A Appendix

A.1 Proofs of lemmas

The proofs of Lemma 3.1 and 3.2 follow.

Lemma 3.1

For a given value of M and $i \leq 73$,

$$\begin{aligned}
A_i(\alpha,\pi_0,\tau) &= \int f(\zeta_i|\tau,W_i)\pi_0^{1-W_i}(1-\pi_0)^{W_i}\prod_{j=1}^{2} f(z_{ij},R_i,S_i|\zeta_i)d\zeta_i dW_i \\
&= \int \left[(1-W_i)+W_i\frac{\zeta_i^2}{\tau^{3/2}\sqrt{2\pi}}\exp\left(-\frac{\zeta_i^2}{2\tau}\right)\right]\pi_0^{1-W_i}(1-\pi_0)^{W_i}\alpha^{1-S_i}\prod_{j=1}^{2}\phi[z_{ij}|\zeta_i,\sigma_{ij}^2]d\zeta_i dW_i \\
&= \pi_0\alpha^{1-S_i}\prod_{j=1}^{2}\phi[z_{ij}|\zeta_i,\sigma_{ij}^2]+(1-\pi_0)\alpha^{1-S_i} \\
&\qquad \times\int\frac{\zeta_i^2}{(2\pi\tau)^{3/2}\sigma_{i1}\sigma_{i2}}\exp\left(-\frac{\zeta_i^2}{2\tau}-\frac{(z_{i1}-\zeta_i)^2}{2\sigma_{i1}^2}-\frac{(z_{i2}-\zeta_i)^2}{2\sigma_{i2}^2}\right)d\zeta_i.
\end{aligned}$$

Let

$$\begin{aligned}
a &= \frac{1}{2\pi\sigma_{i1}\sigma_{i2}\tau^{3/2}}, \quad w = \frac{1}{\sigma_{i1}^2}+\frac{1}{\sigma_{i2}^2}+\frac{1}{\tau}, \\
b &= a\exp\left\{-0.5\left[\frac{z_{i1}^2}{\sigma_{i1}^2}+\frac{z_{i2}^2}{\sigma_{i2}^2}-\frac{1}{w}\left(\frac{z_{i1}}{\sigma_{i1}^2}+\frac{z_{i2}}{\sigma_{i2}^2}\right)^2\right]\right\},
\end{aligned}$$

then

$$\begin{aligned}
&\int\frac{\zeta_i^2}{(2\pi\tau)^{3/2}\sigma_{i1}\sigma_{i2}}\exp\left(-\frac{\zeta_i^2}{2\tau}-\frac{(z_{i1}-\zeta_i)^2}{2\sigma_{i1}^2}-\frac{(z_{i2}-\zeta_i)^2}{2\sigma_{i2}^2}\right)d\zeta_i \\
&= a\exp\left\{-0.5\left[\frac{z_{i1}^2}{\sigma_{i1}^2}+\frac{z_{i2}^2}{\sigma_{i2}^2}-\frac{1}{w}\left(\frac{z_{i1}}{\sigma_{i1}^2}+\frac{z_{i2}}{\sigma_{i2}^2}\right)^2\right]\right\}\int\frac{\zeta_i^2}{\sqrt{2\pi}}\exp\left\{-\frac{w}{2}\left(\zeta_i-\frac{\frac{z_{i1}}{\sigma_{i1}^2}+\frac{z_{i2}}{\sigma_{i2}^2}}{w}\right)^2\right\}d\zeta_i \\
&= b\int\frac{\zeta_i^2}{\sqrt{2\pi}}\exp\left\{-\frac{w}{2}\left(\zeta_i-\frac{\frac{z_{i1}}{\sigma_{i1}^2}+\frac{z_{i2}}{\sigma_{i2}^2}}{w}\right)^2\right\}d\zeta_i \\
&= bw^{-3/2}\left[1+\frac{1}{w}\left(\frac{z_{i1}}{\sigma_{i1}^2}+\frac{z_{i2}}{\sigma_{i2}^2}\right)^2\right].
\end{aligned}$$

Hence,

$$A_i(\alpha,\pi_0,\tau) \;=\; \pi_0\alpha^{1-S_i}\prod_{j=1}^{2}\phi[z_{ij}|\zeta_i,\sigma_{ij}^2] + (1-\pi_0)\alpha^{1-S_i}bw^{-3/2}\left[1+\frac{1}{w}\left(\frac{z_{i1}}{\sigma_{i1}^2}+\frac{z_{i2}}{\sigma_{i2}^2}\right)^2\right].$$

Lemma 3.2

For a given value of M *and* $i \geq 73$,

$$\begin{aligned}
B_i(\alpha,\pi_0,\tau) &= \int f(\zeta_i|\tau,W_i)\pi_0^{1-W_i}(1-\pi_0)^{W_i}\prod_{j=1}^{2} f(z_{ij},R_i,S_i|\zeta_i)dz_{i1}dz_{i2}d\zeta_i dW_i \\
&= \int\left[(1-W_i)+W_i\frac{\zeta_i^2}{\tau^{3/2}\sqrt{2\pi}}\exp\left(-\frac{\zeta_i^2}{2\tau}\right)\right]\pi_0^{1-W_i}(1-\pi_0)^{W_i} \\
&\qquad \times(1-\alpha)\prod_{j=1}^{2}\phi[z_{ij}|\zeta_i,\sigma_{ij}^2]dz_{i1}dz_{i2}d\zeta_i dW_i \\
&= (1-\alpha)\pi_0\int\prod_{j=1}^{2}\phi[z_{ij}|0,\sigma_{ij}^2]dz_{i1}dz_{i2} \\
&\qquad +(1-\alpha)(1-\pi_0)\int\frac{\zeta_i^2}{\tau^{3/2}\sqrt{2\pi}}\exp\left(-\frac{\zeta_i^2}{2\tau}\right)\prod_{j=1}^{2}\phi[z_{ij}|\zeta_i,\sigma_{ij}^2]d\zeta_i dz_{i1}dz_{i2}.
\end{aligned}$$

Let

$$c \;=\; \frac{1}{\sigma_{i1}^2}+\frac{1}{\tau},\quad d \;=\; \frac{1}{\sigma_{i1}^2}-\frac{1}{c\sigma_{i1}^4},\quad f \;=\; \sqrt{d}b_i=\sqrt{d}q_\gamma\sigma_{i1},\quad g \;=\; \Phi(f)-\Phi(-f),$$

$$h \;=\; \frac{1}{\sigma_{i1}\sqrt{\tau^3dc^3}}\left\{\frac{1}{cd\sigma_{i1}^4}\left[g-\sqrt{\frac{2}{\pi}}f\exp\left(-\frac{f^2}{2}\right)\right]+g\right\},$$

then

$$\begin{aligned}
&(1-\alpha)\pi_0\int\prod_{j=1}^{2}\phi[z_{ij}|0,\sigma_{ij}^2]dz_{i1}dz_{i2} \\
&= (1-\alpha)\pi_0\int_{-b_i}^{b_i}\phi[z_{i1}|0,\sigma_{i1}^2]dz_{i1}\int_{-\infty}^{\infty}\phi[z_{i2}|0,\sigma_{i2}^2]dz_{i2} \\
&= (1-\alpha)\pi_0(2\gamma-1),
\end{aligned}$$

$$\begin{aligned}
&(1-\alpha)(1-\pi_0)\int\frac{\zeta_i^2}{\tau^{3/2}\sqrt{2\pi}}\exp\left(-\frac{\zeta_i^2}{2\tau}\right)\prod_{j=1}^{2}\phi[z_{ij}|\zeta_i,\sigma_{ij}^2]d\zeta_i dz_{i1}dz_{i2} \\
&= (1-\alpha)(1-\pi_0)\int_{-b_i}^{b_i}\int_{-\infty}^{\infty}\frac{\zeta_i^2}{\tau^{3/2}\sqrt{2\pi}}\exp\left(-\frac{\zeta_i^2}{2\tau}\right)\phi[z_{i1}|\zeta_i,\sigma_{i1}^2]\int_{-\infty}^{\infty}\phi[z_{i2}|\zeta_i,\sigma_{i2}^2]dz_{i2}d\zeta_i dz_{i1}
\end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha)(1-\pi_0)\int_{-b_i}^{b_i}\int_{-\infty}^{\infty}\frac{\zeta_i^2}{\tau^{3/2}\sqrt{2\pi}}\exp\left(-\frac{\zeta_i^2}{2\tau}\right)\phi[z_{i1}|\zeta_i,\sigma_{i1}^2]d\zeta_i dz_{i1} \\
&= \frac{(1-\alpha)(1-\pi_0)}{2\pi\tau^{3/2}\sigma_{i1}}\int_{-b_i}^{b_i}\exp\left\{-\frac{z_{i1}^2}{2}\left(\frac{1}{\sigma_{i1}^2}-\frac{1}{c\sigma_{i1}^4}\right)\right\}\int_{-\infty}^{\infty}\zeta_i^2\exp\left\{-\frac{c}{2}\left(\zeta_i-\frac{z_{i1}}{c\sigma_{i1}^2}\right)^2\right\}d\zeta_i dz_{i1} \\
&= \frac{(1-\alpha)(1-\pi_0)}{\sqrt{2\pi}\tau^{3/2}c^{3/2}\sigma_{i1}}\int_{-b_i}^{b_i}\left(1+\frac{z_{i1}^2}{c\sigma_{i1}^4}\right)\exp\left(-\frac{z_{i1}^2 d}{2}\right)dz_{i1} \\
&= \frac{(1-\alpha)(1-\pi_0)}{\sqrt{2\pi}\tau^{3/2}c^{3/2}\sigma_{i1}}\left[\int_{-b_i}^{b_i}\exp\left(-\frac{z_{i1}^2 d}{2}\right)dz_{i1}+\int_{-b_i}^{b_i}\frac{z_{i1}^2}{c\sigma_{i1}^4}\exp\left(-\frac{z_{i1}^2 d}{2}\right)dz_{i1}\right] \\
&= \frac{(1-\alpha)(1-\pi_0)}{\sqrt{2\pi}\tau^{3/2}c^{3/2}\sigma_{i1}}\left[\sqrt{\frac{2\pi}{d}}g+\frac{\sqrt{2\pi}}{cd^{3/2}\sigma_{i1}^4}\left(g-\sqrt{\frac{2}{\pi}}f\exp\left(-\frac{f^2}{2}\right)\right)\right] \\
&= \frac{(1-\alpha)(1-\pi_0)}{\sigma_{i1}\sqrt{\tau^3c^3d}}\left\{\frac{1}{cd\sigma_{i1}^4}\left[g-\sqrt{\frac{2}{\pi}}f\exp\left(-\frac{f^2}{2}\right)\right]+g\right\} \\
&= (1-\alpha)(1-\pi_0)h.
\end{aligned}
$$

Hence

$$B_i(\alpha,\pi_0,\tau) = (1-\alpha)[(1-\pi_0)h+\pi_0(2\gamma-1)].$$

The prior density on $\zeta_i, i = 1, ..., M$, given W_i and τ can be expressed as

$$f(\zeta_i|\tau,W_i) = (1-W_i)\delta_0 + W_i\frac{1}{\sqrt{2\pi\tau}}\exp\left(-\frac{\zeta_i^2}{2\tau}\right)$$

where δ_0 denotes a unit mass at 0. The following results are useful in describing the marginal posterior density function.

Lemma 1 *For a given value of M and $i \le 73$,*

$$A_i(\alpha,\pi_0,\tau) = \int f(\zeta_i|\tau,W_i)\pi_0^{1-W_i}(1-\pi_0)^{W_i}\prod_{j=1}^{2}f(z_{ij},R_i,S_i|\zeta_i)d\zeta_i dW_i,$$

and let

$$
\begin{aligned}
a^* &= \frac{1}{2\pi\sigma_{i1}\sigma_{i2}\tau^{1/2}}, \quad w^* = \frac{1}{\sigma_{i1}^2}+\frac{1}{\sigma_{i2}^2}+\frac{1}{\tau}, \\
b^* &= a^*\exp\left\{-0.5\left[\frac{z_{i1}^2}{\sigma_{i1}^2}+\frac{z_{i2}^2}{\sigma_{i2}^2}-\frac{1}{w^*}\left(\frac{z_{i1}}{\sigma_{i1}^2}+\frac{z_{i2}}{\sigma_{i2}^2}\right)^2\right]\right\},
\end{aligned}
$$

then

$$A_i(\alpha,\pi_0,\tau) = \pi_0\alpha^{1-S_i}\prod_{j=1}^{2}\phi[z_{ij}|0,\sigma_{ij}^2]+(1-\pi_0)\alpha^{1-S_i}b^*w^{*-1/2}.$$

Lemma 2 *For a given value of M and $i \geq 73$,*

$$B_i(\alpha, \pi_0, \tau) = \int f(\zeta_i|\tau, W_i)\pi_0^{1-W_i}(1-\pi_0)^{W_i}\prod_{j=1}^{2} f(z_{ij}, R_i, S_i|\zeta_i)dz_{i1}dz_{i2}d\zeta_i dW_i,$$

Let

$$c^* = \frac{1}{\sigma_{i1}^2} + \frac{1}{\tau}, \quad d^* = \frac{1}{\sigma_{i1}^2} - \frac{1}{c^*\sigma_{i1}^4}, \quad f^* = \sqrt{d^*}b_i = \sqrt{d^*}q_\gamma\sigma_{i1}, \quad g^* = \Phi(f^*) - \Phi(-f^*),$$

$$h^* = \frac{g^*}{\sqrt{\tau c^* d^*}\sigma_{i1}}$$

then

$$B_i(\alpha, \pi_0, \tau) = (1-\alpha)[(1-\pi_0)h^* + \pi_0(2\gamma - 1)].$$

Proofs are similar to the proofs for Lemma 3.1 and Lemma 3.2.

Based on the results above, it follows that the marginal posterior distribution on (M, α, π_0, τ) can be expressed in the same form as Eq(11).

A.2 Sensitivity analysis

In this section we examine the sensitivity of our conclusions to changes in the prior on M and the parametric form assumed for the effect sizes. We considered two improper priors on M, namely M^{-1} and M^{-2}, and two parametric forms of the effect sizes, the moment prior (discussed in the body of the paper) and a normal prior with mean 0 and variance τ. For each of the 4 prior combinations, an MCMC chain was run for 10^6 iterations following a 10^5 burn-in period. The posterior densities are plotted in Figure 1, and Tables 1-4 provide the posterior means, medians, and 95% credible intervals for each combination of priors for α, M, π_0, and τ.

Sensitivity to M

The posterior distributions of α, π_0, and τ are essentially insensitive to the prior on M for a given parametric form of the effect sizes. There appears to be a slight shift in the

Effect Size Prior	M Prior	Posterior Mean	Posterior Median	2.5%	97.5%
Moment	M^{-1}	.00540	.00479	.00122	.0131
	M^{-2}	.00545	.00484	.00124	.0131
Normal	M^{-1}	.00575	.00510	.00131	.0139
	M^{-2}	.00584	.00518	.00133	.0141

Table 1: Posterior means, medians, and 95% credible intervals for α with varying priors on effect size and M.

posterior distribution of M, with the posterior shifted towards smaller values for the M^2 prior. However this shift is of little practical importance as the posterior mean and medians only a change at most by 8 and 7 respectively. Overall, the model is insensitive to the choice of prior on M.

Sensitivity to Prior on Effect Sizes

The model is far more sensitive to changes in the prior on the effect sizes. The most striking sensitivities are found in the posterior distribution of τ and π_0. However, it is important to note that τ represents a slightly different parameter in both models (although it is still a measure of the spread of the effect sizes in both models), so this result should be neither surprising nor worrying. The posterior of π_0 is also sensitive, noting moderate changes in both location and scale, however the same general conclusions of π_0 being alarmingly high are still warranted in either model. M and α both appear to be very insensitive to the parametric form of the effect sizes.

Comparison of Distribution of Effect Sizes

Since the posterior distribution of π_0 is particularly sensitive to the choice of prior for the effect sizes, it is of interest to determine which prior provides a better fit to the effect sizes. To assess model fit, a Bayesian χ^2 goodness of fit (GOF) test was performed using pivotal quantities for both the normal and moment priors. After drawing values of ζ_i from the

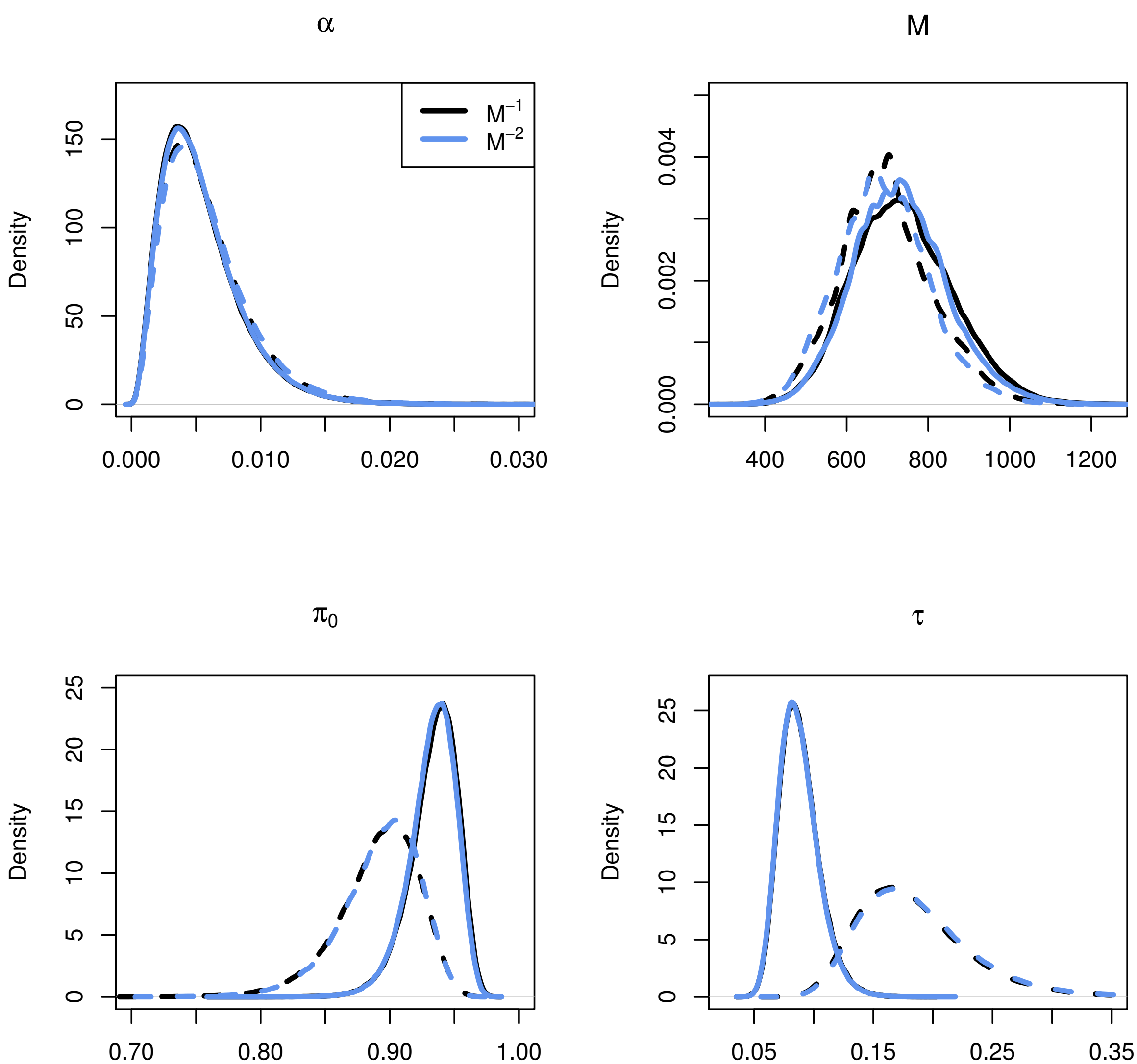

Figure 1: Posterior Densities for various combinations of priors for M and the effect sizes. Moment priors (———), normal priors (- - - - -).

Effect Size Prior	M Prior	Posterior Mean	Posterior Median	2.5%	97.5%
Moment	M^{-1}	739	732	519	995
	M^{-2}	732	728	519	979
Normal	M^{-1}	698	692	487	941
	M^{-2}	690	685	484	922

Table 2: Posterior means, medians, and 95% credible intervals for M with varying priors on effect size and M.

Effect Size Prior	M Prior	Posterior Mean	Posterior Median	2.5%	97.5%
Moment	M^{-1}	.934	.936	.891	.963
	M^{-2}	.933	.935	.892	.962
Normal	M^{-1}	.891	.895	.818	.941
	M^{-2}	.892	.896	.821	.941

Table 3: Posterior means, medians, and 95% credible intervals for π_0 with varying priors on effect size and M.

Effect Size Prior	M Prior	Posterior Mean	Posterior Median	2.5%	97.5%
Moment	M^{-1}	.0881	.0862	.0608	.126
	M^{-2}	.0877	.0859	.0605	.125
Normal	M^{-1}	.184	.177	.112	.296
	M^{-2}	.186	.179	.113	.301

Table 4: Posterior means, medians, and 95% credible intervals for τ with varying priors on effect size and M.

posterior distribution using MCMC, a χ^2 GOF test was performed comparing the values of $|\zeta_{ij}|$ (since the values of ζ_{ij} were arbitrarily signed) to the density $2 \times f(\zeta_i|\tau, W_i)$, $\zeta_i > 0$ where f is the parametric form of the effect sizes (moment or normal). Note that only values of ζ_i were used in which $W_i = 1$ on each MCMC draw since we are only interested in the distribution of effect sizes when the null hypothesis is false. Three bins were chosen using the 1/3 and 2/3 quantiles of $2f$. Thus the χ^2 GOF values calculated from the posterior draws of ζ_i should follow a χ^2_2 distribution if the model has been specified correctly.

The χ^2 values were calculated from a random sample of 10,000 posterior draws for each of the 4 model combinations in the sensitivity analysis. Figure 2 plots the histogram of the calculated χ^2 values and the theoretical χ^2_2 distribution for each of the cases. In each instance, the moment prior appears to be a much better fit than the normal prior, indicating that we have selected an appropriate model for the effect sizes. There does not seem to be any indication that changing the prior for M strongly influences the fit of the effect sizes.

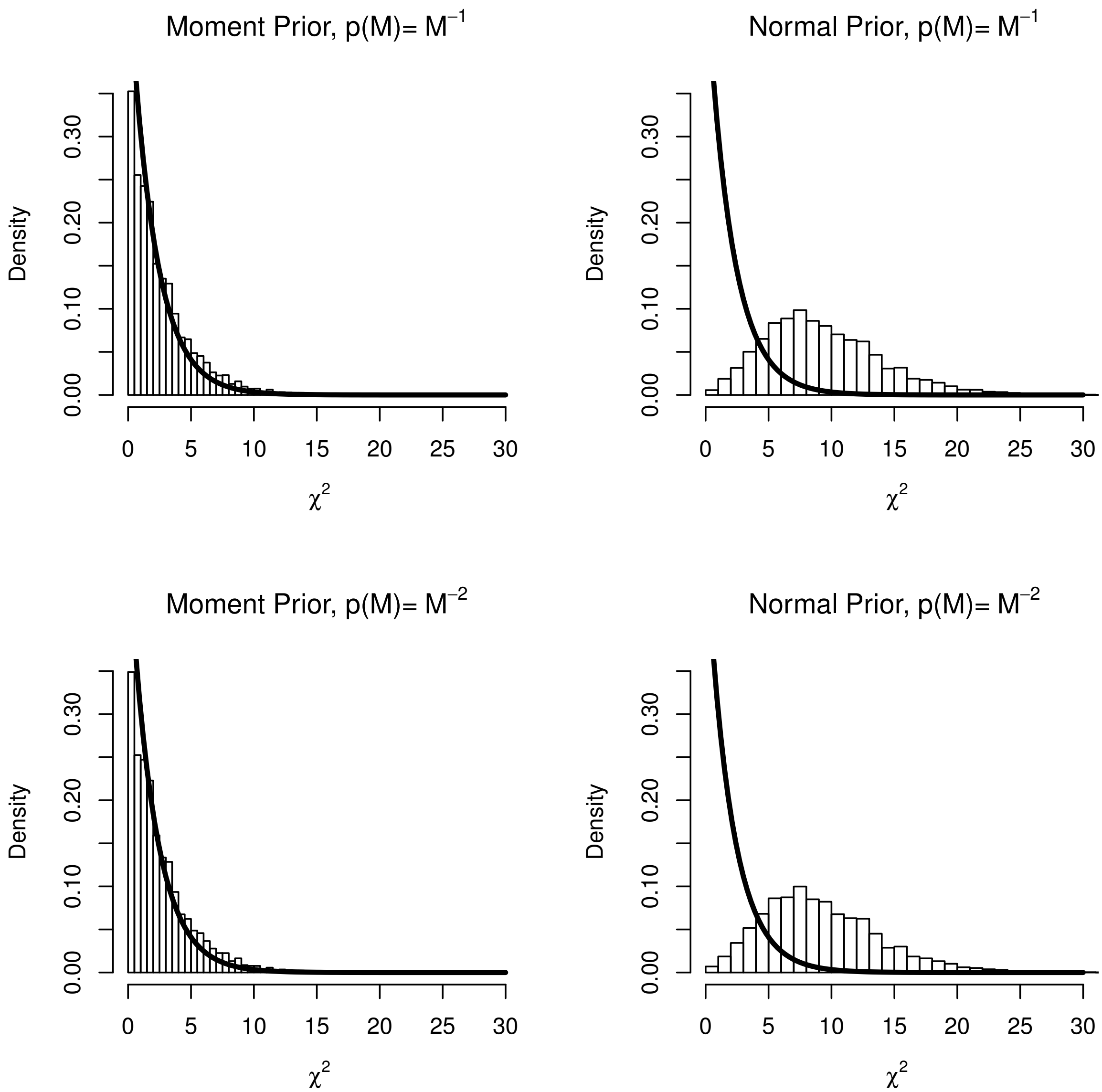

Figure 2: Posterior pivotal quantities (histogram) vs a χ_2^2 distribution (—–). The moment prior appears to be a more appropriate model for the effect sizes.