1, \dots, 5$. Now, we focus on

$$\hat{\delta} - \delta = \left(-\bar{\epsilon}^2, -\bar{\epsilon}^3, -\bar{\epsilon} \cdot \bar{\epsilon}^2, \bar{\epsilon}^2 - \bar{\epsilon}^2, \bar{\epsilon}^3 - \bar{\epsilon}^3 \right)^T.$$

Due to $\bar{\epsilon} = O_p(n^{-1/2})$, we have $\hat{\delta}_1 - \delta_1 = -\bar{\epsilon}^2 = O_p(n^{-1})$, $\hat{\delta}_2 - \delta_2 = -\bar{\epsilon}^3 = O_p(n^{-3/2})$, and $\hat{\delta}_3 - \delta_3 = -\bar{\epsilon} \cdot \bar{\epsilon}^2 = O_p(n^{-1/2})$. Note that

$$\hat{\epsilon}_i = (\alpha - \hat{\alpha}) + (\mu - \hat{\mu})x_i + \epsilon_i,$$

and $\hat{\alpha} - \alpha = O_p(n^{-1/2})$, $\hat{\mu} - \mu = O_p(n^{-1/2})$. Also, by the assumption that x_i 's have finite third moment and the weak law of large numbers, $\frac{1}{n} \sum_{i=1}^n x_i^k = O_p(1)$ for $k = 1, 2, 3$. Consequently, we have

$$\begin{aligned} \hat{\delta}_4 - \delta_4 &= \bar{\hat{\epsilon}}^2 - \bar{\epsilon}^2 \\ &= \frac{1}{n} \sum_{i=1}^n [(\alpha - \hat{\alpha}) + (\mu - \hat{\mu})x_i + \epsilon_i]^2 - \frac{1}{n} \sum_{i=1}^n \epsilon_i^2 \\ &= (\alpha - \hat{\alpha})^2 + (\mu - \hat{\mu})^2 \frac{\sum_{i=1}^n x_i^2}{n} + 2(\alpha - \hat{\alpha})(\mu - \hat{\mu}) \frac{\sum_{i=1}^n x_i}{n} \\ &\quad + 2(\alpha - \hat{\alpha})\bar{\epsilon} + 2(\mu - \hat{\mu}) \frac{\sum_{i=1}^n x_i \epsilon_i}{n} \\ &= O_p(n^{-1}), \end{aligned}$$

and

$$\begin{aligned}
\hat{\delta}_5 - \delta_5 &= \bar{\hat{\epsilon}}^3 - \bar{\epsilon}^3 \\
&= \frac{1}{n} \sum_{i=1}^n [(\alpha - \hat{\alpha}) + (\mu - \hat{\mu})x_i + \epsilon_i]^3 - \frac{1}{n} \sum_{i=1}^n \epsilon_i^3 \\
&= (\alpha - \hat{\alpha})^3 + (\mu - \hat{\mu})^3 \frac{\sum_{i=1}^n x_i^3}{n} + 3(\alpha - \hat{\alpha})^2(\mu - \hat{\mu}) \frac{\sum_{i=1}^n x_i}{n} \\
&\quad + 3(\alpha - \hat{\alpha})(\mu - \hat{\mu})^2 \frac{\sum_{i=1}^n x_i^2}{n} + 3(\alpha - \hat{\alpha})^2 \bar{\epsilon} + 3(\mu - \hat{\mu})^2 \frac{\sum_{i=1}^n x_i^2 \epsilon_i}{n} \\
&\quad + 6(\alpha - \hat{\alpha})(\mu - \hat{\mu}) \frac{\sum_{i=1}^n x_i \epsilon_i}{n} + 3(\alpha - \hat{\alpha}) \bar{\epsilon}^2 + 3(\mu - \hat{\mu}) \frac{\sum_{i=1}^n x_i \epsilon_i^2}{n} \\
&= 3(\alpha - \hat{\alpha}) \bar{\epsilon}^2 + 3(\mu - \hat{\mu}) \frac{\sum_{i=1}^n x_i \epsilon_i^2}{n} + O_p(n^{-1}) \\
&= O_p(n^{-1/2}).
\end{aligned}$$

Therefore, $O_p(\|\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\|^2) = O_p(n^{-1})$, implying

$$\begin{aligned}
f(\hat{\boldsymbol{\delta}}) - f(\boldsymbol{\delta}) &= [\mathbf{h}(\boldsymbol{\delta})]^T (\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}) + O_p(n^{-1}) \\
&= \sum_{j=1}^5 h_j(\boldsymbol{\delta})(\hat{\delta}_j - \delta_j) + O_p(n^{-1}) \\
&= h_3(\boldsymbol{\delta})(\hat{\delta}_3 - \delta_3) + h_5(\boldsymbol{\delta})(\hat{\delta}_5 - \delta_5) + O_p(n^{-1}) \\
&= 3(\delta_4 - \delta_1)^{3/2} \cdot \bar{\epsilon} \cdot \bar{\epsilon}^2 + (\delta_4 - \delta_1)^{3/2} \cdot \left[3(\alpha - \hat{\alpha}) \bar{\epsilon}^2 + 3(\mu - \hat{\mu}) \frac{\sum_{i=1}^n x_i \epsilon_i^2}{n} \right] + O_p(n^{-1}) \\
&= 3(\delta_4 - \delta_1)^{3/2} \left\{ [(\alpha - \hat{\alpha}) + \bar{\epsilon}] \bar{\epsilon}^2 + (\mu - \hat{\mu}) \frac{\sum_{i=1}^n x_i \epsilon_i^2}{n} \right\} + O_p(n^{-1})
\end{aligned}$$

Note that $\sum_{i=1}^n \hat{\epsilon}_i = 0$, so $(\alpha - \hat{\alpha}) + \bar{\epsilon} = (\hat{\mu} - \mu) \frac{\sum_{i=1}^n x_i}{n}$. Consequently,

$$\begin{aligned}
f(\hat{\boldsymbol{\delta}}) - f(\boldsymbol{\delta}) &= 3(\delta_4 - \delta_1)^{3/2} \left\{ (\hat{\mu} - \mu) \frac{\sum_{i=1}^n x_i}{n} \bar{\epsilon}^2 - (\hat{\mu} - \mu) \frac{\sum_{i=1}^n x_i \epsilon_i^2}{n} \right\} + O_p(n^{-1}) \\
&= 3(\delta_4 - \delta_1)^{3/2} (\hat{\mu} - \mu) \left\{ \frac{\sum_{i=1}^n x_i}{n} \bar{\epsilon}^2 - \frac{\sum_{i=1}^n x_i \epsilon_i^2}{n} \right\} + O_p(n^{-1}) \\
&= O_p(n^{-1/2}) \{ [\mathbf{E}(x_1) + O_p(n^{-1/2})][\beta_2 + O_p(n^{-1/2})] - [\mathbf{E}(x_1 \epsilon_1^2) + O_p(n^{-1/2})] \} + O_p(n^{-1}) \\
&= O_p(n^{-1/2}) \{ [\mathbf{E}(x_1) \beta_2 + O_p(n^{-1/2})] - [\mathbf{E}(x_1) \beta_2 + O_p(n^{-1/2})] \} + O_p(n^{-1}) \\
&= O_p(n^{-1}).
\end{aligned}$$

This leads to $\sqrt{n}(f(\hat{\boldsymbol{\delta}}) - f(\boldsymbol{\delta})) \xrightarrow{D} 0$; hence, $\sqrt{n}(T_S - \text{Skew}(\boldsymbol{\epsilon})) \xrightarrow{D} 0$, and $\sqrt{n}(T_S - \gamma) \xrightarrow{D} N(0, v)$.

Finally, we show that $\hat{v} \xrightarrow{P} v$. By continuous mapping theorem, it is sufficient to show that $\hat{m}_k \xrightarrow{P} \beta_k$ for $k = 2, \dots, 6$. Recall that $\beta_k = \mathbf{E}(\epsilon_1^k)$ and $\hat{m}_k = n^{-1} \sum_{i=1}^n \hat{\epsilon}_i^k$. Since $\hat{\epsilon}_i = (\alpha - \hat{\alpha}) + (\mu - \hat{\mu})x_i + \epsilon_i = \epsilon_i + O_p(n^{-1/2})$, we have $\hat{m}_k = n^{-1} \sum_{i=1}^n (\epsilon_i + O_p(n^{-1/2}))^k = n^{-1} \sum_{i=1}^n \epsilon_i^k + o_p(1) = \beta_k + o_p(1)$; that is, $\hat{m}_k \xrightarrow{P} \beta_k$. By Slutsky's theorem, $\sqrt{n}(T_S - \gamma) / \sqrt{\hat{v}} \xrightarrow{D} N(0, 1)$; this completes the proof. \square

Proof of Corollary 1. Under H_0'' , we have $\epsilon_i \sim N(0, \sigma^2)$, so $\beta_{2k} = (2k-1)!!\sigma^{2k}$ and $\beta_{2k-1} = 0$ for $k \geq 1$. Here, $c!! = c \cdot (c-2) \cdot (c-4) \cdots$ is the double factorial. Specifically, $\beta_2 = \sigma^2$, $\beta_4 = 3\sigma^4$, and $\beta_6 = 15\sigma^6$. In the proof of Proposition 1, we showed that $\sqrt{n}(T_S - \gamma) \xrightarrow{D} N(0, v)$. Under H_0'' , v is simplified as $v = 9 - 6(\sigma^2)^{-2} \cdot 3\sigma^4 + (\sigma^2)^{-3} \cdot 15\sigma^6 = 6$. This completes the proof. \square

References

Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics*. John Wiley & Sons, New York, NY.