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Supplementary Materials for

Structure and topology of band structures in the 1651 magnetic space groups

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Section S1. Tables for spinful electrons

Here we include the following tables for spinful electrons:

- Tables S1–S6: d, $X_{\rm BS}$, and $\nu_{\rm BS}$ for MSGs.
- Table S7: d, $X_{\rm BS}$, and $\nu_{\rm BS}$ for MLGs.
- Table S8: $\{\nu\}_{\rm BS}$ and $\{\nu\}_{\rm AI}$ for exceptional MSGs.

Instructions on how to read the tables can be found in Tables I and II of the main text.

We remark that the results on d, $X_{\rm BS}$, and $\nu_{\rm BS}$ for types I and II MSGs, which account for 460 out of the 1651 entries, were already published in Ref. [15]. For the readers' convenience, we reproduce them here alongside with the new results on the remaining 1191 type III and IV MSGs. Similarly, the data on exceptional filling patterns among the type II MSGs (i.e., for time-reversal symmetric systems) were reported in Refs. [9,10], but are also reproduced below.

Table S1. Characterization of MSGs in the triclinic family for spinful electrons.

MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$
1.1 I	1	(1)	1	1.3 IV	1	(1)	2	2.5 II	9	$(2,2,2,4)$	2	2.7 IV	5	(2)	2
1.2 II	1	(1)	2	2.4 I	9	$(2,2,2,4)$	1	2.6 III	1	(1)	2				

d: Rank of the band structure group $\{\rm BS\}$

$X_{\rm BS}$: Symmetry-based indicators of band topology

$\nu_{\rm BS}$: Set of ν bands are symmetry-forbidden from being isolated by band gaps if $\nu \notin \nu_{\rm BS}\,\mathbb{Z}$

Table. S2. Characterization of MSGs in the monoclinic family for spinful electrons.

MSG	d	X_{BS}	ν_{BS}	MSG	d	X_{BS}	ν_{BS}	MSG	d	X_{BS}	ν_{BS}	MSG	d	X_{BS}	ν_{BS}
3.1 I	5	(2)	1	7.24 I	1	(1)	2	10.47 IV	7	(2,2)	2	13.70 IV	4	(2)	4
3.2 II	1	(1)	2	7.25 II	1	(1)	4	10.48 IV	6	(2)	2	13.71 IV	3	(2)	4
3.3 III	1	(1)	1	7.26 III	1	(1)	2	10.49 IV	8	(2)	2	13.72 IV	5	(2)	2
3.4 IV	3	(2)	2	7.27 IV	1	(1)	4	11.50 I	6	(2)	2	13.73 IV	3	(2)	4
3.5 IV	1	(1)	2	7.28 IV	1	(1)	2	11.51 II	5	(2,2,4)	4	13.74 IV	4	(2)	4
3.6 IV	3	(1)	2	7.29 IV	1	(1)	4	11.52 III	2	(1)	2	14.75 I	5	(2)	2
4.7 I	1	(1)	2	7.30 IV	1	(1)	4	11.53 III	1	(1)	2	14.76 II	5	(2,4)	4
4.8 II	1	(1)	4	7.31 IV	1	(1)	2	11.54 III	5	(2,2,4)	2	14.77 III	1	(1)	4
4.9 III	1	(1)	2	8.32 I	2	(1)	1	11.55 IV	3	(2)	4	14.78 III	1	(1)	4
4.10 IV	1	(1)	4	8.33 II	1	(1)	2	11.56 IV	5	(2)	2	14.79 III	5	(2,4)	2
4.11 IV	1	(1)	2	8.34 III	1	(1)	1	11.57 IV	3	(2)	4	14.80 IV	3	(2)	4
4.12 IV	1	(1)	2	8.35 IV	1	(1)	2	12.58 I	10	(2,2)	1	14.81 IV	3	(2)	4
5.13 I	3	(1)	1	8.36 IV	1	(1)	2	12.59 II	7	(2,2,4)	2	14.82 IV	3	(2)	4
5.14 II	1	(1)	2	9.37 I	1	(1)	2	12.60 III	1	(1)	2	14.83 IV	5	(2)	2
5.15 III	1	(1)	1	9.38 II	1	(1)	4	12.61 III	1	(1)	2	14.84 IV	3	(2)	4
5.16 IV	2	(1)	2	9.39 III	1	(1)	2	12.62 III	7	(2,2,4)	1	15.85 I	6	(2,2)	2
5.17 IV	1	(1)	2	9.40 IV	1	(1)	2	12.63 IV	5	(2)	2	15.86 II	5	(2,4)	4
6.18 I	3	(1)	1	9.41 IV	1	(1)	4	12.64 IV	5	(2)	2	15.87 III	1	(1)	2
6.19 II	1	(1)	2	10.42 I	15	(2,2,2)	1	13.65 I	7	(2,2)	2	15.88 III	2	(1)	2
6.20 III	1	(1)	1	10.43 II	9	(2,2,2,4)	2	13.66 II	5	(2,2,4)	4	15.89 III	5	(2,4)	2
6.21 IV	1	(1)	2	10.44 III	1	(1)	2	13.67 III	1	(1)	2	15.90 IV	4	(2)	2
6.22 IV	2	(1)	2	10.45 III	1	(1)	2	13.68 III	3	(1)	2	15.91 IV	3	(2)	4
6.23 IV	2	(1)	2	10.46 III	9	(2,2,2,4)	1	13.69 III	5	(2,2,4)	2				

d: Rank of the band structure group $\{\mathrm{BS}\}$
X_{BS}: Symmetry-based indicators of band topology
ν_{BS}: Set of ν bands are symmetry-forbidden from being isolated by band gaps if $\nu \notin \nu_{\mathrm{BS}}\,\mathbb{Z}$

Table S3. Characterization of MSGs in the orthorhombic family for spinful electrons.

MSG	d	X_{BS}	ν_{BS}	MSG	d	X_{BS}	ν_{BS}	MSG	d	X_{BS}	ν_{BS}	MSG	d	X_{BS}	ν_{BS}
16.1 I	1	(1)	2	23.51 III	3	(1)	1	29.101 III	1	(1)	4	33.151 IV	1	(1)	4
16.2 II	1	(1)	2	23.52 IV	1	(1)	2	29.102 III	1	(1)	4	33.152 IV	1	(1)	4
16.3 III	5	(2)	1	24.53 I	4	(1)	2	29.103 III	1	(1)	4	33.153 IV	1	(1)	4
16.4 IV	1	(1)	2	24.54 II	1	(1)	4	29.104 IV	1	(1)	4	33.154 IV	1	(1)	4
16.5 IV	1	(1)	4	24.55 III	2	(1)	2	29.105 IV	1	(1)	8	33.155 IV	1	(1)	4
16.6 IV	1	(1)	4	24.56 IV	3	(1)	2	29.106 IV	1	(1)	4	34.156 I	3	(1)	2
17.7 I	5	(1)	2	25.57 I	1	(1)	2	29.107 IV	1	(1)	4	34.157 II	1	(1)	4
17.8 II	1	(1)	4	25.58 II	1	(1)	2	29.108 IV	1	(1)	4	34.158 III	1	(1)	2
17.9 III	1	(1)	2	25.59 III	3	(1)	1	29.109 IV	1	(1)	4	34.159 III	3	(1)	2
17.10 III	3	(2)	2	25.60 III	5	(2)	1	29.110 IV	1	(1)	4	34.160 IV	2	(1)	4
17.11 IV	2	(1)	4	25.61 IV	1	(1)	4	30.111 I	3	(1)	2	34.161 IV	1	(1)	4
17.12 IV	5	(1)	2	25.62 IV	1	(1)	2	30.112 II	1	(1)	4	34.162 IV	2	(1)	4
17.13 IV	4	(1)	2	25.63 IV	1	(1)	4	30.113 III	1	(1)	2	34.163 IV	3	(2)	4
17.14 IV	3	(1)	4	25.64 IV	1	(1)	4	30.114 III	1	(1)	2	34.164 IV	3	(1)	2
17.15 IV	3	(1)	4	25.65 IV	1	(1)	4	30.115 III	3	(1)	2	35.165 I	2	(1)	2
18.16 I	3	(1)	2	26.66 I	1	(1)	2	30.116 IV	2	(1)	4	35.166 II	1	(1)	2
18.17 II	1	(1)	4	26.67 II	1	(1)	4	30.117 IV	3	(2)	4	35.167 III	2	(1)	1
18.18 III	3	(2)	2	26.68 III	1	(1)	2	30.118 IV	1	(1)	4	35.168 III	4	(2)	1
18.19 III	1	(1)	2	26.69 III	3	(1)	2	30.119 IV	3	(1)	2	35.169 IV	1	(1)	4
18.20 IV	2	(1)	4	26.70 III	1	(1)	2	30.120 IV	2	(1)	4	35.170 IV	2	(1)	2
18.21 IV	1	(1)	4	26.71 IV	1	(1)	4	30.121 IV	2	(1)	4	35.171 IV	2	(1)	4
18.22 IV	2	(1)	4	26.72 IV	1	(1)	4	30.122 IV	2	(1)	4	36.172 I	1	(1)	2
18.23 IV	3	(1)	2	26.73 IV	1	(1)	2	31.123 I	2	(1)	2	36.173 II	1	(1)	4
18.24 IV	3	(1)	2	26.74 IV	1	(1)	2	31.124 II	1	(1)	4	36.174 III	1	(1)	2
19.25 I	1	(1)	4	26.75 IV	1	(1)	4	31.125 III	1	(1)	2	36.175 III	2	(1)	2
19.26 II	1	(1)	8	26.76 IV	1	(1)	4	31.126 III	2	(1)	2	36.176 III	1	(1)	2
19.27 III	1	(1)	4	26.77 IV	1	(1)	4	31.127 III	1	(1)	2	36.177 IV	1	(1)	2
19.28 IV	1	(1)	4	27.78 I	1	(1)	2	31.128 IV	2	(1)	4	36.178 IV	1	(1)	4
19.29 IV	1	(1)	4	27.79 II	1	(1)	4	31.129 IV	1	(1)	4	36.179 IV	1	(1)	2
19.30 IV	1	(1)	4	27.80 III	1	(1)	2	31.130 IV	1	(1)	4	37.180 I	2	(1)	2
20.31 I	3	(1)	2	27.81 III	5	(2)	2	31.131 IV	1	(1)	4	37.181 II	1	(1)	4
20.32 II	1	(1)	4	27.82 IV	1	(1)	2	31.132 IV	2	(1)	2	37.182 III	1	(1)	2
20.33 III	1	(1)	2	27.83 IV	1	(1)	4	31.133 IV	2	(1)	4	37.183 III	4	(2)	2
20.34 III	2	(1)	2	27.84 IV	1	(1)	4	31.134 IV	2	(1)	2	37.184 IV	1	(1)	2
20.35 IV	3	(1)	2	27.85 IV	1	(1)	4	32.135 I	3	(1)	2	37.185 IV	2	(2)	4
20.36 IV	1	(1)	4	27.86 IV	1	(1)	4	32.136 II	1	(1)	4	37.186 IV	2	(1)	2
20.37 IV	3	(1)	2	28.87 I	4	(1)	2	32.137 III	1	(1)	2	38.187 I	1	(1)	2
21.38 I	2	(1)	2	28.88 II	1	(1)	4	32.138 III	3	(2)	2	38.188 II	1	(1)	2
21.39 II	1	(1)	2	28.89 III	1	(1)	2	32.139 IV	1	(1)	4	38.189 III	2	(1)	1
21.40 III	4	(2)	1	28.90 III	2	(1)	2	32.140 IV	2	(1)	4	38.190 III	3	(1)	1
21.41 III	3	(1)	1	28.91 III	3	(2)	2	32.141 IV	3	(1)	2	38.191 III	3	(1)	1
21.42 IV	1	(1)	2	28.92 IV	4	(1)	2	32.142 IV	2	(1)	4	38.192 IV	1	(1)	2
21.43 IV	2	(1)	2	28.93 IV	2	(1)	4	32.143 IV	3	(1)	4	38.193 IV	1	(1)	2
21.44 IV	2	(1)	2	28.94 IV	1	(1)	4	33.144 I	1	(1)	4	38.194 IV	1	(1)	4
22.45 I	1	(1)	2	28.95 IV	2	(1)	4	33.145 II	1	(1)	8	39.195 I	1	(1)	2
22.46 II	1	(1)	2	28.96 IV	4	(1)	4	33.146 III	1	(1)	4	39.196 II	1	(1)	4
22.47 III	3	(1)	1	28.97 IV	3	(1)	2	33.147 III	1	(1)	4	39.197 III	2	(1)	2
22.48 IV	1	(1)	2	28.98 IV	3	(1)	4	33.148 III	1	(1)	4	39.198 III	1	(1)	2
23.49 I	1	(1)	2	29.99 I	1	(1)	4	33.149 IV	1	(1)	4	39.199 III	3	(2)	2
23.50 II	1	(1)	2	29.100 II	1	(1)	8	33.150 IV	1	(1)	8	39.200 IV	1	(1)	4

(Continued from the previous page)

39.201	IV	1	(1)	2	47.251	III	1	(1)	2	51.301	IV	4	(2)	4	54.351	IV	3	(2)	4
39.202	IV	1	(1)	2	47.252	III	15	(2, 2, 2)	1	51.302	IV	4	(2, 2)	4	54.352	IV	3	(2)	8
40.203	I	3	(1)	2	47.253	III	1	(1)	2	51.303	IV	5	(2)	4	55.353	I	7	(2, 2)	2
40.204	II	1	(1)	4	47.254	IV	5	(2, 2, 4)	4	51.304	IV	4	(2)	4	55.354	II	5	(2, 4)	4
40.205	III	1	(1)	2	47.255	IV	5	(2, 4)	4	52.305	I	5	(2)	4	55.355	III	1	(1)	4
40.206	III	2	(1)	2	47.256	IV	5	(4)	4	52.306	II	3	(4)	8	55.356	III	1	(1)	4
40.207	III	2	(1)	2	48.257	I	3	(2)	4	52.307	III	2	(1)	4	55.357	III	9	(2, 2, 2)	2
40.208	IV	3	(1)	2	48.258	II	3	(2, 4)	4	52.308	III	1	(1)	4	55.358	III	5	(2)	2
40.209	IV	1	(1)	4	48.259	III	3	(1)	2	52.309	III	2	(1)	4	55.359	III	1	(1)	4
40.210	IV	3	(1)	2	48.260	III	5	(2)	2	52.310	III	4	(2)	4	55.360	IV	4	(2, 2)	4
41.211	I	2	(1)	2	48.261	III	1	(1)	4	52.311	III	4	(2)	4	55.361	IV	3	(2)	4
41.212	II	1	(1)	4	48.262	IV	2	(2)	4	52.312	III	3	(2)	4	55.362	IV	4	(2)	4
41.213	III	1	(1)	2	48.263	IV	3	(2)	4	52.313	III	3	(1)	4	55.363	IV	7	(2, 2)	2
41.214	III	1	(1)	2	48.264	IV	2	(2)	4	52.314	IV	3	(2)	4	55.364	IV	5	(2)	4
41.215	III	2	(2)	2	49.265	I	6	(2)	2	52.315	IV	4	(2)	4	56.365	I	3	(2)	4
41.216	IV	2	(1)	4	49.266	II	5	(2, 2, 4)	4	52.316	IV	3	(2)	8	56.366	II	3	(4)	8
41.217	IV	1	(1)	4	49.267	III	3	(1)	2	52.317	IV	4	(2)	4	56.367	III	1	(1)	4
41.218	IV	2	(1)	2	49.268	III	1	(1)	2	52.318	IV	3	(2)	4	56.368	III	1	(1)	4
42.219	I	1	(1)	2	49.269	III	10	(2, 2)	2	52.319	IV	4	(2)	4	56.369	III	5	(2, 2)	4
42.220	II	1	(1)	2	49.270	III	7	(2, 2)	2	52.320	IV	4	(2)	4	56.370	III	3	(2)	4
42.221	III	2	(1)	1	49.271	III	1	(1)	4	53.321	I	9	(2)	2	56.371	III	3	(1)	4
42.222	III	3	(2)	1	49.272	IV	3	(2)	4	53.322	II	5	(2, 4)	4	56.372	IV	2	(2)	8
42.223	IV	1	(1)	2	49.273	IV	6	(2)	2	53.323	III	1	(1)	4	56.373	IV	2	(2)	4
43.224	I	2	(1)	2	49.274	IV	4	(2)	4	53.324	III	2	(1)	4	56.374	IV	3	(2)	4
43.225	II	1	(1)	4	49.275	IV	3	(2)	4	53.325	III	1	(1)	4	56.375	IV	3	(2)	4
43.226	III	1	(1)	2	49.276	IV	4	(2)	4	53.326	III	5	(2)	2	56.376	IV	2	(2)	4
43.227	III	2	(1)	2	50.277	I	3	(2)	4	53.327	III	8	(2, 2)	2	57.377	I	4	(2)	4
43.228	IV	1	(1)	4	50.278	II	3	(2, 4)	4	53.328	III	6	(2, 2)	2	57.378	II	3	(2, 4)	8
44.229	I	1	(1)	2	50.279	III	3	(1)	2	53.329	III	2	(1)	4	57.379	III	3	(1)	4
44.230	II	1	(1)	2	50.280	III	3	(1)	2	53.330	IV	5	(2)	4	57.380	III	1	(1)	4
44.231	III	2	(1)	1	50.281	III	5	(2, 2)	2	53.331	IV	4	(2)	4	57.381	III	1	(1)	4
44.232	III	3	(1)	1	50.282	III	5	(2)	2	53.332	IV	6	(2)	4	57.382	III	4	(2)	4
44.233	IV	1	(1)	4	50.283	III	1	(1)	4	53.333	IV	5	(2)	4	57.383	III	4	(2, 2)	4
44.234	IV	1	(1)	2	50.284	IV	3	(2)	4	53.334	IV	8	(2)	2	57.384	III	3	(2)	4
45.235	I	1	(1)	2	50.285	IV	2	(2)	4	53.335	IV	6	(2)	4	57.385	III	2	(1)	4
45.236	II	1	(1)	4	50.286	IV	2	(2)	8	53.336	IV	5	(2)	4	57.386	IV	2	(2)	8
45.237	III	1	(1)	2	50.287	IV	3	(2)	4	54.337	I	4	(2)	4	57.387	IV	3	(2)	4
45.238	III	3	(2)	2	50.288	IV	2	(2)	8	54.338	II	3	(2, 4)	8	57.388	IV	4	(2)	4
45.239	IV	1	(1)	2	51.289	I	7	(2, 2)	2	54.339	III	1	(1)	4	57.389	IV	4	(2)	4
45.240	IV	1	(1)	4	51.290	II	5	(2, 2, 4)	4	54.340	III	2	(1)	4	57.390	IV	3	(2)	8
46.241	I	2	(1)	2	51.291	III	1	(1)	2	54.341	III	1	(1)	4	57.391	IV	2	(2)	4
46.242	II	1	(1)	4	51.292	III	2	(1)	2	54.342	III	5	(2, 2)	4	57.392	IV	3	(2)	4
46.243	III	1	(1)	2	51.293	III	1	(1)	4	54.343	III	3	(2)	4	58.393	I	8	(2)	2
46.244	III	2	(1)	2	51.294	III	7	(2, 2)	2	54.344	III	4	(2, 2)	4	58.394	II	5	(4)	4
46.245	III	2	(1)	2	51.295	III	6	(2)	2	54.345	III	4	(1)	4	58.395	III	1	(1)	4
46.246	IV	1	(1)	2	51.296	III	9	(2, 2, 2)	2	54.346	IV	4	(2)	4	58.396	III	1	(1)	4
46.247	IV	2	(1)	2	51.297	III	3	(1)	2	54.347	IV	2	(2)	8	58.397	III	8	(2, 2)	2
46.248	IV	2	(1)	4	51.298	IV	7	(2, 2)	2	54.348	IV	3	(2)	4	58.398	III	5	(2)	2
47.249	I	9	(2, 2, 2, 4)	2	51.299	IV	3	(2)	4	54.349	IV	2	(2)	8	58.399	III	1	(1)	4
47.250	II	9	(2, 2, 2, 4)	2	51.300	IV	4	(2, 2)	4	54.350	IV	4	(2)	4	58.400	IV	4	(2)	4

(Continued from the previous page)

58.401	IV	4	(2)	4	62.442	II	3	(4)	8	65.483	III	1	(1)	2	69.524	III	9	(2, 2)	1
58.402	IV	5	(2)	4	62.443	III	1	(1)	4	65.484	III	1	(1)	2	69.525	III	1	(1)	2
58.403	IV	5	(2)	4	62.444	III	1	(1)	4	65.485	III	12	(2, 2, 2)	1	69.526	IV	5	(4)	2
58.404	IV	8	(2)	2	62.445	III	2	(1)	4	65.486	III	10	(2, 2)	1	70.527	I	3	(2)	4
59.405	I	3	(2)	4	62.446	III	3	(2)	4	65.487	III	1	(1)	2	70.528	II	3	(4)	4
59.406	II	3	(2, 4)	4	62.447	III	3	(2)	4	65.488	IV	4	(4)	4	70.529	III	2	(1)	2
59.407	III	2	(1)	2	62.448	III	4	(2)	4	65.489	IV	6	(2, 4)	2	70.530	III	4	(2)	2
59.408	III	1	(1)	4	62.449	III	1	(1)	4	65.490	IV	5	(4)	4	70.531	III	1	(1)	4
59.409	III	5	(2, 2)	2	62.450	IV	2	(2)	4	66.491	I	7	(2)	2	70.532	IV	2	(2)	4
59.410	III	4	(2)	2	62.451	IV	3	(2)	4	66.492	II	5	(2, 4)	4	71.533	I	6	(4)	2
59.411	III	3	(1)	2	62.452	IV	2	(2)	8	66.493	III	2	(1)	2	71.534	II	6	(2, 4)	2
59.412	IV	3	(2)	4	62.453	IV	2	(2)	4	66.494	III	1	(1)	2	71.535	III	1	(1)	2
59.413	IV	2	(2)	8	62.454	IV	3	(2)	4	66.495	III	9	(2, 2)	2	71.536	III	9	(2, 2)	1
59.414	IV	2	(2)	8	62.455	IV	2	(2)	8	66.496	III	6	(2, 2)	2	71.537	III	1	(1)	2
59.415	IV	3	(2)	4	62.456	IV	3	(2)	4	66.497	III	1	(1)	4	71.538	IV	4	(4)	4
59.416	IV	2	(2)	4	63.457	I	5	(2)	2	66.498	IV	5	(2)	2	72.539	I	4	(2)	2
60.417	I	4	(2)	4	63.458	II	4	(2, 4)	4	66.499	IV	4	(2)	4	72.540	II	4	(2, 4)	4
60.418	II	3	(4)	8	63.459	III	2	(1)	2	66.500	IV	6	(2)	2	72.541	III	2	(1)	2
60.419	III	1	(1)	4	63.460	III	1	(1)	4	67.501	I	5	(2)	2	72.542	III	1	(1)	2
60.420	III	2	(1)	4	63.461	III	1	(1)	2	67.502	II	5	(2, 2, 4)	4	72.543	III	7	(2, 2)	2
60.421	III	1	(1)	4	63.462	III	5	(2)	2	67.503	III	1	(1)	2	72.544	III	5	(2, 2)	2
60.422	III	3	(2)	4	63.463	III	6	(2, 2)	2	67.504	III	2	(1)	2	72.545	III	1	(1)	4
60.423	III	3	(2)	4	63.464	III	5	(2, 2)	2	67.505	III	7	(2, 2)	2	72.546	IV	4	(2)	2
60.424	III	4	(2, 2)	4	63.465	III	2	(1)	2	67.506	III	8	(2, 2)	2	72.547	IV	3	(2)	4
60.425	III	2	(1)	4	63.466	IV	5	(2)	2	67.507	III	2	(1)	2	73.548	I	3	(2)	4
60.426	IV	3	(2)	8	63.467	IV	3	(2)	4	67.508	IV	3	(2)	4	73.549	II	3	(2, 4)	8
60.427	IV	2	(2)	8	63.468	IV	4	(2)	4	67.509	IV	5	(2)	2	73.550	III	1	(1)	4
60.428	IV	3	(2)	4	64.469	I	5	(2)	2	67.510	IV	3	(2)	4	73.551	III	4	(2, 2)	4
60.429	IV	2	(2)	8	64.470	II	4	(2, 4)	4	68.511	I	3	(2)	4	73.552	III	4	(1)	4
60.430	IV	3	(2)	4	64.471	III	1	(1)	4	68.512	II	3	(2, 4)	4	73.553	IV	3	(2)	4
60.431	IV	4	(2)	4	64.472	III	1	(1)	4	68.513	III	2	(1)	2	74.554	I	7	(2)	2
60.432	IV	4	(2)	4	64.473	III	1	(1)	4	68.514	III	2	(1)	2	74.555	II	5	(2, 4)	4
61.433	I	3	(2)	4	64.474	III	4	(2)	2	68.515	III	5	(2, 2)	2	74.556	III	1	(1)	2
61.434	II	3	(4)	8	64.475	III	6	(2, 2)	2	68.516	III	4	(2, 2)	2	74.557	III	1	(1)	4
61.435	III	1	(1)	8	64.476	III	5	(2, 2)	2	68.517	III	2	(1)	4	74.558	III	6	(2, 2)	2
61.436	III	3	(2)	4	64.477	III	2	(1)	4	68.518	IV	3	(2)	4	74.559	III	7	(2, 2)	2
61.437	III	1	(1)	8	64.478	IV	4	(2)	4	68.519	IV	3	(2)	4	74.560	III	2	(1)	2
61.438	IV	2	(2)	8	64.479	IV	3	(2)	4	68.520	IV	3	(2)	4	74.561	IV	5	(2)	4
61.439	IV	3	(2)	4	64.480	IV	5	(2)	2	69.521	I	6	(4)	2	74.562	IV	5	(2)	2
61.440	IV	2	(2)	8	65.481	I	8	(2, 4)	2	69.522	II	6	(2, 2, 4)	2					
62.441	I	3	(2)	4	65.482	II	7	(2, 2, 4)	2	69.523	III	1	(1)	2					

d: Rank of the band structure group $\{\mathrm{BS}\}$
X_{BS}: Symmetry-based indicators of band topology
ν_{BS}: Set of ν bands are symmetry-forbidden from being isolated by band gaps if $\nu \notin \nu_{\mathrm{BS}}\,\mathbb{Z}$

Table S4. Characterization of MSGs in the tetragonal family for spinful electrons.

MSG	d	X_{BS}	ν_{BS}	MSG	d	X_{BS}	ν_{BS}	MSG	d	X_{BS}	ν_{BS}	MSG	d	X_{BS}	ν_{BS}
75.1 I	8	(4)	1	84.51 I	13	(2, 4)	2	90.101 IV	4	(1)	2	97.151 I	2	(1)	2
75.2 II	3	(1)	2	84.52 II	7	(2, 4)	4	90.102 IV	4	(1)	2	97.152 II	2	(1)	2
75.3 III	2	(1)	2	84.53 III	7	(2)	2	91.103 I	4	(1)	4	97.153 III	1	(1)	2
75.4 IV	3	(1)	2	84.54 III	1	(1)	4	91.104 II	1	(1)	8	97.154 III	5	(2)	1
75.5 IV	4	(2)	2	84.55 III	5	(2)	2	91.105 III	3	(1)	4	97.155 III	1	(1)	2
75.6 IV	4	(1)	2	84.56 IV	6	(4)	4	91.106 III	1	(1)	4	97.156 IV	2	(1)	2
76.7 I	1	(1)	4	84.57 IV	6	(4)	4	91.107 III	2	(1)	4	98.157 I	2	(1)	4
76.8 II	1	(1)	8	84.58 IV	7	(4)	4	91.108 IV	4	(1)	4	98.158 II	1	(1)	4
76.9 III	1	(1)	4	85.59 I	11	(2, 4)	2	91.109 IV	2	(1)	8	98.159 III	3	(1)	2
76.10 IV	1	(1)	4	85.60 II	6	(2, 4)	4	91.110 IV	3	(1)	4	98.160 III	2	(1)	2
76.11 IV	1	(1)	8	85.61 III	3	(2)	4	92.111 I	2	(1)	4	98.161 III	1	(1)	4
76.12 IV	1	(1)	4	85.62 III	4	(1)	2	92.112 II	1	(1)	8	98.162 IV	2	(1)	4
77.13 I	4	(2)	2	85.63 III	6	(2)	2	92.113 III	1	(1)	4	99.163 I	3	(1)	2
77.14 II	1	(1)	4	85.64 IV	5	(2)	4	92.114 III	1	(1)	4	99.164 II	3	(1)	2
77.15 III	2	(1)	2	85.65 IV	6	(2)	4	92.115 III	2	(1)	4	99.165 III	2	(1)	2
77.16 IV	1	(1)	4	85.66 IV	5	(2)	4	92.116 IV	2	(1)	4	99.166 III	1	(1)	2
77.17 IV	2	(2)	4	86.67 I	9	(2, 2)	2	92.117 IV	1	(1)	8	99.167 III	8	(4)	1
77.18 IV	2	(1)	4	86.68 II	5	(2, 4)	4	92.118 IV	2	(1)	4	99.168 IV	3	(1)	4
78.19 I	1	(1)	4	86.69 III	3	(2)	4	93.119 I	1	(1)	4	99.169 IV	2	(1)	4
78.20 II	1	(1)	8	86.70 III	2	(1)	4	93.120 II	1	(1)	4	99.170 IV	2	(1)	4
78.21 III	1	(1)	4	86.71 III	6	(2)	2	93.121 III	1	(1)	2	100.171 I	4	(1)	2
78.22 IV	1	(1)	4	86.72 IV	4	(2)	4	93.122 III	4	(2)	2	100.172 II	2	(1)	4
78.23 IV	1	(1)	8	86.73 IV	5	(2)	4	93.123 III	2	(1)	2	100.173 III	1	(1)	4
78.24 IV	1	(1)	4	86.74 IV	4	(2)	4	93.124 IV	1	(1)	4	100.174 III	2	(1)	2
79.25 I	5	(2)	1	87.75 I	16	(4, 4)	1	93.125 IV	1	(1)	4	100.175 III	5	(4)	2
79.26 II	2	(1)	2	87.76 II	9	(2, 8)	2	93.126 IV	1	(1)	4	100.176 IV	2	(1)	4
79.27 III	1	(1)	2	87.77 III	5	(4)	2	94.127 I	2	(1)	4	100.177 IV	4	(1)	2
79.28 IV	2	(1)	2	87.78 III	2	(1)	2	94.128 II	1	(1)	4	100.178 IV	4	(1)	4
80.29 I	2	(1)	2	87.79 III	5	(2)	2	94.129 III	2	(1)	2	101.179 I	1	(1)	4
80.30 II	1	(1)	4	87.80 IV	8	(4)	2	94.130 III	3	(2)	2	101.180 II	1	(1)	4
80.31 III	2	(1)	2	88.81 I	8	(2, 2)	2	94.131 III	1	(1)	4	101.181 III	2	(1)	4
80.32 IV	1	(1)	4	88.82 II	5	(4)	4	94.132 IV	1	(1)	4	101.182 III	1	(1)	2
81.33 I	12	(2, 2, 4)	1	88.83 III	3	(2)	4	94.133 IV	2	(1)	4	101.183 III	4	(2)	2
81.34 II	5	(2)	2	88.84 III	1	(1)	4	94.134 IV	2	(1)	4	101.184 IV	1	(1)	4
81.35 III	2	(1)	2	88.85 III	6	(2)	2	95.135 I	4	(1)	4	101.185 IV	1	(1)	4
81.36 IV	5	(2)	2	88.86 IV	4	(2)	4	95.136 II	1	(1)	8	101.186 IV	1	(1)	8
81.37 IV	6	(2, 2)	2	89.87 I	3	(1)	2	95.137 III	3	(1)	4	102.187 I	2	(1)	4
81.38 IV	6	(2)	2	89.88 II	3	(1)	2	95.138 III	1	(1)	4	102.188 II	1	(1)	4
82.39 I	11	(2, 2, 2)	1	89.89 III	1	(1)	2	95.139 III	2	(1)	4	102.189 III	1	(1)	4
82.40 II	5	(2)	2	89.90 III	8	(4)	1	95.140 IV	4	(1)	4	102.190 III	2	(1)	2
82.41 III	1	(1)	2	89.91 III	2	(1)	2	95.141 IV	2	(1)	8	102.191 III	3	(2)	2
82.42 IV	5	(2)	2	89.92 IV	3	(1)	2	95.142 IV	3	(1)	4	102.192 IV	1	(1)	8
83.43 I	24	(4, 4, 4)	1	89.93 IV	2	(1)	4	96.143 I	2	(1)	4	102.193 IV	2	(1)	4
83.44 II	13	(2, 4, 8)	2	89.94 IV	2	(1)	4	96.144 II	1	(1)	8	102.194 IV	2	(1)	4
83.45 III	8	(2, 4)	2	90.95 I	4	(1)	2	96.145 III	1	(1)	4	103.195 I	3	(1)	2
83.46 III	3	(1)	2	90.96 II	2	(1)	4	96.146 III	1	(1)	4	103.196 II	3	(1)	4
83.47 III	5	(2)	2	90.97 III	2	(1)	2	96.147 III	2	(1)	4	103.197 III	2	(1)	4
83.48 IV	11	(4)	2	90.98 III	5	(4)	2	96.148 IV	2	(1)	4	103.198 III	1	(1)	4
83.49 IV	12	(4, 4)	2	90.99 III	1	(1)	4	96.149 IV	1	(1)	8	103.199 III	8	(4)	2
83.50 IV	12	(4)	2	90.100 IV	2	(1)	4	96.150 IV	2	(1)	4	103.200 IV	3	(1)	2

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103.201	IV	2	(1)	4	111.251	I	5	(2)	2	117.301	III	1	(1)	4	124.351	I	11	(4)	2
103.202	IV	2	(1)	4	111.252	II	5	(2)	2	117.302	III	2	(1)	2	124.352	II	8	$(2,8)$	4
104.203	I	4	(1)	2	111.253	III	2	(1)	2	117.303	III	7	$(2,2,4)$	2	124.353	III	3	(1)	2
104.204	II	2	(1)	4	111.254	III	1	(1)	2	117.304	IV	3	(2)	4	124.354	III	5	(4)	4
104.205	III	1	(1)	4	111.255	III	12	$(2,2,4)$	1	117.305	IV	6	(2)	2	124.355	III	4	(4)	4
104.206	III	2	(1)	4	111.256	IV	3	(2)	4	117.306	IV	4	(2)	4	124.356	III	3	(2)	4
104.207	III	5	(2)	2	111.257	IV	3	(2)	4	118.307	I	6	(2)	2	124.357	III	16	$(4,4)$	2
104.208	IV	2	(1)	4	111.258	IV	3	(2)	4	118.308	II	3	(2)	4	124.358	III	3	(2)	4
104.209	IV	4	(2)	4	112.259	I	5	(2)	2	118.309	III	1	(1)	4	124.359	III	3	(1)	4
104.210	IV	4	(1)	2	112.260	II	3	(2)	4	118.310	III	2	(1)	2	124.360	IV	11	(4)	2
105.211	I	1	(1)	4	112.261	III	2	(1)	2	118.311	III	7	$(2,2,2)$	2	124.361	IV	6	(4)	4
105.212	II	1	(1)	4	112.262	III	1	(1)	4	118.312	IV	3	(2)	4	124.362	IV	7	(4)	4
105.213	III	2	(1)	2	112.263	III	8	$(2,2,2)$	2	118.313	IV	4	(2)	4	125.363	I	6	(2)	4
105.214	III	1	(1)	4	112.264	IV	5	(2)	2	118.314	IV	6	(2)	2	125.364	II	6	$(2,4)$	4
105.215	III	4	(2)	2	112.265	IV	3	(2)	4	119.315	I	5	(2)	2	125.365	III	4	(1)	2
105.216	IV	1	(1)	4	112.266	IV	3	(2)	4	119.316	II	5	(2)	2	125.366	III	3	(2)	4
105.217	IV	1	(1)	8	113.267	I	6	(2)	2	119.317	III	1	(1)	2	125.367	III	3	(2)	4
105.218	IV	1	(1)	4	113.268	II	3	(2)	4	119.318	III	1	(1)	2	125.368	III	3	(2)	4
106.219	I	2	(1)	4	113.269	III	1	(1)	4	119.319	III	11	$(2,2,2)$	1	125.369	III	11	$(2,4)$	2
106.220	II	1	(1)	8	113.270	III	2	(1)	2	119.320	IV	3	(2)	4	125.370	III	6	(2)	2
106.221	III	1	(1)	4	113.271	III	7	$(2,2,4)$	2	120.321	I	5	(2)	2	125.371	III	2	(1)	4
106.222	III	2	(1)	4	113.272	IV	3	(2)	4	120.322	II	3	(2)	4	125.372	IV	4	(2)	4
106.223	III	3	(2)	4	113.273	IV	6	(2)	2	120.323	III	1	(1)	4	125.373	IV	6	(2)	4
106.224	IV	1	(1)	4	113.274	IV	4	(2)	4	120.324	III	1	(1)	2	125.374	IV	4	(2)	8
106.225	IV	2	(1)	4	114.275	I	6	(2)	2	120.325	III	7	$(2,2,2)$	2	126.375	I	5	(2)	4
106.226	IV	2	(1)	4	114.276	II	3	(2)	4	120.326	IV	5	(2)	2	126.376	II	4	(4)	4
107.227	I	2	(1)	2	114.277	III	1	(1)	4	121.327	I	5	(2)	2	126.377	III	4	(1)	2
107.228	II	2	(1)	2	114.278	III	2	(1)	4	121.328	II	4	(2)	2	126.378	III	2	(2)	4
107.229	III	1	(1)	2	114.279	III	7	$(2,2,2)$	2	121.329	III	1	(1)	2	126.379	III	2	(2)	4
107.230	III	1	(1)	2	114.280	IV	3	(2)	4	121.330	III	1	(1)	2	126.380	III	3	(2)	4
107.231	III	5	(2)	1	114.281	IV	4	(2)	4	121.331	III	9	$(2,2,2)$	1	126.381	III	8	$(2,2)$	2
107.232	IV	2	(1)	4	114.282	IV	6	(2)	2	121.332	IV	4	(2)	2	126.382	III	4	(2)	4
108.233	I	2	(1)	2	115.283	I	5	(2)	2	122.333	I	7	(2)	2	126.383	III	2	(1)	4
108.234	II	2	(1)	4	115.284	II	5	(2)	2	122.334	II	3	(2)	4	126.384	IV	5	(2)	4
108.235	III	1	(1)	4	115.285	III	2	(1)	2	122.335	III	1	(1)	4	126.385	IV	4	(2)	8
108.236	III	1	(1)	2	115.286	III	1	(1)	2	122.336	III	2	(1)	4	126.386	IV	5	(2)	4
108.237	III	5	(4)	2	115.287	III	12	$(2,2,4)$	1	122.337	III	6	$(2,2)$	2	127.387	I	12	$(4,4)$	2
108.238	IV	2	(1)	2	115.288	IV	3	(2)	4	122.338	IV	4	(2)	4	127.388	II	8	$(4,8)$	4
109.239	I	1	(1)	4	115.289	IV	3	(2)	4	123.339	I	13	$(2,4,8)$	2	127.389	III	2	(1)	4
109.240	II	1	(1)	4	115.290	IV	3	(2)	4	123.340	II	13	$(2,4,8)$	2	127.390	III	5	$(2,4)$	4
109.241	III	2	(1)	2	116.291	I	5	(2)	2	123.341	III	3	(1)	2	127.391	III	6	$(2,4)$	2
109.242	III	1	(1)	4	116.292	II	3	(2)	4	123.342	III	8	$(2,4)$	2	127.392	III	3	(2)	4
109.243	III	2	(1)	2	116.293	III	2	(1)	4	123.343	III	7	$(2,2,4)$	2	127.393	III	15	$(4,4,4)$	2
109.244	IV	1	(1)	8	116.294	III	1	(1)	2	123.344	III	5	(2)	2	127.394	III	3	(2)	4
110.245	I	1	(1)	4	116.295	III	8	$(2,2,2)$	2	123.345	III	24	$(4,4,4)$	1	127.395	III	2	(1)	4
110.246	II	1	(1)	8	116.296	IV	5	(2)	2	123.346	III	5	(2)	2	127.396	IV	6	(4)	4
110.247	III	2	(1)	4	116.297	IV	3	(2)	4	123.347	III	3	(1)	2	127.397	IV	12	$(4,4)$	2
110.248	III	1	(1)	4	116.298	IV	3	(2)	4	123.348	IV	8	$(2,8)$	4	127.398	IV	8	(4)	4
110.249	III	2	(2)	4	117.299	I	6	(2)	2	123.349	IV	8	$(4,8)$	4	128.399	I	12	(4)	2
110.250	IV	1	(1)	4	117.300	II	3	(2)	4	123.350	IV	7	(8)	4	128.400	II	7	(8)	4

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128.401	III	2	(1)	4	131.444	IV	6	(2, 4)	4	135.487	III	4	(4)	4	138.530	IV	3	(2)	8
128.402	III	4	(4)	4	131.445	IV	4	(4)	8	135.488	III	3	(2)	4	139.531	I	9	(8)	2
128.403	III	5	(4)	4	131.446	IV	5	(4)	4	135.489	III	8	(2, 4)	4	139.532	II	9	(2, 8)	2
128.404	III	3	(2)	4	132.447	I	6	(4)	4	135.490	III	3	(2)	4	139.533	III	2	(1)	2
128.405	III	13	(4, 4)	2	132.448	II	6	(2, 4)	4	135.491	III	1	(1)	8	139.534	III	5	(4)	2
128.406	III	3	(2)	4	132.449	III	1	(1)	4	135.492	IV	4	(4)	4	139.535	III	5	(4)	2
128.407	III	2	(1)	4	132.450	III	5	(4)	4	135.493	IV	6	(4)	4	139.536	III	5	(2)	2
128.408	IV	7	(4)	4	132.451	III	5	(2)	2	135.494	IV	6	(4)	4	139.537	III	16	(4, 4)	1
128.409	IV	8	(4)	4	132.452	III	3	(2)	4	136.495	I	7	(4)	4	139.538	III	4	(2)	2
128.410	IV	12	(4)	2	132.453	III	12	(2, 4)	2	136.496	II	5	(4)	4	139.539	III	2	(1)	2
129.411	I	6	(2)	4	132.454	III	5	(2)	2	136.497	III	1	(1)	4	139.540	IV	7	(8)	4
129.412	II	6	(2, 4)	4	132.455	III	1	(1)	4	136.498	III	4	(4)	4	140.541	I	8	(4)	2
129.413	III	2	(1)	4	132.456	IV	6	(4)	4	136.499	III	6	(2)	2	140.542	II	7	(2, 8)	4
129.414	III	3	(2)	4	132.457	IV	4	(4)	4	136.500	III	3	(2)	4	140.543	III	2	(1)	2
129.415	III	3	(2)	4	132.458	IV	4	(4)	8	136.501	III	9	(2, 4)	2	140.544	III	4	(4)	4
129.416	III	6	(2)	2	133.459	I	4	(2)	4	136.502	III	3	(2)	4	140.545	III	4	(4)	2
129.417	III	11	(2, 4)	2	133.460	II	3	(4)	8	136.503	III	1	(1)	4	140.546	III	3	(2)	4
129.418	III	3	(2)	4	133.461	III	2	(1)	4	136.504	IV	4	(4)	8	140.547	III	13	(4, 4)	2
129.419	III	4	(1)	2	133.462	III	2	(2)	4	136.505	IV	6	(4)	4	140.548	III	4	(2)	2
129.420	IV	4	(2)	8	133.463	III	2	(2)	4	136.506	IV	7	(4)	4	140.549	III	2	(1)	4
129.421	IV	6	(2)	4	133.464	III	3	(2)	4	137.507	I	4	(2)	4	140.550	IV	8	(4)	2
129.422	IV	4	(2)	4	133.465	III	6	(2, 2)	4	137.508	II	4	(4)	4	141.551	I	6	(2)	4
130.423	I	5	(2)	4	133.466	III	4	(2)	4	137.509	III	1	(1)	4	141.552	II	5	(4)	4
130.424	II	4	(4)	8	133.467	III	1	(1)	8	137.510	III	2	(2)	4	141.553	III	1	(1)	4
130.425	III	2	(1)	4	133.468	IV	4	(2)	4	137.511	III	2	(2)	4	141.554	III	3	(2)	4
130.426	III	2	(2)	8	133.469	IV	4	(2)	4	137.512	III	6	(2)	2	141.555	III	4	(2)	4
130.427	III	2	(2)	4	133.470	IV	3	(2)	8	137.513	III	8	(2, 2)	2	141.556	III	6	(2)	2
130.428	III	4	(2)	4	134.471	I	5	(2)	4	137.514	III	3	(2)	4	141.557	III	8	(2, 2)	2
130.429	III	8	(2, 4)	4	134.472	II	5	(2, 4)	4	137.515	III	2	(1)	4	141.558	III	3	(2)	4
130.430	III	3	(2)	4	134.473	III	2	(1)	4	137.516	IV	3	(2)	8	141.559	III	1	(1)	4
130.431	III	4	(1)	4	134.474	III	3	(2)	4	137.517	IV	3	(2)	8	141.560	IV	4	(2)	8
130.432	IV	5	(2)	4	134.475	III	3	(2)	4	137.518	IV	4	(2)	4	142.561	I	4	(2)	4
130.433	IV	4	(2)	4	134.476	III	3	(2)	4	138.519	I	5	(2)	4	142.562	II	3	(4)	8
130.434	IV	5	(2)	4	134.477	III	9	(2, 2)	2	138.520	II	4	(4)	8	142.563	III	1	(1)	4
131.435	I	7	(2, 4)	4	134.478	III	6	(2)	2	138.521	III	1	(1)	4	142.564	III	2	(2)	8
131.436	II	7	(2, 4)	4	134.479	III	1	(1)	4	138.522	III	3	(2)	4	142.565	III	2	(2)	4
131.437	III	1	(1)	4	134.480	IV	3	(2)	8	138.523	III	3	(2)	4	142.566	III	5	(2)	4
131.438	III	7	(2)	2	134.481	IV	4	(2)	4	138.524	III	4	(2)	4	142.567	III	5	(2, 2)	4
131.439	III	5	(2, 4)	4	134.482	IV	4	(2)	4	138.525	III	7	(2, 2)	4	142.568	III	3	(2)	4
131.440	III	5	(2)	2	135.483	I	6	(4)	4	138.526	III	3	(2)	4	142.569	III	2	(1)	8
131.441	III	13	(2, 4)	2	135.484	II	4	(4)	8	138.527	III	2	(1)	4	142.570	IV	4	(2)	4
131.442	III	3	(2)	4	135.485	III	1	(1)	4	138.528	IV	4	(2)	4					
131.443	III	1	(1)	4	135.486	III	4	(2)	4	138.529	IV	5	(2)	4					

d: Rank of the band structure group $\{\mathrm{BS}\}$

X_{BS}: Symmetry-based indicators of band topology

ν_{BS}: Set of ν bands are symmetry-forbidden from being isolated by band gaps if $\nu \notin \nu_{\mathrm{BS}}\,\mathbb{Z}$

Table S5. Characterization of MSGs in the hexagonal family for spinful electrons.

MSG	d	X_{BS}	ν_{BS}	MSG	d	X_{BS}	ν_{BS}	MSG	d	X_{BS}	ν_{BS}	MSG	d	X_{BS}	ν_{BS}
143.1 I	7	(3)	1	156.51 III	7	(3)	1	166.101 III	11	(2, 4)	1	177.151 III	5	(1)	1
143.2 II	4	(1)	2	156.52 IV	4	(1)	2	166.102 IV	6	(2)	2	177.152 III	6	(1)	1
143.3 IV	4	(1)	2	157.53 I	5	(1)	1	167.103 I	7	(2)	2	177.153 III	9	(6)	1
144.4 I	1	(1)	3	157.54 II	3	(1)	2	167.104 II	5	(4)	4	177.154 IV	4	(1)	2
144.5 II	1	(1)	6	157.55 III	5	(3)	1	167.105 III	2	(1)	2	178.155 I	3	(1)	6
144.6 IV	1	(1)	6	157.56 IV	3	(1)	2	167.106 III	3	(1)	2	178.156 II	1	(1)	12
145.7 I	1	(1)	3	158.57 I	4	(1)	2	167.107 III	7	(4)	2	178.157 III	2	(1)	6
145.8 II	1	(1)	6	158.58 II	4	(1)	4	167.108 IV	6	(2)	2	178.158 III	2	(1)	6
145.9 IV	1	(1)	6	158.59 III	7	(3)	2	168.109 I	9	(6)	1	178.159 III	1	(1)	6
146.10 I	3	(1)	1	158.60 IV	4	(1)	2	168.110 II	4	(1)	2	178.160 IV	3	(1)	6
146.11 II	2	(1)	2	159.61 I	4	(1)	2	168.111 III	4	(1)	1	179.161 I	3	(1)	6
146.12 IV	2	(1)	2	159.62 II	3	(1)	4	168.112 IV	4	(1)	2	179.162 II	1	(1)	12
147.13 I	13	(2, 12)	1	159.63 III	5	(3)	2	169.113 I	1	(1)	6	179.163 III	2	(1)	6
147.14 II	9	(2, 4)	2	159.64 IV	3	(1)	2	169.114 II	1	(1)	12	179.164 III	2	(1)	6
147.15 III	4	(1)	2	160.65 I	3	(1)	1	169.115 III	1	(1)	6	179.165 III	1	(1)	6
147.16 IV	7	(2)	2	160.66 II	2	(1)	2	169.116 IV	1	(1)	6	179.166 IV	3	(1)	6
148.17 I	11	(2, 4)	1	160.67 III	3	(1)	1	170.117 I	1	(1)	6	180.167 I	1	(1)	6
148.18 II	8	(2, 4)	2	160.68 IV	2	(1)	2	170.118 II	1	(1)	12	180.168 II	1	(1)	6
148.19 III	2	(1)	2	161.69 I	2	(1)	2	170.119 III	1	(1)	6	180.169 III	3	(1)	3
148.20 IV	6	(2)	2	161.70 II	2	(1)	4	170.120 IV	1	(1)	6	180.170 III	3	(1)	3
149.21 I	6	(1)	1	161.71 III	3	(1)	2	171.121 I	3	(2)	3	180.171 III	3	(2)	3
149.22 II	4	(1)	2	161.72 IV	2	(1)	2	171.122 II	1	(1)	6	180.172 IV	1	(1)	6
149.23 III	7	(3)	1	162.73 I	12	(2)	1	171.123 III	1	(1)	3	181.173 I	1	(1)	6
149.24 IV	5	(1)	2	162.74 II	9	(2, 4)	2	171.124 IV	1	(1)	6	181.174 II	1	(1)	6
150.25 I	6	(1)	1	162.75 III	4	(1)	2	172.125 I	3	(2)	3	181.175 III	3	(1)	3
150.26 II	3	(1)	2	162.76 III	3	(1)	2	172.126 II	1	(1)	6	181.176 III	3	(1)	3
150.27 III	5	(3)	1	162.77 III	13	(2, 12)	1	172.127 III	1	(1)	3	181.177 III	3	(2)	3
150.28 IV	4	(1)	2	162.78 IV	7	(2)	2	172.128 IV	1	(1)	6	181.178 IV	1	(1)	6
151.29 I	3	(1)	3	163.79 I	8	(2)	2	173.129 I	5	(3)	2	182.179 I	5	(1)	2
151.30 II	1	(1)	6	163.80 II	6	(4)	4	173.130 II	3	(1)	4	182.180 II	3	(1)	4
151.31 III	1	(1)	3	163.81 III	4	(1)	2	173.131 III	4	(1)	2	182.181 III	4	(1)	2
151.32 IV	2	(1)	6	163.82 III	4	(1)	2	173.132 IV	3	(1)	2	182.182 III	5	(1)	2
152.33 I	3	(1)	3	163.83 III	9	(12)	2	174.133 I	21	(3, 3, 3)	1	182.183 III	5	(3)	2
152.34 II	1	(1)	6	163.84 IV	7	(2)	2	174.134 II	10	(3, 3)	2	182.184 IV	5	(1)	2
152.35 III	1	(1)	3	164.85 I	12	(2)	1	174.135 III	4	(1)	1	183.185 I	4	(1)	2
152.36 IV	2	(1)	6	164.86 II	9	(2, 4)	2	174.136 IV	11	(3)	2	183.186 II	4	(1)	2
153.37 I	3	(1)	3	164.87 III	3	(1)	2	175.137 I	27	(6, 6, 6)	1	183.187 III	5	(1)	1
153.38 II	1	(1)	6	164.88 III	4	(1)	2	175.138 II	14	(6, 12)	2	183.188 III	4	(1)	1
153.39 III	1	(1)	3	164.89 III	13	(2, 12)	1	175.139 III	10	(3, 3)	2	183.189 III	9	(6)	1
153.40 IV	2	(1)	6	164.90 IV	7	(2)	2	175.140 III	4	(1)	2	183.190 IV	4	(1)	4
154.41 I	3	(1)	3	165.91 I	8	(2)	2	175.141 III	9	(2, 4)	1	184.191 I	4	(1)	2
154.42 II	1	(1)	6	165.92 II	6	(4)	4	175.142 IV	13	(6)	2	184.192 II	4	(1)	4
154.43 III	1	(1)	3	165.93 III	3	(1)	2	176.143 I	16	(3, 6)	2	184.193 III	4	(1)	2
154.44 IV	2	(1)	6	165.94 III	5	(1)	2	176.144 II	9	(12)	4	184.194 III	3	(1)	2
155.45 I	4	(1)	1	165.95 III	9	(12)	2	176.145 III	11	(3)	2	184.195 III	9	(6)	2
155.46 II	2	(1)	2	165.96 IV	7	(2)	2	176.146 III	3	(1)	2	184.196 IV	4	(1)	2
155.47 III	3	(1)	1	166.97 I	11	(2)	1	176.147 III	7	(4)	2	185.197 I	3	(1)	2
155.48 IV	3	(1)	2	166.98 II	8	(2, 4)	2	176.148 IV	9	(6)	2	185.198 II	3	(1)	4
156.49 I	5	(1)	1	166.99 III	2	(1)	2	177.149 I	4	(1)	2	185.199 III	5	(1)	2
156.50 II	4	(1)	2	166.100 III	2	(1)	2	177.150 II	4	(1)	2	185.200 III	3	(1)	2

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185.201	III	5	(3)	2	188.219	III	14	(3, 3)	2	191.237	III	7	(3, 3)	2	193.255	III	3	(1)	2
185.202	IV	3	(1)	2	188.220	IV	12	(3)	2	191.238	III	12	(2)	1	193.256	III	7	(3)	4
186.203	I	3	(1)	2	189.221	I	10	(3, 3)	2	191.239	III	12	(2)	1	193.257	III	8	(3)	2
186.204	II	3	(1)	4	189.222	II	7	(3, 3)	2	191.240	III	27	(6, 6, 6)	1	193.258	III	8	(2)	2
186.205	III	4	(1)	2	189.223	III	4	(1)	1	191.241	III	4	(1)	2	193.259	III	8	(2)	2
186.206	III	4	(1)	2	189.224	III	5	(1)	1	191.242	IV	9	(12)	4	193.260	III	14	(3, 6)	2
186.207	III	5	(3)	2	189.225	III	15	(3, 3, 3)	1	192.243	I	13	(6)	2	193.261	III	4	(1)	2
186.208	IV	3	(1)	2	189.226	IV	6	(3)	2	192.244	II	9	(12)	4	193.262	IV	10	(6)	2
187.209	I	10	(3, 3)	2	190.227	I	12	(3)	2	192.245	III	4	(1)	2	194.263	I	10	(6)	2
187.210	II	10	(3, 3)	2	190.228	II	6	(3)	4	192.246	III	8	(3)	2	194.264	II	9	(12)	4
187.211	III	6	(1)	1	190.229	III	3	(1)	2	192.247	III	7	(3)	2	194.265	III	3	(1)	2
187.212	III	5	(1)	1	190.230	III	4	(1)	2	192.248	III	7	(2)	2	194.266	III	11	(3)	2
187.213	III	21	(3, 3, 3)	1	190.231	III	12	(3, 3)	2	192.249	III	7	(2)	2	194.267	III	6	(3)	4
187.214	IV	7	(3)	2	190.232	IV	9	(3)	2	192.250	III	18	(6, 6)	2	194.268	III	8	(2)	2
188.215	I	12	(3)	2	191.233	I	14	(6, 12)	2	192.251	III	4	(1)	4	194.269	III	8	(2)	2
188.216	II	7	(3)	4	191.234	II	14	(6, 12)	2	192.252	IV	13	(6)	2	194.270	III	16	(3, 6)	2
188.217	III	5	(1)	2	191.235	III	4	(1)	2	193.253	I	10	(6)	2	194.271	III	4	(1)	2
188.218	III	4	(1)	2	191.236	III	10	(3, 3)	2	193.254	II	8	(12)	4	194.272	IV	9	(6)	2

d: Rank of the band structure group $\{\mathrm{BS}\}$

X_{BS}: Symmetry-based indicators of band topology

ν_{BS}: Set of ν bands are symmetry-forbidden from being isolated by band gaps if $\nu \notin \nu_{\mathrm{BS}}\,\mathbb{Z}$

Table S6. Characterization of MSGs in the cubic family for spinful electrons.

MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$
195.1 I	3	(1)	2	206.39 III	3	(1)	4^*	216.77 IV	4	(2)	4	224.115 IV	6	(2)	4
195.2 II	2	(1)	2	207.40 I	4	(1)	2	217.78 I	6	(2)	2	225.116 I	11	(8)	2
195.3 IV	2	(1)	4	207.41 II	4	(1)	2	217.79 II	5	(2)	2	225.117 II	11	(8)	2
196.4 I	3	(1)	2	207.42 III	3	(1)	2	217.80 III	3	(1)	2	225.118 III	5	(2)	2
196.5 II	2	(1)	2	207.43 IV	3	(1)	4	218.81 I	6	(2)	2^*	225.119 III	10	(4)	2
196.6 IV	2	(1)	2^*	208.44 I	2	(1)	4	218.82 II	4	(2)	4	225.120 III	3	(1)	2
197.7 I	3	(1)	2	208.45 II	2	(1)	4	218.83 III	3	(1)	4	225.121 IV	8	(8)	4
197.8 II	2	(1)	2	208.46 III	3	(1)	2^*	218.84 IV	4	(2)	4	226.122 I	10	(4)	2^*
198.9 I	3	(1)	4	208.47 IV	2	(1)	4	219.85 I	6	(2)	2^*	226.123 II	8	(8)	4
198.10 II	2	(1)	8	209.48 I	3	(1)	2	219.86 II	4	(2)	4	226.124 III	5	(2)	2^*
198.11 IV	2	(1)	4^*	209.49 II	3	(1)	2	219.87 III	3	(1)	4	226.125 III	7	(4)	4
199.12 I	4	(1)	2^*	209.50 III	3	(1)	2	219.88 IV	6	(2)	2^*	226.126 III	3	(1)	4
199.13 II	2	(1)	4^*	209.51 IV	3	(1)	2^*	220.89 I	7	(2)	2^*	226.127 IV	10	(4)	2^*
200.14 I	11	$(2,4)$	2	210.52 I	3	(1)	4	220.90 II	4	(2)	4^*	227.128 I	9	(2)	4
200.15 II	8	$(2,4)$	2	210.53 II	2	(1)	4	220.91 III	3	(1)	4^*	227.129 II	8	(4)	4
200.16 III	2	(1)	2	210.54 III	3	(1)	4	221.92 I	14	$(4,8)$	2	227.130 III	4	(2)	4
200.17 IV	6	(4)	4	210.55 IV	3	(1)	4^*	221.93 II	14	$(4,8)$	2	227.131 III	9	(2)	4
201.18 I	9	(2)	4	211.56 I	3	(1)	2	221.94 III	6	(2)	2	227.132 III	2	(1)	4
201.19 II	6	$(2,4)$	4	211.57 II	3	(1)	2	221.95 III	11	$(2,4)$	2	227.133 IV	6	(2)	8
201.20 III	2	(1)	4	211.58 III	3	(1)	2	221.96 III	4	(1)	2	228.134 I	7	(2)	4^*
201.21 IV	5	(2)	4	212.59 I	3	(1)	4	221.97 IV	8	(8)	4	228.135 II	5	(4)	8
202.22 I	10	(4)	2	212.60 II	2	(1)	8	222.98 I	8	(2)	4	228.136 III	4	(2)	4^*
202.23 II	7	(4)	2	212.61 III	3	(1)	4	222.99 II	6	(4)	4	228.137 III	6	(2)	8
202.24 III	2	(1)	2	212.62 IV	3	(1)	4^*	222.100 III	4	(2)	4	228.138 III	3	(1)	8
202.25 IV	6	(4)	2^*	213.63 I	3	(1)	4	222.101 III	6	(2)	4	228.139 IV	7	(2)	4^*
203.26 I	9	(2)	4	213.64 II	2	(1)	8	222.102 III	3	(1)	4	229.140 I	11	(8)	2
203.27 II	6	(4)	4	213.65 III	3	(1)	4	222.103 IV	8	(2)	4	229.141 II	11	$(2,8)$	2
203.28 III	2	(1)	4	213.66 IV	3	(1)	4^*	223.104 I	8	(4)	4	229.142 III	6	(2)	2
203.29 IV	5	(2)	4^*	214.67 I	2	(1)	4	223.105 II	7	(4)	4	229.143 III	10	(4)	2
204.30 I	10	(4)	2	214.68 II	2	(1)	4^*	223.106 III	6	(2)	2^*	229.144 III	3	(1)	2
204.31 II	7	$(2,4)$	2	214.69 III	4	(1)	2^*	223.107 III	7	(4)	4	230.145 I	7	(2)	4^*
204.32 III	2	(1)	2	215.70 I	6	(2)	2	223.108 III	2	(1)	4	230.146 II	5	(4)	8^*
205.33 I	9	(2)	4	215.71 II	6	(2)	2	223.109 IV	7	(4)	4	230.147 III	5	(2)	4^*
205.34 II	6	(4)	8	215.72 III	3	(1)	2	224.110 I	8	(2)	4	230.148 III	6	(2)	4^*
205.35 III	2	(1)	8	215.73 IV	4	(2)	4	224.111 II	8	$(2,4)$	4	230.149 III	2	(1)	8
205.36 IV	5	(2)	8	216.74 I	6	(2)	2	224.112 III	4	(2)	4				
206.37 I	9	(2)	4	216.75 II	6	(2)	2	224.113 III	9	(2)	4				
206.38 II	6	$(2,4)$	8	216.76 III	3	(1)	2	224.114 III	2	(1)	4				

d: Rank of the band structure group $\{\mathrm{BS}\}$
$X_{\rm BS}$: Symmetry-based indicators of band topology
$\nu_{\rm BS}$: Set of ν bands are symmetry-forbidden from being isolated by band gaps if $\nu \notin \nu_{\rm BS}\,\mathbb{Z}$
$*$: Exhibiting exceptional filling pattern; see table S8

Table S7. Characterization of the projections of MSGs that correspond to MLG, assuming spinful electrons.

MSG		d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG		d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG		d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG		d	$X_{\rm BS}$	$\nu_{\rm BS}$
1.1 (1)	I	1	(1)	1	6.19 (1)	II	1	(1)	2	11.52 (1)	III	2	(1)	2	16.5	IV	1	(1)	4
1.1 (2)	I	1	(1)	1	6.19 (2)	II	1	(1)	2	11.52 (2)	III	2	(1)	2	17.7 (1)	I	4	(1)	2
1.1 (3)	I	1	(1)	1	6.19 (3)	II	1	(1)	2	11.53 (1)	III	1	(1)	2	17.7 (2)	I	4	(1)	2
1.2 (1)	II	1	(1)	2	6.20 (1)	III	1	(1)	1	11.53 (2)	III	1	(1)	2	17.8 (1)	II	1	(1)	4
1.2 (2)	II	1	(1)	2	6.20 (2)	III	1	(1)	1	11.54 (1)	III	3	(2)	2	17.8 (2)	II	1	(1)	4
1.2 (3)	II	1	(1)	2	6.20 (3)	III	1	(1)	1	11.54 (2)	III	3	(2)	2	17.9 (1)	III	1	(1)	2
1.3 (1)	IV	1	(1)	2	6.21 (1)	IV	1	(1)	2	11.55	IV	2	(1)	4	17.9 (2)	III	1	(1)	2
1.3 (2)	IV	1	(1)	2	6.21 (2)	IV	1	(1)	2	11.56 (1)	IV	3	(1)	2	17.10 (1)	III	3	(2)	2
2.4 (1)	I	5	(2)	1	6.22 (1)	IV	2	(1)	2	11.56 (2)	IV	3	(1)	2	17.10 (2)	III	2	(1)	2
2.4 (2)	I	5	(2)	1	6.22 (2)	IV	2	(1)	2	11.57	IV	2	(1)	4	17.11	IV	2	(1)	4
2.4 (3)	I	5	(2)	1	6.23	IV	2	(1)	2	12.58	I	6	(1)	1	17.12 (1)	IV	4	(1)	2
2.5 (1)	II	5	(2)	2	7.24 (1)	I	1	(1)	2	12.59	II	4	(2)	2	17.12 (2)	IV	4	(1)	2
2.5 (2)	II	5	(2)	2	7.24 (2)	I	1	(1)	2	12.60	III	1	(1)	2	17.13	IV	3	(1)	2
2.5 (3)	II	5	(2)	2	7.25 (1)	II	1	(1)	4	12.61	III	1	(1)	2	18.16	I	3	(1)	2
2.6 (1)	III	1	(1)	2	7.25 (2)	II	1	(1)	4	12.62	III	4	(2)	1	18.17	II	1	(1)	4
2.6 (2)	III	1	(1)	2	7.26 (1)	III	1	(1)	2	12.64	IV	3	(1)	2	18.18	III	3	(2)	2
2.6 (3)	III	1	(1)	2	7.26 (2)	III	1	(1)	2	13.65 (1)	I	4	(1)	2	18.19	III	1	(1)	2
2.7 (1)	IV	3	(1)	2	7.27	IV	1	(1)	4	13.65 (2)	I	5	(2)	2	18.20	IV	2	(1)	4
2.7 (2)	IV	3	(1)	2	7.28 (1)	IV	1	(1)	2	13.66 (1)	II	3	(2)	4	18.23	IV	3	(1)	2
3.1 (1)	I	3	(1)	1	7.28 (2)	IV	1	(1)	2	13.66 (2)	II	3	(2)	4	21.38	I	2	(1)	2
3.1 (2)	I	5	(2)	1	7.29	IV	1	(1)	4	13.67 (1)	III	1	(1)	2	21.39	II	1	(1)	2
3.1 (3)	I	3	(1)	1	7.31	IV	1	(1)	2	13.67 (2)	III	1	(1)	2	21.40	III	4	(2)	1
3.2 (1)	II	1	(1)	2	8.32	I	2	(1)	1	13.68 (1)	III	2	(1)	2	21.41	III	2	(1)	1
3.2 (2)	II	1	(1)	2	8.33	II	1	(1)	2	13.68 (2)	III	3	(1)	2	21.43	IV	2	(1)	2
3.2 (3)	II	1	(1)	2	8.34	III	1	(1)	1	13.69 (1)	III	3	(2)	2	25.57 (1)	I	1	(1)	2
3.3 (1)	III	1	(1)	1	8.36	IV	1	(1)	2	13.69 (2)	III	3	(2)	2	25.57 (2)	I	1	(1)	2
3.3 (2)	III	1	(1)	1	10.42 (1)	I	9	(1)	1	13.70	IV	3	(1)	4	25.57 (3)	I	1	(1)	2
3.3 (3)	III	1	(1)	1	10.42 (2)	I	10	(2, 2)	1	13.71	IV	2	(1)	4	25.58 (1)	II	1	(1)	2
3.4 (1)	IV	3	(2)	2	10.42 (3)	I	9	(1)	1	13.72 (1)	IV	3	(1)	2	25.58 (2)	II	1	(1)	2
3.4 (2)	IV	2	(1)	2	10.43 (1)	II	5	(2)	2	13.72 (2)	IV	3	(1)	2	25.58 (3)	II	1	(1)	2
3.5 (1)	IV	1	(1)	2	10.43 (2)	II	5	(2)	2	13.73	IV	2	(1)	4	25.59 (1)	III	3	(1)	1
3.5 (2)	IV	1	(1)	2	10.43 (3)	II	5	(2)	2	14.75	I	3	(1)	2	25.59 (2)	III	2	(1)	1
3.6	IV	2	(1)	2	10.44 (1)	III	1	(1)	2	14.76	II	3	(2)	4	25.59 (3)	III	3	(1)	1
4.7 (1)	I	1	(1)	2	10.44 (2)	III	1	(1)	2	14.77	III	1	(1)	4	25.60 (1)	III	3	(1)	1
4.7 (2)	I	1	(1)	2	10.44 (3)	III	1	(1)	2	14.78	III	1	(1)	4	25.60 (2)	III	3	(1)	1
4.8 (1)	II	1	(1)	4	10.45 (1)	III	1	(1)	2	14.79	III	3	(2)	2	25.60 (3)	III	5	(2)	1
4.8 (2)	II	1	(1)	4	10.45 (2)	III	1	(1)	2	14.81	IV	2	(1)	4	25.61 (1)	IV	1	(1)	4
4.9 (1)	III	1	(1)	2	10.45 (3)	III	1	(1)	2	14.82	IV	2	(1)	4	25.61 (2)	IV	1	(1)	4
4.9 (2)	III	1	(1)	2	10.46 (1)	III	5	(2)	1	14.83	IV	3	(1)	2	25.62 (1)	IV	1	(1)	2
4.10	IV	1	(1)	4	10.46 (2)	III	5	(2)	1	16.1 (1)	I	1	(1)	2	25.62 (2)	IV	1	(1)	2
4.11 (1)	IV	1	(1)	2	10.46 (3)	III	5	(2)	1	16.1 (2)	I	1	(1)	2	25.63	IV	1	(1)	4
4.11 (2)	IV	1	(1)	2	10.47 (1)	IV	5	(2)	2	16.1 (3)	I	1	(1)	2	25.64	IV	1	(1)	4
4.12	IV	1	(1)	2	10.47 (2)	IV	4	(1)	2	16.2 (1)	II	1	(1)	2	26.66 (1)	I	1	(1)	2
5.13	I	2	(1)	1	10.48 (1)	IV	4	(1)	2	16.2 (2)	II	1	(1)	2	26.66 (2)	I	1	(1)	2
5.14	II	1	(1)	2	10.48 (2)	IV	4	(1)	2	16.2 (3)	II	1	(1)	2	26.67 (1)	II	1	(1)	4
5.15	III	1	(1)	1	10.49	IV	5	(1)	2	16.3 (1)	III	3	(1)	1	26.67 (2)	II	1	(1)	4
5.17	IV	1	(1)	2	11.50 (1)	I	4	(1)	2	16.3 (2)	III	3	(1)	1	26.68 (1)	III	1	(1)	2
6.18 (1)	I	3	(1)	1	11.50 (2)	I	4	(1)	2	16.3 (3)	III	5	(2)	1	26.68 (2)	III	1	(1)	2
6.18 (2)	I	2	(1)	1	11.51 (1)	II	3	(2)	4	16.4 (1)	IV	1	(1)	2	26.69 (1)	III	2	(1)	2
6.18 (3)	I	3	(1)	1	11.51 (2)	II	3	(2)	4	16.4 (2)	IV	1	(1)	2	26.69 (2)	III	3	(1)	2

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26.70 (1)	III	1	(1)	2	30.118	IV	1	(1)	4	47.255	IV	3	(2)	4	51.302	IV	3	(2)	4
26.70 (2)	III	1	(1)	2	30.119	IV	2	(1)	2	49.265 (1)	I	4	(1)	2	51.303	IV	3	(1)	4
26.71	IV	1	(1)	4	31.123	I	2	(1)	2	49.265 (2)	I	4	(1)	2	53.321	I	6	(1)	2
26.72	IV	1	(1)	4	31.124	II	1	(1)	4	49.266 (1)	II	3	(2)	4	53.322	II	3	(2)	4
26.73 (1)	IV	1	(1)	2	31.125	III	1	(1)	2	49.266 (2)	II	3	(2)	4	53.323	III	1	(1)	4
26.73 (2)	IV	1	(1)	2	31.126	III	2	(1)	2	49.267 (1)	III	3	(1)	2	53.324	III	2	(1)	4
26.74	IV	1	(1)	2	31.127	III	1	(1)	2	49.267 (2)	III	2	(1)	2	53.325	III	1	(1)	4
26.75	IV	1	(1)	4	31.128	IV	2	(1)	4	49.268 (1)	III	1	(1)	2	53.326	III	3	(1)	2
27.78 (1)	I	1	(1)	2	31.130	IV	1	(1)	4	49.268 (2)	III	1	(1)	2	53.327	III	5	(1)	2
27.78 (2)	I	1	(1)	2	31.132	IV	2	(1)	2	49.269 (1)	III	6	(1)	2	53.328	III	4	(2)	2
27.79 (1)	II	1	(1)	4	32.135	I	3	(1)	2	49.269 (2)	III	6	(1)	2	53.329	III	2	(1)	4
27.79 (2)	II	1	(1)	4	32.136	II	1	(1)	4	49.270 (1)	III	4	(1)	2	53.330	IV	4	(1)	4
27.80 (1)	III	1	(1)	2	32.137	III	1	(1)	2	49.270 (2)	III	5	(2)	2	53.332	IV	4	(1)	4
27.80 (2)	III	1	(1)	2	32.138	III	3	(2)	2	49.271 (1)	III	1	(1)	4	53.334	IV	5	(1)	2
27.81 (1)	III	3	(1)	2	32.140	IV	2	(1)	4	49.271 (2)	III	1	(1)	4	54.337	I	3	(1)	4
27.81 (2)	III	3	(1)	2	32.141	IV	3	(1)	2	49.272	IV	2	(1)	4	54.338	II	2	(2)	8
27.82 (1)	IV	1	(1)	2	35.165	I	2	(1)	2	49.273 (1)	IV	4	(1)	2	54.339	III	1	(1)	4
27.82 (2)	IV	1	(1)	2	35.166	II	1	(1)	2	49.273 (2)	IV	4	(1)	2	54.340	III	2	(1)	4
27.83	IV	1	(1)	4	35.167	III	2	(1)	1	49.274	IV	3	(1)	4	54.341	III	1	(1)	4
27.85	IV	1	(1)	4	35.168	III	4	(2)	1	50.277	I	2	(1)	4	54.342	III	3	(1)	4
28.87 (1)	I	3	(1)	2	35.170	IV	2	(1)	2	50.278	II	2	(2)	4	54.343	III	2	(1)	4
28.87 (2)	I	4	(1)	2	38.187	I	1	(1)	2	50.279	III	2	(1)	2	54.344	III	3	(2)	4
28.88 (1)	II	1	(1)	4	38.188	II	1	(1)	2	50.280	III	3	(1)	2	54.345	III	3	(1)	4
28.88 (2)	II	1	(1)	4	38.189	III	2	(1)	1	50.281	III	4	(2)	2	54.346	IV	3	(1)	4
28.89 (1)	III	1	(1)	2	38.190	III	2	(1)	1	50.282	III	3	(1)	2	54.348	IV	2	(1)	4
28.89 (2)	III	1	(1)	2	38.191	III	2	(1)	1	50.283	III	1	(1)	4	54.350	IV	3	(1)	4
28.90 (1)	III	2	(1)	2	38.193	IV	1	(1)	2	50.284	IV	2	(1)	4	55.353	I	5	(2)	2
28.90 (2)	III	2	(1)	2	39.195	I	1	(1)	2	50.287	IV	2	(1)	4	55.354	II	3	(2)	4
28.91 (1)	III	2	(1)	2	39.196	II	1	(1)	4	51.289 (1)	I	5	(2)	2	55.355	III	1	(1)	4
28.91 (2)	III	3	(2)	2	39.197	III	2	(1)	2	51.289 (2)	I	4	(1)	2	55.356	III	1	(1)	4
28.92 (1)	IV	3	(1)	2	39.198	III	1	(1)	2	51.290 (1)	II	3	(2)	4	55.357	III	6	(2, 2)	2
28.92 (2)	IV	4	(1)	2	39.199	III	2	(1)	2	51.290 (2)	II	3	(2)	4	55.358	III	3	(1)	2
28.93	IV	2	(1)	4	39.201	IV	1	(1)	2	51.291 (1)	III	1	(1)	2	55.359	III	1	(1)	4
28.94	IV	1	(1)	4	47.249 (1)	I	5	(2)	2	51.291 (2)	III	1	(1)	2	55.360	IV	3	(2)	4
28.96	IV	3	(1)	4	47.249 (2)	I	5	(2)	2	51.292 (1)	III	2	(1)	2	55.363	IV	5	(2)	2
28.97	IV	3	(1)	2	47.249 (3)	I	5	(2)	2	51.292 (2)	III	2	(1)	2	57.377	I	3	(1)	4
29.99	I	1	(1)	4	47.250 (1)	II	5	(2)	2	51.293 (1)	III	1	(1)	4	57.378	II	2	(2)	8
29.100	II	1	(1)	8	47.250 (2)	II	5	(2)	2	51.293 (2)	III	1	(1)	4	57.379	III	3	(1)	4
29.101	III	1	(1)	4	47.250 (3)	II	5	(2)	2	51.294 (1)	III	4	(1)	2	57.380	III	1	(1)	4
29.102	III	1	(1)	4	47.251 (1)	III	1	(1)	2	51.294 (2)	III	5	(2)	2	57.381	III	1	(1)	4
29.103	III	1	(1)	4	47.251 (2)	III	1	(1)	2	51.295 (1)	III	4	(1)	2	57.382	III	3	(1)	4
29.104	IV	1	(1)	4	47.251 (3)	III	1	(1)	2	51.295 (2)	III	4	(1)	2	57.383	III	3	(2)	4
29.106	IV	1	(1)	4	47.252 (1)	III	9	(1)	1	51.296 (1)	III	6	(2, 2)	2	57.384	III	2	(1)	4
29.108	IV	1	(1)	4	47.252 (2)	III	9	(1)	1	51.296 (2)	III	6	(1)	2	57.385	III	2	(1)	4
30.111	I	2	(1)	2	47.252 (3)	III	10	(2, 2)	1	51.297 (1)	III	2	(1)	2	57.387	IV	2	(1)	4
30.112	II	1	(1)	4	47.253 (1)	III	1	(1)	2	51.297 (2)	III	3	(1)	2	57.388	IV	3	(1)	4
30.113	III	1	(1)	2	47.253 (2)	III	1	(1)	2	51.298 (1)	IV	5	(2)	2	57.389	IV	3	(1)	4
30.114	III	1	(1)	2	47.253 (3)	III	1	(1)	2	51.298 (2)	IV	4	(1)	2	59.405	I	2	(1)	4
30.115	III	2	(1)	2	47.254 (1)	IV	3	(2)	4	51.299	IV	2	(1)	4	59.406	II	2	(2)	4
30.117	IV	2	(1)	4	47.254 (2)	IV	3	(2)	4	51.300	IV	3	(2)	4	59.407	III	2	(1)	2

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59.408	III	1	(1)	4	90.97	III	2	(1)	2	125.363	I	4	(1)	4	162.76	III	3	(1)	2
59.409	III	4	(2)	2	90.98	III	5	(4)	2	125.364	II	4	(2)	4	162.77	III	9	(6)	1
59.410	III	3	(1)	2	90.99	III	1	(1)	4	125.365	III	4	(1)	2	164.85	I	8	(1)	1
59.411	III	3	(1)	2	90.101	IV	4	(1)	2	125.366	III	2	(1)	4	164.86	II	6	(2)	2
59.412	IV	2	(1)	4	99.163	I	3	(1)	2	125.367	III	2	(1)	4	164.87	III	3	(1)	2
59.415	IV	2	(1)	4	99.164	II	3	(1)	2	125.368	III	2	(1)	4	164.88	III	4	(1)	2
65.481	I	5	(2)	2	99.165	III	2	(1)	2	125.369	III	8	(4)	2	164.89	III	9	(6)	1
65.482	II	4	(2)	2	99.166	III	1	(1)	2	125.370	III	4	(1)	2	168.109	I	9	(6)	1
65.483	III	1	(1)	2	99.167	III	8	(4)	1	125.371	III	2	(1)	4	168.110	II	4	(1)	2
65.484	III	1	(1)	2	99.169	IV	2	(1)	4	125.373	IV	4	(1)	4	168.111	III	4	(1)	1
65.485	III	8	(2, 2)	1	100.171	I	4	(1)	2	127.387	I	8	(4)	2	174.133	I	14	(3, 3)	1
65.486	III	6	(1)	1	100.172	II	2	(1)	4	127.388	II	5	(4)	4	174.134	II	7	(3)	2
65.487	III	1	(1)	2	100.173	III	1	(1)	4	127.389	III	2	(1)	4	174.135	III	4	(1)	1
65.489	IV	4	(2)	2	100.174	III	2	(1)	2	127.390	III	3	(2)	4	175.137	I	18	(6, 6)	1
67.501	I	3	(1)	2	100.175	III	5	(4)	2	127.391	III	4	(2)	2	175.138	II	9	(6)	2
67.502	II	3	(2)	4	100.177	IV	4	(1)	2	127.392	III	2	(1)	4	175.139	III	7	(3)	2
67.503	III	1	(1)	2	111.251	I	3	(1)	2	127.393	III	10	(4, 4)	2	175.140	III	4	(1)	2
67.504	III	2	(1)	2	111.252	II	3	(1)	2	127.394	III	2	(1)	4	175.141	III	6	(2)	1
67.505	III	5	(2)	2	111.253	III	2	(1)	2	127.395	III	2	(1)	4	177.149	I	4	(1)	2
67.506	III	5	(1)	2	111.254	III	1	(1)	2	127.397	IV	8	(4)	2	177.150	II	4	(1)	2
67.507	III	2	(1)	2	111.255	III	8	(4)	1	129.411	I	4	(1)	4	177.151	III	4	(1)	1
67.509	IV	3	(1)	2	111.257	IV	2	(1)	4	129.412	II	4	(2)	4	177.152	III	5	(1)	1
75.1	I	8	(4)	1	113.267	I	4	(1)	2	129.413	III	2	(1)	4	177.153	III	9	(6)	1
75.2	II	3	(1)	2	113.268	II	2	(1)	4	129.414	III	2	(1)	4	183.185	I	4	(1)	2
75.3	III	2	(1)	2	113.269	III	1	(1)	4	129.415	III	2	(1)	4	183.186	II	4	(1)	2
75.5	IV	4	(2)	2	113.270	III	2	(1)	2	129.416	III	4	(1)	2	183.187	III	5	(1)	1
81.33	I	8	(4)	1	113.271	III	5	(4)	2	129.417	III	8	(4)	2	183.188	III	4	(1)	1
81.34	II	3	(1)	2	113.273	IV	4	(1)	2	129.418	III	2	(1)	4	183.189	III	9	(6)	1
81.35	III	2	(1)	2	115.283	I	3	(1)	2	129.419	III	4	(1)	2	187.209	I	7	(3)	2
81.37	IV	4	(2)	2	115.284	II	3	(1)	2	129.421	IV	4	(1)	4	187.210	II	7	(3)	2
83.43	I	16	(4, 4)	1	115.285	III	2	(1)	2	143.1	I	7	(3)	1	187.211	III	5	(1)	1
83.44	II	8	(4)	2	115.286	III	1	(1)	2	143.2	II	4	(1)	2	187.212	III	5	(1)	1
83.45	III	5	(2)	2	115.287	III	8	(4)	1	147.13	I	9	(6)	1	187.213	III	14	(3, 3)	1
83.46	III	3	(1)	2	115.289	IV	2	(1)	4	147.14	II	6	(2)	2	189.221	I	7	(3)	2
83.47	III	3	(1)	2	117.299	I	4	(1)	2	147.15	III	4	(1)	2	189.222	II	5	(3)	2
83.49	IV	8	(4)	2	117.300	II	2	(1)	4	149.21	I	5	(1)	1	189.223	III	4	(1)	1
85.59	I	8	(4)	2	117.301	III	1	(1)	4	149.22	II	4	(1)	2	189.224	III	4	(1)	1
85.60	II	4	(2)	4	117.302	III	2	(1)	2	149.23	III	7	(3)	1	189.225	III	10	(3, 3)	1
85.61	III	2	(1)	4	117.303	III	5	(4)	2	150.25	I	5	(1)	1	191.233	I	9	(6)	2
85.62	III	4	(1)	2	117.305	IV	4	(1)	2	150.26	II	3	(1)	2	191.234	II	9	(6)	2
85.63	III	4	(1)	2	123.339	I	8	(4)	2	150.27	III	5	(3)	1	191.235	III	4	(1)	2
85.65	IV	4	(1)	4	123.340	II	8	(4)	2	156.49	I	5	(1)	1	191.236	III	7	(3)	2
89.87	I	3	(1)	2	123.341	III	3	(1)	2	156.50	II	4	(1)	2	191.237	III	5	(3)	2
89.88	II	3	(1)	2	123.342	III	5	(2)	2	156.51	III	7	(3)	1	191.238	III	8	(1)	1
89.89	III	1	(1)	2	123.343	III	4	(2)	2	157.53	I	5	(1)	1	191.239	III	8	(1)	1
89.90	III	8	(4)	1	123.344	III	3	(1)	2	157.54	II	3	(1)	2	191.240	III	18	(6, 6)	1
89.91	III	2	(1)	2	123.345	III	16	(4, 4)	1	157.55	III	5	(3)	1	191.241	III	4	(1)	2
89.93	IV	2	(1)	4	123.346	III	3	(1)	2	162.73	I	8	(1)	1					
90.95	I	4	(1)	2	123.347	III	3	(1)	2	162.74	II	6	(2)	2					
90.96	II	2	(1)	4	123.349	IV	5	(4)	4	162.75	III	4	(1)	2					

d: Rank of the band structure group $\{\mathrm{BS}\}$

X_{BS}: Symmetry-based indicators of band topology

ν_{BS}: Set of ν bands can only be isolated by band gaps if $\nu \in \nu_{\mathrm{BS}}\,\mathbb{Z}$

Table S8. MSGs for which spinful electrons exhibit exceptional filling patterns.

MSG	$\{\nu\}_{\rm AI}$	$\{\nu\}_{\rm BS}$	MSG	$\{\nu\}_{\rm AI}$	$\{\nu\}_{\rm BS}$	MSG	$\{\nu\}_{\rm AI}$	$\{\nu\}_{\rm BS}$	MSG	$\{\nu\}_{\rm AI}$	$\{\nu\}_{\rm BS}$
196.6 IV	$2\mathbb{N} \setminus \{2\}$	$2\mathbb{N}$	209.51 IV	$2\mathbb{N} \setminus \{2\}$	$2\mathbb{N}$	219.88 IV	$2\mathbb{N} \setminus \{2\}$	$2\mathbb{N}$	228.134 I	$4\mathbb{N} \setminus \{4\}$	$4\mathbb{N}$
198.11 IV	$4\mathbb{N} \setminus \{4\}$	$4\mathbb{N}$	210.55 IV	$4\mathbb{N} \setminus \{4\}$	$4\mathbb{N}$	220.89 I	$2\mathbb{N} \setminus \{2, 4, 10\}$	$2\mathbb{N} \setminus \{2\}$	228.136 III	$4\mathbb{N} \setminus \{4\}$	$4\mathbb{N}$
199.12 I	$2\mathbb{N} \setminus \{2\}$	–	212.62 IV	$4\mathbb{N} \setminus \{4\}$	$4\mathbb{N}$	220.90 II	$4\mathbb{N} \setminus \{4, 8, 20\}$	$4\mathbb{N} \setminus \{4\}$	228.139 IV	$4\mathbb{N} \setminus \{4\}$	$4\mathbb{N}$
199.13 II	$4\mathbb{N} \setminus \{4\}$	$4\mathbb{N}$	213.66 IV	$4\mathbb{N} \setminus \{4\}$	$4\mathbb{N}$	220.91 III	$4\mathbb{N} \setminus \{4\}$	–	230.145 I	$4\mathbb{N} \setminus \{4\}$	$4\mathbb{N}$
202.25 IV	$2\mathbb{N} \setminus \{2\}$	$2\mathbb{N}$	214.68 II	$4\mathbb{N} \setminus \{4\}$	$4\mathbb{N}$	223.106 III	$2\mathbb{N} \setminus \{2\}$	$2\mathbb{N}$	230.146 II	$8\mathbb{N} \setminus \{8\}$	$8\mathbb{N}$
203.29 IV	$4\mathbb{N} \setminus \{4\}$	$4\mathbb{N}$	214.69 III	$2\mathbb{N} \setminus \{2\}$	–	226.122 I	$2\mathbb{N} \setminus \{2\}$	$2\mathbb{N}$	230.147 III	$4\mathbb{N} \setminus \{4\}$	–
206.39 III	$4\mathbb{N} \setminus \{4\}$	–	218.81 I	$2\mathbb{N} \setminus \{2\}$	$2\mathbb{N}$	226.124 III	$2\mathbb{N} \setminus \{2\}$	$2\mathbb{N}$	230.148 III	$4\mathbb{N} \setminus \{4\}$	–
208.46 III	$2\mathbb{N} \setminus \{2\}$	$2\mathbb{N}$	219.85 I	$2\mathbb{N} \setminus \{2\}$	$2\mathbb{N}$	226.127 IV	$2\mathbb{N} \setminus \{2\}$	$2\mathbb{N}$			

$\mathbb{N}$: The set of natural numbers

$\{\nu\}_{\rm AI}$: Set of fillings for which physical atomic insulators are possible

$\{\nu\}_{\rm BS}$: Set of fillings for which physical band structure are possible; a dash (–) indicates $\{\nu\}_{\rm BS} = \{\nu\}_{\rm AI}$ for that MSG

Section S2. feSMs in stoichiometric compounds

While the electron filling ν of a physical system can, in principle, take any real value upon doping, in perfectly ordered stoichiometric crystals the realizable fillings are further constrained by crystalline symmetries. As the atoms in such a crystal form $\mathcal{M}$-symmetric lattices, the allowed fillings for an MSG $\mathcal{M}$ are determined by the multiplicities of the Wyckoff positions [33]. Mathematically, this is captured by demanding $\nu \in \nu_{\rm P}\,\mathbb{N}$, where $\nu_{\rm P}$ is the gcd of the multiplicities of the Wyckoff positions and $\mathbb{N}$ denotes the set of natural number. Whenever $\nu_{\rm P} < \nu_{\rm BS}$, one can find systems with a filling ν that is incompatible with a band gap, and therefore $\mathcal{M}$ admits feSMs. We also remark that for the four 'Wyckoff-mismatched' SGs identified in Ref. [9], as well as their magnetic descendants, the set of physically realizable filling does not take the simple form of $\nu_{\rm P}\mathbb{N}$. The exhaustive list of such Wyckoff-mismatched MSGs can be found in table S19, since the set physically realizable fillings coincides with the $\{\nu\}_{\rm AI}$ of *spinless* fermions.

For TR symmetric systems described by the 230 type II MSGs, $\nu_{\rm P} < \nu_{\rm BS}$ holds generally, and the filling criterion for filling-enforced (semi-)metals simplifies into whether or not the filling is an odd multiple of $\nu_{\rm P}$. For a general MSG, however, $\nu_{\rm P} \leq \nu_{\rm BS}$, and to identify magnetic settings which can host filling-enforced (semi-)metals one must first isolate the list of MSGs where $\nu_{\rm P} \neq \nu_{\rm BS}$. This is easily achieved using our theory, and we found that 647 out of the 1651 MSGs can host filling-enforced (semi-)metals (this includes the 230 type II MSGs, i.e., there are are 417 intrinsically magnetic MSGs). For all these MSGs, $\nu_{\rm BS} = 2\nu_{\rm P}$, and therefore gaplessness near the Fermi energy is enforced whenever the filling of the system is an odd multiple of $\nu_{\rm P}$.

Some of these MSGs, however, are still incompatible with feSMs due to 'band-sticking.' For instance, whenever the MSG contains the combination of spatial inversion and TR as a symmetry, each band is doubly degenerate. For such MSGs, if $\nu_{\rm P}$ is also odd, the filling-enforced gaplessness always manifests in the form of a large Fermi-surface enclosing half of the Brillouin zone, resulting in a conventional metal. A similar situation arise for band-sticking along high-symmetry line. We refer to such systems as filling-enforced metals. We found that 226 of the identified MSGs fall into this category, and they are listed in table S11.

This leaves behind 421 MSGs, 250 of which are intrinsically magnetic, which can potentially host feSMs. Among them, we further characterize their minimal fermiology into two types. If the irrep dimensions at any high-symmetry point is incompatible with a band gap at filling $\nu_{\rm P}$, the fermi surface of the feSM consists of pinned nodal points. These MSGs are tabulated in table S9. If all the high-symmetry momenta can be in principle gapped, then the semimetallic behavior must be enforced by a more intricate band connectivity, resulting in nodal features that are movable, but irremovable. (Such fermiology is exemplified by that of simple nonsymmorphic SGs, as discussed in, e.g., Ref. [5].) These MSGs are tabulated in table S10. Note that these are not mutually exclusive characterization of feSMs — certain MSGs in table S9 can also display such movable nodal features.

Lastly, we also include results specific to TR-symmetric systems (type II MSGs) in table S12, which describes the effect of spin-orbit coupling on the fermiology of the feSMs.

Table S9. MSGs that can host feSMs with nodal-point Fermi surfaces pinned at high-symmetry momenta.

MSG	ν_P	FS	MSG	ν_P	FS	MSG	ν_P	FS	MSG	ν_P	FS	MSG	ν_P	FS	MSG	ν_P	FS
1.2 II	1	2	59.415 IV	2	4	100.173 III	2	4	125.366 III	2	4	139.540 IV	2	4	195.3 IV	2	4
3.2 II	1	2	63.468 IV	2	4	101.180 II	2	4	125.368 III	2	4	140.542 II	2	4	196.4 I	1	2
5.14 II	1	2	66.492 II	2	4	101.181 III	2	4	125.371 III	2	4	140.544 III	2	4	196.5 II	1	2
6.19 II	1	2	66.497 III	2	4	101.184 IV	2	4	125.374 IV	4	8	140.546 III	2	4	197.7 I	1	2
8.33 II	1	2	67.502 II	2	4	102.188 II	2	4	126.375 I	2	4	140.549 III	2	4	197.8 II	1	2
11.51 II	2	4	68.512 II	2	4	102.189 III	2	4	126.376 II	2	4	141.551 I	2	4	201.19 II	2	4
11.57 IV	2	4	68.517 III	2	4	102.194 IV	2	4	126.378 III	2	4	141.552 II	2	4	201.20 III	2	4
13.66 II	2	4	68.520 IV	2	4	103.196 II	2	4	126.379 III	2	4	141.555 III	2	4	203.27 II	2	4
13.73 IV	2	4	70.528 II	2	4	103.197 III	2	4	126.380 III	2	4	141.558 III	2	4	203.28 III	2	4
14.76 II	2	4	70.531 III	2	4	103.198 III	2	4	126.382 III	2	4	141.559 III	2	4	207.40 I	1	2
15.86 II	2	4	71.538 IV	2	4	104.204 II	2	4	126.383 III	2	4	143.2 II	1	2	207.41 II	1	2
16.1 I	1	2	72.540 II	2	4	104.205 III	2	4	126.385 IV	4	8	144.5 II	3	2	207.42 III	1	2
16.2 II	1	2	72.545 III	2	4	104.206 III	2	4	127.392 III	2	4	145.8 II	3	2	207.43 IV	2	4
16.5 IV	2	4	74.555 II	2	4	105.212 II	2	4	127.395 III	2	4	146.11 II	1	2	208.47 IV	2	4
16.6 IV	2	4	75.2 II	1	2	105.214 III	2	4	128.403 III	2	4	149.22 II	1	2	209.48 I	1	2
18.17 II	2	4	75.3 III	1	2	105.216 IV	2	4	128.404 III	2	4	150.26 II	1	2	209.49 II	1	2
21.38 I	1	2	79.26 II	1	2	105.218 IV	2	4	128.407 III	2	4	151.30 II	3	2	209.50 III	1	2
21.39 II	1	2	79.27 III	1	2	107.232 IV	2	4	129.411 I	2	4	152.34 II	3	2	211.56 I	1	2
22.45 I	1	2	84.52 II	2	4	108.234 II	2	4	129.420 IV	4	8	153.38 II	3	2	211.57 II	1	2
22.46 II	1	2	84.56 IV	2	4	108.235 III	2	4	129.421 IV	2	4	154.42 II	3	2	211.58 III	1	2
23.49 I	1	2	84.58 IV	2	4	109.240 II	2	4	130.424 II	4	8	155.46 II	1	2	218.82 II	2	4
23.50 II	1	2	85.60 II	2	4	109.242 III	2	4	131.435 I	2	4	156.50 II	1	2	218.83 III	2	4
25.61 IV	2	4	85.65 IV	2	4	111.256 IV	2	4	131.436 II	2	4	157.54 II	1	2	220.90 II	2	4
25.63 IV	2	4	85.66 IV	2	4	112.260 II	2	4	131.439 III	2	4	160.66 II	1	2	220.91 III	2	4
25.64 IV	2	4	86.68 II	2	4	112.262 III	2	4	131.442 III	2	4	163.80 II	2	4	222.98 I	2	4
25.65 IV	2	4	86.74 IV	2	4	113.274 IV	2	4	131.443 III	2	4	165.92 II	2	4	222.99 II	2	4
26.67 II	2	4	88.82 II	2	4	114.278 III	2	4	131.445 IV	4	8	167.104 II	2	4	222.100 III	2	4
27.79 II	2	4	89.87 I	1	2	115.288 IV	2	4	132.447 I	2	4	168.110 II	1	2	222.101 III	2	4
30.112 II	2	4	89.88 II	1	2	115.289 IV	2	4	132.448 II	2	4	171.122 II	3	2	222.102 III	2	4
31.124 II	2	4	89.89 III	1	2	115.290 IV	2	4	132.450 III	2	4	172.126 II	3	2	223.104 I	2	4
32.136 II	2	4	89.91 III	1	2	116.292 II	2	4	132.452 III	2	4	174.134 II	1	2	223.105 II	2	4
34.157 II	2	4	89.93 IV	2	4	116.293 III	2	4	132.455 III	2	4	176.144 II	2	4	223.107 III	2	4
35.169 IV	2	4	89.94 IV	2	4	117.300 II	2	4	132.458 IV	4	8	177.149 I	1	2	223.108 III	2	4
35.171 IV	2	4	90.96 II	2	4	117.301 III	2	4	134.471 I	2	4	177.150 II	1	2	224.110 I	2	4
36.173 II	2	4	90.99 III	2	4	118.308 II	2	4	134.472 II	2	4	180.167 I	3	2	224.111 II	2	4
37.181 II	2	4	93.126 IV	2	4	118.309 III	2	4	134.474 III	2	4	180.168 II	3	2	224.112 III	2	4
43.225 II	2	4	94.128 II	2	4	122.334 II	2	4	134.476 III	2	4	181.173 I	3	2	224.114 III	2	4
48.258 II	2	4	94.131 III	2	4	122.336 III	2	4	134.479 III	2	4	181.174 II	3	2	225.121 IV	2	4
48.261 III	2	4	97.151 I	1	2	124.352 II	2	4	135.484 II	4	8	183.190 IV	2	4	226.123 II	2	4
49.266 II	2	4	97.152 II	1	2	124.354 III	2	4	136.495 I	2	4	184.192 II	2	4	226.125 III	2	4
49.271 III	2	4	97.153 III	1	2	124.355 III	2	4	136.500 III	2	4	185.198 II	2	4	226.126 III	2	4
50.278 II	2	4	97.155 III	1	2	124.356 III	2	4	136.503 III	2	4	186.204 II	2	4	227.128 I	2	4
50.283 III	2	4	99.168 IV	2	4	124.358 III	2	4	136.504 IV	4	8	192.244 II	2	4	227.129 II	2	4
53.329 III	2	4	99.169 IV	2	4	124.359 III	2	4	137.507 I	2	4	192.251 III	2	4	227.130 III	2	4
55.359 III	2	4	99.170 IV	2	4	125.363 I	2	4	137.511 III	2	4	195.1 I	1	2	227.132 III	2	4
58.399 III	2	4	100.172 II	2	4	125.364 II	2	4	137.515 III	2	4	195.2 II	1	2			

ν_P: The physically achievable fillings are quantized in units of ν_P

FS: Dimension of the irrep(s) dissected by the Fermi energy when the physical filling is an odd-integer-multiple of ν_P

Table S10. MSGs that can host feSMs with movable nodal Fermi surfaces.

MSG		ν_P	MSG		ν_P	MSG		ν_P	MSG		ν_P	MSG		ν_P	MSG		ν_P
4.8	II	2	53.325	III	2	77.16	IV	2	102.192	IV	4	138.520	II	4	206.38	II	4
7.25	II	2	54.338	II	4	77.18	IV	2	105.211	I	2	138.530	IV	4	208.44	I	2
9.38	II	2	54.349	IV	4	78.20	II	4	105.217	IV	4	141.553	III	2	208.45	II	2
14.77	III	2	56.366	II	4	80.30	II	2	106.220	II	4	141.554	III	2	210.52	I	2
14.78	III	2	57.378	II	4	84.54	III	2	109.239	I	2	141.560	IV	4	210.53	II	2
17.8	II	2	57.390	IV	4	85.61	III	2	109.244	IV	4	142.562	II	4	210.54	III	2
19.26	II	4	59.405	I	2	86.69	III	2	110.246	II	4	142.564	III	4	212.60	II	4
20.32	II	2	59.413	IV	4	86.70	III	2	119.320	IV	2	142.569	III	4	213.64	II	4
24.54	II	2	59.414	IV	4	88.83	III	2	120.322	II	2	158.58	II	2	214.67	I	2
28.88	II	2	60.418	II	4	88.84	III	2	120.323	III	2	159.62	II	2	214.68	II	2
29.100	II	4	60.429	IV	4	91.104	II	4	122.335	III	2	161.70	II	2	216.77	IV	2
33.145	II	4	61.434	II	4	92.112	II	4	125.367	III	2	169.114	II	6	219.86	II	2
38.194	IV	2	61.435	III	4	93.119	I	2	129.415	III	2	170.118	II	6	219.87	III	2
39.196	II	2	61.437	III	4	93.120	II	2	130.426	III	4	173.130	II	2	224.113	III	2
40.204	II	2	62.442	II	4	93.124	IV	2	131.437	III	2	178.156	II	6	227.131	III	2
41.212	II	2	62.455	IV	4	94.127	I	2	132.449	III	2	179.162	II	6	227.133	IV	4
44.233	IV	2	64.471	III	2	94.134	IV	2	133.460	II	4	182.180	II	2	228.135	II	4
45.236	II	2	64.473	III	2	95.136	II	4	133.467	III	4	188.216	II	2	228.137	III	4
46.242	II	2	64.477	III	2	96.144	II	4	133.470	IV	4	190.228	II	2	228.138	III	4
48.257	I	2	68.511	I	2	98.157	I	2	134.473	III	2	198.10	II	4	230.146	II	4
50.277	I	2	70.527	I	2	98.158	II	2	134.475	III	2	199.13	II	2	230.149	III	4
50.286	IV	4	73.549	II	4	98.161	III	2	134.480	IV	4	201.18	I	2			
50.288	IV	4	74.557	III	2	101.179	I	2	135.491	III	4	203.26	I	2			
52.306	II	4	76.8	II	4	101.186	IV	4	137.516	IV	4	205.34	II	4			
53.323	III	2	77.14	II	2	102.187	I	2	137.517	IV	4	205.35	III	4			

ν_P: The physically achievable fillings are quantized in units of ν_P

Table S11. MSGs that can host filling-enforced metals.

MSG	ν_P	Deg	MSG	ν_P	Deg	MSG	ν_P	Deg	MSG	ν_P	Deg	MSG	ν_P	Deg	MSG	ν_P	Deg
2.5 II	1	3	59.406 II	2	1	99.166 III	1	1	123.350 IV	2	1	147.14 II	1	3	202.22 I	1	1
2.6 III	1	3	59.408 III	2	1	107.227 I	1	1	125.373 IV	2	1	147.15 III	1	3	202.23 II	1	3
10.43 II	1	3	59.416 IV	2	1	107.228 II	1	1	126.386 IV	2	1	148.18 II	1	3	202.24 III	1	3
10.44 III	1	3	63.458 II	2	1	107.229 III	1	1	127.388 II	2	1	148.19 III	1	3	204.30 I	1	1
10.45 III	1	3	63.460 III	2	1	107.230 III	1	1	127.389 III	2	1	162.74 II	1	3	204.31 II	1	3
12.59 II	1	3	64.470 II	2	1	111.251 I	1	1	127.390 III	2	1	162.75 III	1	3	204.32 III	1	3
12.60 III	1	3	64.472 III	2	1	111.252 II	1	1	127.394 III	2	1	162.76 III	1	3	215.70 I	1	1
12.61 III	1	3	65.481 I	1	1	111.253 III	1	1	128.400 II	2	1	164.86 II	1	3	215.71 II	1	1
25.57 I	1	1	65.482 II	1	3	111.254 III	1	1	128.401 III	2	1	164.87 III	1	3	215.72 III	1	1
25.58 II	1	1	65.483 III	1	3	111.257 IV	2	1	128.402 III	2	1	164.88 III	1	3	215.73 IV	2	1
35.165 I	1	1	65.484 III	1	3	111.258 IV	2	1	128.406 III	2	1	166.98 II	1	3	216.74 I	1	1
35.166 II	1	1	65.487 III	1	3	112.266 IV	2	1	129.412 II	2	1	166.99 III	1	3	216.75 II	1	1
38.187 I	1	1	65.488 IV	2	1	113.268 II	2	1	129.413 III	2	1	166.100 III	1	3	216.76 III	1	1
38.188 II	1	1	65.490 IV	2	1	113.269 III	2	1	129.414 III	2	1	175.138 II	1	3	217.78 I	1	1
42.219 I	1	1	67.510 IV	2	1	114.276 II	2	1	129.418 III	2	1	175.139 III	1	3	217.79 II	1	1
42.220 II	1	1	69.521 I	1	1	114.277 III	2	1	129.422 IV	2	1	175.140 III	1	3	217.80 III	1	1
44.229 I	1	1	69.522 II	1	3	115.283 I	1	1	131.444 IV	2	1	183.185 I	1	1	218.84 IV	2	1
44.230 II	1	1	69.523 III	1	3	115.284 II	1	1	131.446 IV	2	1	183.186 II	1	1	221.92 I	1	1
47.249 I	1	1	69.525 III	1	3	115.285 III	1	1	132.456 IV	2	1	187.209 I	1	1	221.93 II	1	3
47.250 II	1	3	71.533 I	1	1	115.286 III	1	1	134.482 IV	2	1	187.210 II	1	1	221.94 III	1	3
47.251 III	1	3	71.534 II	1	3	119.315 I	1	1	136.496 II	2	1	189.221 I	1	1	221.95 III	1	1
47.253 III	1	3	71.535 III	1	3	119.316 II	1	1	136.497 III	2	1	189.222 II	1	1	221.96 III	1	3
47.254 IV	2	1	71.537 III	1	3	119.317 III	1	1	136.498 III	2	1	191.233 I	1	1	221.97 IV	2	1
47.255 IV	2	1	81.34 II	1	1	119.318 III	1	1	136.502 III	2	1	191.234 II	1	3	222.103 IV	2	1
47.256 IV	2	1	81.35 III	1	1	121.327 I	1	1	136.506 IV	2	1	191.235 III	1	3	223.109 IV	2	1
48.264 IV	2	1	82.40 II	1	1	121.328 II	1	1	137.508 II	2	1	191.236 III	1	3	224.115 IV	2	1
50.287 IV	2	1	82.41 III	1	1	121.329 III	1	1	137.509 III	2	1	191.237 III	1	3	225.116 I	1	1
51.290 II	2	1	83.44 II	1	3	121.330 III	1	1	137.510 III	2	1	191.241 III	1	3	225.117 II	1	3
51.293 III	2	1	83.45 III	1	1	123.339 I	1	1	137.514 III	2	1	191.242 IV	2	1	225.118 III	1	3
51.302 IV	2	1	83.46 III	1	3	123.340 II	1	3	137.518 IV	2	1	193.254 II	2	1	225.119 III	1	1
53.322 II	2	1	83.47 III	1	3	123.341 III	1	3	139.531 I	1	1	193.256 III	2	1	225.120 III	1	3
53.324 III	2	1	87.76 II	1	3	123.342 III	1	1	139.532 II	1	3	194.264 II	2	1	229.140 I	1	1
55.354 II	2	1	87.77 III	1	1	123.343 III	1	1	139.533 III	1	3	194.267 III	2	1	229.141 II	1	3
55.355 III	2	1	87.78 III	1	3	123.344 III	1	3	139.534 III	1	1	200.14 I	1	1	229.142 III	1	3
55.356 III	2	1	87.79 III	1	3	123.346 III	1	3	139.535 III	1	1	200.15 II	1	3	229.143 III	1	1
58.394 II	2	1	99.163 I	1	1	123.347 III	1	3	139.536 III	1	3	200.16 III	1	3	229.144 III	1	3
58.395 III	2	1	99.164 II	1	1	123.348 IV	2	1	139.538 III	1	3	200.17 IV	2	1			
58.396 III	2	1	99.165 III	1	1	123.349 IV	2	1	139.539 III	1	3	201.21 IV	2	1			

ν_P: The physically achievable fillings are quantized in units of ν_P

Deg: Dimension of the manifold on which band sticking occurs, which enforces metallic behavior at fillings which are odd-integer multiples of ν_P

Table S12. Effect of turning on spin-orbit coupling on the fermiology of TR-symmetric filling-enforced (semi)metals.

SG	ν_P	FS	SG	ν_P	FS	SG	ν_P	FS	SG	ν_P	FS	SG	ν_P	FS	SG	ν_P	FS
1	1	$M_3 \to p$	40	2	$M_1 \to m$	79	1	$M_3 \to p$	118	2	$M_1 \to p$	157	1	$M_3 \to p$	196	1	$M_3 \to p$
2	1	$M_3 \to M_3$	41	2	$M_1 \to m$	80	2	$p \to m$	119	1	$M_3 \to M_1$	158	2	$M_1 \to m$	197	1	$M_3 \to p$
3	1	$M_3 \to p$	42	1	$M_3 \to M_1$	81	1	$M_3 \to M_1$	120	2	$M_1 \to m$	159	2	$M_1 \to m$	198	4	$p \to m$
4	2	$M_2 \to m$	43	2	$M_1 \to p$	82	1	$M_3 \to M_1$	121	1	$M_3 \to M_1$	160	1	$M_3 \to p$	199	2	$p \to m$
5	1	$M_3 \to p$	44	1	$M_3 \to M_1$	83	1	$M_3 \to M_3$	122	2	$M_1 \to p$	161	2	$M_1 \to m$	200	1	$M_3 \to M_3$
6	1	$M_3 \to p$	45	2	$M_1 \to m$	84	2	$p \to p$	123	1	$M_3 \to M_3$	162	1	$M_3 \to M_3$	201	2	$M_1 \to p$
7	2	$M_1 \to m$	46	2	$M_1 \to m$	85	2	$M_1 \to p$	124	2	$M_1 \to p$	163	2	$M_1 \to p$	202	1	$M_3 \to M_3$
8	1	$M_3 \to p$	47	1	$M_3 \to M_3$	86	2	$M_1 \to p$	125	2	$M_1 \to p$	164	1	$M_3 \to M_3$	203	2	$M_1 \to p$
9	2	$M_1 \to m$	48	2	$M_1 \to p$	87	1	$M_3 \to M_3$	126	2	$M_1 \to p$	165	2	$M_1 \to p$	204	1	$M_3 \to M_3$
10	1	$M_3 \to M_3$	49	2	$M_1 \to p$	88	2	$M_1 \to p$	127	2	$M_2 \to M_1$	166	1	$M_3 \to M_3$	205	4	$M_1 \to m$
11	2	$M_2 \to p$	50	2	$M_1 \to p$	89	1	$M_3 \to p$	128	2	$M_2 \to M_1$	167	2	$M_1 \to p$	206	4	$p \to m$
12	1	$M_3 \to M_3$	51	2	$M_2 \to M_1$	90	2	$M_2 \to p$	129	2	$M_2 \to M_1$	168	1	$M_3 \to p$	207	1	$M_3 \to p$
13	2	$M_1 \to p$	52	4	$p \to m$	91	4	$m \to m$	130	4	$p \to p$	169	6	$M_2 \to m$	208	2	$m \to m$
14	2	$M_2 \to p$	53	2	$M_2 \to M_1$	92	4	$p \to m$	131	2	$M_1 \to p$	170	6	$M_2 \to m$	209	1	$M_3 \to p$
15	2	$M_1 \to p$	54	4	$p \to m$	93	2	$m \to m$	132	2	$M_1 \to p$	171	3	$M_3 \to p$	210	2	$p \to m$
16	1	$M_3 \to p$	55	2	$M_2 \to M_1$	94	2	$M_2 \to p$	133	4	$m \to m$	172	3	$M_3 \to p$	211	1	$M_3 \to p$
17	2	$M_2 \to m$	56	4	$p \to m$	95	4	$m \to m$	134	2	$M_1 \to p$	173	2	$M_2 \to m$	212	4	$p \to m$
18	2	$M_2 \to p$	57	4	$M_1 \to m$	96	4	$p \to m$	135	4	$p \to p$	174	1	$M_3 \to p$	213	4	$p \to m$
19	4	$p \to m$	58	2	$M_2 \to M_1$	97	1	$M_3 \to p$	136	2	$M_2 \to M_1$	175	1	$M_3 \to M_3$	214	2	$p \to m$
20	2	$M_2 \to m$	59	2	$M_2 \to M_1$	98	2	$p \to m$	137	2	$M_2 \to M_1$	176	2	$M_2 \to p$	215	1	$M_3 \to M_1$
21	1	$M_3 \to p$	60	4	$M_1 \to m$	99	1	$M_3 \to M_1$	138	4	$p \to m$	177	1	$M_3 \to p$	216	1	$M_3 \to M_1$
22	1	$M_3 \to p$	61	4	$M_1 \to m$	100	2	$M_1 \to p$	139	1	$M_3 \to M_3$	178	6	$M_2 \to m$	217	1	$M_3 \to M_1$
23	1	$M_3 \to p$	62	4	$M_1 \to m$	101	2	$M_1 \to p$	140	2	$M_1 \to p$	179	6	$M_2 \to m$	218	2	$M_1 \to p$
24	2	$p \to m$	63	2	$M_2 \to M_1$	102	2	$M_1 \to p$	141	2	$M_1 \to p$	180	3	$M_3 \to p$	219	2	$M_1 \to m$
25	1	$M_3 \to M_1$	64	2	$M_2 \to M_1$	103	2	$M_1 \to p$	142	4	$p \to m$	181	3	$M_3 \to p$	220	2	$M_1 \to p$
26	2	$M_2 \to p$	65	1	$M_3 \to M_3$	104	2	$M_1 \to p$	143	1	$M_3 \to p$	182	2	$M_2 \to m$	221	1	$M_3 \to M_3$
27	2	$M_1 \to p$	66	2	$M_1 \to p$	105	2	$M_1 \to p$	144	3	$M_3 \to p$	183	1	$M_3 \to M_1$	222	2	$M_1 \to p$
28	2	$M_1 \to m$	67	2	$M_1 \to p$	106	4	$m \to m$	145	3	$M_3 \to p$	184	2	$M_1 \to p$	223	2	$M_1 \to p$
29	4	$p \to m$	68	2	$M_1 \to p$	107	1	$M_3 \to M_1$	146	1	$M_3 \to p$	185	2	$M_2 \to p$	224	2	$M_1 \to p$
30	2	$M_1 \to p$	69	1	$M_3 \to M_3$	108	2	$M_1 \to p$	147	1	$M_3 \to M_3$	186	2	$M_2 \to p$	225	1	$M_3 \to M_3$
31	2	$M_2 \to p$	70	2	$M_1 \to p$	109	2	$M_1 \to p$	148	1	$M_3 \to M_3$	187	1	$M_3 \to M_1$	226	2	$M_1 \to p$
32	2	$M_1 \to p$	71	1	$M_3 \to M_3$	110	4	$p \to m$	149	1	$M_3 \to p$	188	2	$M_1 \to m$	227	2	$M_1 \to p$
33	4	$p \to m$	72	2	$M_1 \to p$	111	1	$M_3 \to M_1$	150	1	$M_3 \to p$	189	1	$M_3 \to M_1$	228	4	$p \to m$
34	2	$M_1 \to p$	73	4	$p \to m$	112	2	$M_1 \to p$	151	3	$M_3 \to p$	190	2	$M_1 \to m$	229	1	$M_3 \to M_3$
35	1	$M_3 \to M_1$	74	2	$M_1 \to p$	113	2	$M_2 \to M_1$	152	3	$M_3 \to p$	191	1	$M_3 \to M_3$	230	4	$p \to m$
36	2	$M_2 \to p$	75	1	$M_3 \to p$	114	2	$M_2 \to M_1$	153	3	$M_3 \to p$	192	2	$M_1 \to p$			
37	2	$M_1 \to p$	76	4	$m \to m$	115	1	$M_3 \to M_1$	154	3	$M_3 \to p$	193	2	$M_2 \to M_1$			
38	1	$M_3 \to M_1$	77	2	$m \to m$	116	2	$M_1 \to p$	155	1	$M_3 \to p$	194	2	$M_2 \to M_1$			
39	2	$M_1 \to m$	78	4	$m \to m$	117	2	$M_1 \to p$	156	1	$M_3 \to p$	195	1	$M_3 \to p$			

SG: Space group describing the spatial symmetry of a material with time-reversal symmetry
ν_P: The physically achievable fillings are quantized in units of ν_P
FS: Fermiology for a material with a filling being an odd-integer multiple of ν_P, dubbed a filling-enforced (semi-)metal
M_d: Filling-enforced metals due to band sticking on d-dimensional manifolds
p: Filling-enforced semimetals with pinned, nodal-point Fermi surfaces
m: Filling-enforced semimetals with movable nodal Fermi surfaces

Section S3. Notations

For a MSG $\mathcal{M} = \mathcal{G} + \mathcal{A}$, $g \in \mathcal{G}$ maps the position $\boldsymbol{x}$ and the momentum $\boldsymbol{k}$ to

$$g(\boldsymbol{x}) \equiv p_g \boldsymbol{x} + \boldsymbol{t}_g, \;\; g(\boldsymbol{k}) \equiv p_g(\boldsymbol{k}), \;\; p_g \in O(3), \tag{1}$$

while $a \in \mathcal{A}$ acts on them as

$$a(\boldsymbol{x}) = p_a \boldsymbol{x} + \boldsymbol{t}_a, \quad a(\boldsymbol{k}) = -p_a \boldsymbol{k}, \;\; p_a \in O(3). \tag{2}$$

The little group of $\boldsymbol{k}$ is defined by

$$\mathcal{M}_{\boldsymbol{k}} = \{m \in \mathcal{M} \,|\, m(\boldsymbol{k}) = \boldsymbol{k}\} = \mathcal{G}_{\boldsymbol{k}} + \mathcal{A}_{\boldsymbol{k}}, \tag{3}$$

where $\mathcal{G}_{\boldsymbol{k}}$ is the unitary part and $\mathcal{A}_{\boldsymbol{k}}$ is the anti-unitary part:

$$\mathcal{G}_{\boldsymbol{k}} = \{g \in \mathcal{G} \,|\, g(\boldsymbol{k}) = \boldsymbol{k} + \boldsymbol{G}\}, \tag{4}$$

$$\mathcal{A}_{\boldsymbol{k}} = \{a \in \mathcal{A} \,|\, a(\boldsymbol{k}) = \boldsymbol{k} + \boldsymbol{G}\} \tag{5}$$

and $\boldsymbol{G}$ is a reciprocal lattice vector. $\mathcal{G}_{\boldsymbol{k}}$ is a subgroup of $\mathcal{M}_{\boldsymbol{k}}$ but $\mathcal{A}_{\boldsymbol{k}}$ is not even a group.

Similarly, the site symmetry group of $\boldsymbol{x}$ is defined by

$$\mathcal{M}_{\boldsymbol{x}} = \{m \in \mathcal{G} \,|\, m(\boldsymbol{x}) = \boldsymbol{x}\} = \mathcal{G}_{\boldsymbol{x}} + \mathcal{A}_{\boldsymbol{x}}. \tag{6}$$

where $\mathcal{G}_{\boldsymbol{x}}$ is the unitary part and $\mathcal{A}_{\boldsymbol{x}}$ is the anti-unitary part:

$$\mathcal{G}_{\boldsymbol{x}} = \{g \in \mathcal{G} \,|\, g(\boldsymbol{x}) = \boldsymbol{x}\}, \tag{7}$$

$$\mathcal{A}_{\boldsymbol{x}} = \{a \in \mathcal{A} \,|\, a(\boldsymbol{x}) = \boldsymbol{x}\}. \tag{8}$$

Since we are interested in systems of spinful electrons with spin-orbit coupling, we assume a projective representation, i.e.,

$$u(g)u(g') = z_{g,g'} u(gg'), \tag{9}$$

$$u(g)u(a') = z_{g,a'} u(ga'), \tag{10}$$

$$u(a)u(g')^* = z_{a,g'} u(ag'), \tag{11}$$

$$u(a)u(a')^* = z_{a,a'} u(aa'). \tag{12}$$

For example, $z_{\mathcal{T},\mathcal{T}} = -1$ for the time-reversal symmetry $\mathcal{T}$. For spinless electrons, all z should be set to be unity in the following discussions.

Section S4. Compatibility relations

Here we review the compatibility relations among the integers $\boldsymbol{n} = \{n_{\boldsymbol{k}}^{\alpha}\}$ to be consistent with $\mathcal{M} = \mathcal{G} + \mathcal{A}$.

A. Basics

Suppose that $\{|\boldsymbol{k}, i\rangle\}$ is a basis of an irrep $u_{\boldsymbol{k}}^{\alpha}$ of $\mathcal{G}_{\boldsymbol{k}}$, i.e.,

$$\hat{h}|\boldsymbol{k}, i\rangle = |\boldsymbol{k}, j\rangle [u_{\boldsymbol{k}}^{\alpha}(h)]_{ji}, \quad h \in \mathcal{G}_{\boldsymbol{k}}. \tag{13}$$

Here we show that $\{\hat{g}|\boldsymbol{k}, i\rangle\}$ ($g \in \mathcal{G}$) forms a basis of the irrep

$$u_{g(\boldsymbol{k})}^{\alpha'}(h') \equiv \rho_{h',g} u_{\boldsymbol{k}}^{\alpha}(g^{-1}h'g), \quad h' \in \mathcal{G}_{g(\boldsymbol{k})}. \tag{14}$$

Here, $\rho_{h,g}$ is related to the projective factor $z_{g,g'} = \pm 1$ as

$$\rho_{h,g} = \frac{z_{h,g}}{z_{g,g^{-1}hg}} = \pm 1. \tag{15}$$

To see this, note that $h' \in \mathcal{G}_{g(\boldsymbol{k})}$ can be written as $h' = ghg^{-1}$ using $h \in \mathcal{G}_{\boldsymbol{k}}$. Then,

$$\begin{aligned} \hat{h}'(\hat{g}|\boldsymbol{k}, i\rangle) &= z_{h',g} (\widehat{h'g})|\boldsymbol{k}, i\rangle \\ &= z_{h',g} (\widehat{gh})|\boldsymbol{k}, i\rangle = \frac{z_{h',g}}{z_{g,h}} \hat{g}(\hat{h}|\boldsymbol{k}, i\rangle) \\ &= \rho_{h',g} \hat{g}(|\boldsymbol{k}, j\rangle [u_{\boldsymbol{k}}^{\alpha}(h)]_{ji}) \\ &= (\hat{g}|\boldsymbol{k}, j\rangle) \rho_{h',g} [u_{\boldsymbol{k}}^{\alpha}(g^{-1}h'g)]_{ji}. \end{aligned} \tag{16}$$

Similarly, $\{\hat{a}|\boldsymbol{k}, i\rangle\}$ ($a \in \mathcal{A}$) forms a basis of the irrep

$$u_{a(\boldsymbol{k})}^{\alpha'}(h') \equiv \rho_{h',a} [u_{\boldsymbol{k}}^{\alpha}(a^{-1}h'a)]_{ji}^*, \quad h' \in \mathcal{G}_{g(\boldsymbol{k})}. \tag{17}$$

where

$$\rho_{h,a} = \frac{z_{h,a}}{z_{a,a^{-1}ha}} = \pm 1. \tag{18}$$

Again writing $h' \in \mathcal{G}_{a(\boldsymbol{k})}$ as $h' = aha^{-1}$ using $h \in \mathcal{G}_{\boldsymbol{k}}$,

$$\begin{aligned} \hat{h}'(\hat{a}|\boldsymbol{k}, i\rangle) &= z_{h',a} (\widehat{h'a})|\boldsymbol{k}, i\rangle \\ &= z_{h',a} (\widehat{ah})|\boldsymbol{k}, i\rangle = \frac{z_{h',a}}{z_{a,h}} \hat{a}(\hat{h}|\boldsymbol{k}, i\rangle) \\ &= \rho_{h',a} \hat{a}(|\boldsymbol{k}, j\rangle [u_{\boldsymbol{k}}^{\alpha}(h)]_{ji}) \\ &= (\hat{a}|\boldsymbol{k}, j\rangle) \rho_{h',a} [u_{\boldsymbol{k}}^{\alpha}(a^{-1}h'a)]_{ji}^*. \end{aligned} \tag{19}$$

B. Compatibility relations from the unitary part $\mathcal{C}_{\mathcal{G}}$

Here we summarize the compatibility relations arising from the unitary part $\mathcal{G}$ of $\mathcal{M}$. There are three major categories:

(i) The representations used at symmetry-related momenta are related to each other. Suppose that an irrep $u_{\boldsymbol{k}}^{\alpha}$ of $\mathcal{G}_{\boldsymbol{k}}$ is used $n_{\boldsymbol{k}}^{\alpha}$ times. When $g \notin \mathcal{G}_{\boldsymbol{k}}$, $\boldsymbol{k}$ and $g(\boldsymbol{k})$ are symmetry-related but distinct momenta. Let $\{|\boldsymbol{k}, i\rangle\}$ be the basis of the representation. As explained above, the g-transformed copy, $\{\hat{g}|\boldsymbol{k}, i\rangle\}$ forms a basis of the irrep $u_{g(\boldsymbol{k})}^{\alpha'}$ in Eq. (13). It implies

$$n_{g(\boldsymbol{k})}^{\alpha'} = n_{\boldsymbol{k}}^{\alpha}. \tag{20}$$

(ii) At a high-symmetry momenta slightly off from a higher-symmetry point $\boldsymbol{k}$, the continuity of the band structure requires that

$$n_{\boldsymbol{k}+\delta(\boldsymbol{k})}^{\beta} = \sum_{\alpha} n_{\boldsymbol{k}}^{\alpha} m^{\alpha\beta}, \tag{21}$$

where nonnegative integers $m^{\alpha\beta}$ are the coefficients appearing in the decomposition $u_{\boldsymbol{k}}^{\alpha} = \sum_{\beta} c^{\alpha\beta} u_{\boldsymbol{k}+\delta(\boldsymbol{k})}^{\beta}$ regarding $u_{\boldsymbol{k}}^{\alpha}$ as a representation of $\mathcal{G}_{\boldsymbol{k}+\delta(\boldsymbol{k})} < \mathcal{G}_{\boldsymbol{k}}$.

(iii) For nonsymmorphic operations g with a 'fractional translation', such as glide reflections or screw rotations,

$$\sum_{\alpha} n_{\boldsymbol{k}}^{\alpha} \operatorname{tr} u_{\boldsymbol{k}}^{\alpha}(g) = 0, \tag{22}$$

again due to the continuity of the band structure.

C. Compatibility relations from the anti-unitary part $\tilde{\mathcal{C}}_{\mathcal{A}}$

The anti-unitary part $\mathcal{A}$ adds two more constraints.

(iv) $\mathcal{A}$ may relate two momenta that were not related by $\mathcal{G}$. This occurs when the anti-unitary part $\mathcal{A}_{\boldsymbol{k}}$ is empty. Suppose that an irrep $u_{\boldsymbol{k}}^{\alpha}$ of $\mathcal{G}_{\boldsymbol{k}}$ is used $n_{\boldsymbol{k}}^{\alpha}$ times. Let $\{|\boldsymbol{k},i\rangle\}$ be the basis of the representation. The a-transformed copy, $\{\hat{a}|\boldsymbol{k},i\rangle\}$ forms a basis of the irrep $u_{a(\boldsymbol{k})}^{\alpha'}$ in Eq. (17). It requires a constraint

$$n_{a(\boldsymbol{k})}^{\alpha'} = n_{\boldsymbol{k}}^{\alpha}. \tag{23}$$

(v) When the anti-unitary part $\mathcal{A}_{\boldsymbol{k}}$ is not empty, $u_{a(\boldsymbol{k})}^{\alpha'}$ in Eq. (17) is also an irrep at $\boldsymbol{k}$ if $a \in \mathcal{A}_{\boldsymbol{k}}$. The nature of paring due to $\mathcal{A}_{\boldsymbol{k}}$ can be judged by the Herring rule [2]:

$$\xi_{\boldsymbol{k}}^{\alpha} = \frac{1}{|\mathcal{A}_{\boldsymbol{k}}/T|}\sum_{a\in\mathcal{A}_{\boldsymbol{k}}/T}\delta_{a(\boldsymbol{k}),\boldsymbol{k}}\, z_{a,a}\,\chi_{\boldsymbol{k}}^{\alpha}(a^2), \tag{24}$$

where $\chi_{\boldsymbol{k}}^{\alpha}(h) \equiv \operatorname{tr} u_{\boldsymbol{k}}^{\alpha}(h)$ is the character of the irrep. When $\xi_{\boldsymbol{k}}^{\alpha} = 0$, $u_{\boldsymbol{k}}^{\alpha'}$ and $u_{\boldsymbol{k}}^{\alpha}$ are inequivalent and $n_{\boldsymbol{k}}^{\alpha'} = n_{\boldsymbol{k}}^{\alpha}$ ($\alpha' \neq \alpha$) is required. When $\xi_{\boldsymbol{k}}^{\alpha} = -1$ or $+1$, $u_{\boldsymbol{k}}^{\alpha'}$ and $u_{\boldsymbol{k}}^{\alpha}$ are equivalent ($\alpha' = \alpha$). When $\xi_{\boldsymbol{k}}^{\alpha} = -1$, $\{\hat{a}|\boldsymbol{k},i\rangle\}$ and $\{|\boldsymbol{k},i\rangle\}$ are orthogonal to each other and $n_{\boldsymbol{k}}^{\alpha}$ must be an even integer. When $\xi_{\boldsymbol{k}}^{\alpha} = +1$, $\{\hat{a}|\boldsymbol{k},i\rangle\}$ and $\{|\boldsymbol{k},i\rangle\}$ are linearly dependent and there is no extra constraint on $n_{\boldsymbol{k}}^{\alpha}$.

Section S5. Details of AIs

As explained in the main text, an AI can be specified by choosing the position $\boldsymbol{x}$ in real space at which electrons are localized and the type of orbital on the site. The type of orbital is a (co-)representation of the site symmetry group $\mathcal{M}_{\boldsymbol{x}}$ in Eq. (6).

Let $u_{\boldsymbol{x}}^{r}$ ($r = 1, 2, \ldots$) be an irrep of $\mathcal{G}_{\boldsymbol{x}}$. When $\mathcal{A}_{\boldsymbol{x}}$ is empty, $u_{\boldsymbol{x}}^{r}$ is already an irrep of $\mathcal{M}_{\boldsymbol{x}}$. We put an orbital $u_{\boldsymbol{x}}^{r}$ on the site $\boldsymbol{x}$. One can construct an AI by placing a symmetry-related copy on each site of an $\mathcal{M}$-symmetric lattice $\mathcal{M}\boldsymbol{x} \equiv \{m\boldsymbol{x}\,|\,m \in \mathcal{M}\}$. In this case the lattice $\mathcal{M}\boldsymbol{x}$ contains twice as many number of site as $\mathcal{G}\boldsymbol{x}$.

On the other hand, when $\mathcal{A}_{\boldsymbol{x}}$ is not empty, $\mathcal{A}_{\boldsymbol{x}}$ dictates a certain pairing among $u_{\boldsymbol{x}}^{r}$'s, leading to a co-irrep of $\mathcal{M}_{\boldsymbol{x}}$. The details of paring can be again judged by the Herring rule [2]:

$$\xi_{\boldsymbol{x}}^{r} = \frac{1}{|\mathcal{A}_{\boldsymbol{x}}|}\sum_{a\in\mathcal{A}_{\boldsymbol{x}}} z_{a,a}\,\chi_{\boldsymbol{x}}^{r}(a^2). \tag{25}$$

When $\xi_{\boldsymbol{x}}^{r} = 0$, $u_{\boldsymbol{x}}^{r'}$ and $u_{\boldsymbol{k}}^{r}$ are inequivalent ($r' \neq r$) and must be paired to form a co-irrep of $\mathcal{M}_{\boldsymbol{x}}$. When $\xi_{\boldsymbol{x}}^{r} = -1$ or $+1$, $u_{\boldsymbol{x}}^{r'}$ and $u_{\boldsymbol{x}}^{r}$ are equivalent ($r' = r$). When $\xi_{\boldsymbol{x}}^{r} = -1$, $\{\hat{a}|\boldsymbol{x},i\rangle\}$ and $\{|\boldsymbol{x},i\rangle\}$ are orthogonal to each other, and $u_{\boldsymbol{x}}^{r}$ is 'paired with itself'. Finally, when $\xi_{\boldsymbol{x}}^{r} = +1$, $\{\hat{a}|\boldsymbol{x},i\rangle\}$ and $\{|\boldsymbol{x},i\rangle\}$ are linearly dependent and $u_{\boldsymbol{x}}^{r}$ alone stands a co-irrep of $\mathcal{M}_{\boldsymbol{x}}$. We put a co-irrep on the site $\boldsymbol{x}$ and generate AI by placing symmetry-related copy of the orbital on each site of the lattice $\mathcal{M}\boldsymbol{x} = \mathcal{G}\boldsymbol{x}$.

Section S6. Details of $\tilde{\mathcal{T}}b$

Let us write $\boldsymbol{b} = \{n_{\boldsymbol{k}}^{\alpha}\}$ and $\tilde{\mathcal{T}}\boldsymbol{b} = \{\tilde{n}_{\boldsymbol{k}}^{\alpha}\}$. By definition, $\boldsymbol{b}$ has the representation $\oplus_{\alpha} n_{\boldsymbol{k}}^{\alpha} u_{\boldsymbol{k}}^{\alpha}$ of $\mathcal{G}_{\boldsymbol{k}}$ at $\boldsymbol{k}$. Then $a = \tilde{\mathcal{T}}$ maps it to the representation $\oplus_{\alpha} n_{\boldsymbol{k}}^{\alpha} u_{a(\boldsymbol{k})}^{\alpha'}$ of $\mathcal{G}_{a(\boldsymbol{k})}$ at $a(\boldsymbol{k})$ [see Eq. (17)]. By definition of $\tilde{\mathcal{T}}\boldsymbol{b}$, this coincides with $\oplus_{\alpha} \tilde{n}_{a(\boldsymbol{k})}^{\alpha} u_{a(\boldsymbol{k})}^{\alpha}$, i.e.,

$$\oplus_{\alpha} \tilde{n}_{a(\boldsymbol{k})}^{\alpha} u_{a(\boldsymbol{k})}^{\alpha}(h) = \oplus_{\alpha} n_{\boldsymbol{k}}^{\alpha} u_{a(\boldsymbol{k})}^{\alpha'}(h). \tag{26}$$

Solving for $\tilde{n}_{\boldsymbol{k}}^{\alpha}$, we get

$$\tilde{n}_{\boldsymbol{k}}^{\beta} = \frac{1}{|\mathcal{G}_{\boldsymbol{k}}/T|}\sum_{h\in\mathcal{G}_{\boldsymbol{k}}/T}\chi_{\boldsymbol{k}}^{\beta}(h)^{*}\sum_{\alpha} n_{a^{-1}(\boldsymbol{k})}^{\alpha}\,\chi_{\boldsymbol{k}}^{\alpha'}(h). \tag{27}$$

Section S7. Magnetic layer groups

For each MLG, there is a corresponding MSG that can be achieved by stacking of 2D layers. Conversely, if an MSG $\mathcal{M}$ is such that every element of $m \in \mathcal{M}$ maps, say, z to either $n \pm z$ ($n \in \mathbb{Z}$) (i.e., no mixing of z with x or y or no fractional translation along z), the MSG can be reduced to an MLG by projecting down to xy plane. (In the actual calculation, it suffices to set $k_z = 0$). The projection down to yz and zx planes, if allowed, may also lead to an MSG. These three possible ways of projections lead to sometimes distinct and sometimes the identical MLGs. We present the computation results in Sec. I in a redundant way — for each 'parent' MSG, we list the computed numbers for all possible directions of projections among the three, regardless of their dependence/independence as MLGs. For example, for MSG 3.4, two projections down to xy and xz planes are possible and correspondingly we have two rows 3.4 (1) and 3.4 (2). For this MSG, the two projections correspond to two different MLGs. On the other hand, for the type II MSG 2.5, the three ways of projection lead to an identical MLG. Even in this case the redundant presentation has an advantage: e.g. it tells us that the three $\mathbb{Z}_2$ factor of $X_{\mathrm{BS}} = (\mathbb{Z}_2)^3 \times \mathbb{Z}_4$ for MSG 2.5 can actually be attributed to the thee orthogonal 2D planes, implying the presence of 'weak' phases.

Section S8. Tight-binding model for reSM

Here we present a tight-binding model that realizes the 2D reSM described by the main text. We assume the MLG p_a112. There are two sites in a unit cell. The A-sublattice is at $\boldsymbol{x} = n_1\boldsymbol{a}_1 + n_2\boldsymbol{a}_2$ ($n_1, n_2 \in \mathbb{Z}$) and the B-sublattice is at $\boldsymbol{x} = (n_1 + \frac{1}{2})\boldsymbol{a}_1 + n_2\boldsymbol{a}_2$ ($n, m \in \mathbb{Z}$). For simplicity let us set $\boldsymbol{a}_1 = (2, 0)$, $\boldsymbol{a}_2 = (-1, 1)$. In the following, four by four matrices are in the basis $(|A,\uparrow\rangle, |A,\downarrow\rangle, |B,\uparrow\rangle, |B,\downarrow\rangle)$. We assume the standard symmetry representation:

$$U_{\boldsymbol{k}}(C_{2z}) = i\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & e^{-ik_1} & 0\\ 0 & 0 & 0 & -e^{-ik_1}\end{pmatrix}, \tag{28}$$

$$U_{\boldsymbol{k}}(\mathcal{T}T_{\frac{1}{2}\boldsymbol{a}_1}) = \begin{pmatrix}0 & 0 & 0 & -e^{ik_1}\\ 0 & 0 & e^{ik_1} & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix} \tag{29}$$

satisfying

$$U_{-\boldsymbol{k}}(C_{2z})U_{\boldsymbol{k}}(C_{2z}) = -1, \tag{30}$$

$$U_{-\boldsymbol{k}}(\mathcal{T}T_{\frac{1}{2}\boldsymbol{a}_1})U_{\boldsymbol{k}}(\mathcal{T}T_{\frac{1}{2}\boldsymbol{a}_1})^* = -e^{-ik_1}. \tag{31}$$

An example of the tight-binding Hamiltonian in the momentum space reads

$$h(\boldsymbol{k}) = \sum_{\mu,\nu=0,4} g_{\mu,\nu}(\boldsymbol{k})\,\sigma_\mu\otimes\sigma_\nu, \tag{32}$$

$$g_{1,0}(\boldsymbol{k}) = t[1+\cos k_1+\cos k_2+\cos(k_1+k_2)], \tag{33}$$

$$g_{2,0}(\boldsymbol{k}) = t[\sin k_1-\sin k_2+\sin(k_1+k_2)] \tag{34}$$

$$g_{1,1}(\boldsymbol{k}) = t_J[1-\cos k_1+\cos k_2-\cos(k_1+k_2)], \tag{35}$$

$$g_{2,1}(\boldsymbol{k}) = -t_J[\sin k_1+\sin k_2+\sin(k_1+k_2)], \tag{36}$$

$$g_{3,3}(\boldsymbol{k}) = J. \tag{37}$$

It is easy to check the symmetry of the Hamiltonian:

$$U_{\boldsymbol{k}}(C_{2z})h(\boldsymbol{k}) = h(-\boldsymbol{k})U_{\boldsymbol{k}}(C_{2z}), \tag{38}$$

$$U_{\boldsymbol{k}}(\mathcal{T}T_{\frac{1}{2}\boldsymbol{a}_1})h(\boldsymbol{k})^* = h(-\boldsymbol{k})U_{\boldsymbol{k}}(\mathcal{T}T_{\frac{1}{2}\boldsymbol{a}_1}). \tag{39}$$

When $|t_J| < \frac{1}{2}J$, $\eta = +1$ and the system is a band insulator. When $|t_J| > \frac{1}{2}J$, $\eta = -1$ and the system realizes a Dirac semimetal.

The MLG corresponding to the type III MSG 13.69 ($P112'/a'$) can also host a Dirac semimetal through an almost identical mechanism. The MLG is generated by $T_{\boldsymbol{a}_2}$, I (the 3D inversion), and $\mathcal{T}G$ ($G \equiv T_{\frac{1}{2}\boldsymbol{a}_1}IC_{2z}$ is the glide symmetry). The corresponding topological invariant η is defined by

$$\eta = \prod_{n:\text{occupied}}\ \prod_{\boldsymbol{k}=(0,0),(0,\pi)} \eta_{n,\boldsymbol{k}}, \tag{40}$$

where $\eta_{n,\boldsymbol{k}}$ represents the inversion eigenvalue ± 1.

Section S9. Tables for spinless electrons

Here we include the following tables for spinless electrons:

- Tables S13–S18: d, X_{BS}, and ν_{BS} for MSGs.
- Table S19: $\{\nu\}_{\mathrm{BS}}$ and $\{\nu\}_{\mathrm{AI}}$ for exceptional MSGs.

Similar to the remarks in Sec. I, some previously published results are reproduced below for completeness.

Table S13. Characterization of MSGs in the triclinic family for spinless electrons.

MSG	d	X_{BS}	ν_{BS}	MSG	d	X_{BS}	ν_{BS}	MSG	d	X_{BS}	ν_{BS}	MSG	d	X_{BS}	ν_{BS}
1.1 I	1	(1)	1	1.3 IV	1	(1)	2	2.5 II	9	$(2,2,2,4)$	1	2.7 IV	5	$(2,2,4)$	2
1.2 II	1	(1)	1	2.4 I	9	$(2,2,2,4)$	1	2.6 III	1	(1)	1				

d: Rank of the band structure group $\{\mathrm{BS}\}$
X_{BS}: Symmetry-based indicators of band topology
ν_{BS}: Set of ν bands are symmetry-forbidden from being isolated by band gaps if $\nu \notin \nu_{\mathrm{BS}}\,\mathbb{Z}$

Table S14. Characterization of MSGs in the monoclinic family for spinless electrons.

MSG	d	X_{BS}	ν_{BS}	MSG	d	X_{BS}	ν_{BS}	MSG	d	X_{BS}	ν_{BS}	MSG	d	X_{BS}	ν_{BS}
3.1 I	5	(2)	1	7.24 I	1	(1)	2	10.47 IV	9	$(2,2,2)$	2	13.70 IV	4	$(2,2)$	4
3.2 II	5	(2)	1	7.25 II	1	(1)	2	10.48 IV	10	$(2,2)$	2	13.71 IV	5	$(2,2)$	4
3.3 III	1	(1)	1	7.26 III	1	(1)	2	10.49 IV	8	$(2,2)$	2	13.72 IV	7	$(2,2)$	2
3.4 IV	3	(2)	2	7.27 IV	1	(1)	4	11.50 I	6	(2)	2	13.73 IV	5	(2)	2
3.5 IV	5	(2)	2	7.28 IV	1	(1)	2	11.51 II	6	(2)	2	13.74 IV	4	(2)	4
3.6 IV	3	(1)	2	7.29 IV	1	(1)	4	11.52 III	2	(1)	2	14.75 I	5	(2)	2
4.7 I	1	(1)	2	7.30 IV	1	(1)	4	11.53 III	1	(1)	2	14.76 II	5	(2)	2
4.8 II	1	(1)	2	7.31 IV	1	(1)	2	11.54 III	5	$(2,2,4)$	2	14.77 III	1	(1)	2
4.9 III	1	(1)	2	8.32 I	2	(1)	1	11.55 IV	4	(2)	4	14.78 III	1	(1)	2
4.10 IV	1	(1)	4	8.33 II	2	(1)	1	11.56 IV	6	(2)	2	14.79 III	5	$(2,4)$	2
4.11 IV	1	(1)	2	8.34 III	1	(1)	1	11.57 IV	4	(2)	2	14.80 IV	3	(2)	4
4.12 IV	1	(1)	2	8.35 IV	2	(1)	2	12.58 I	10	$(2,2)$	1	14.81 IV	3	(2)	4
5.13 I	3	(1)	1	8.36 IV	2	(1)	2	12.59 II	10	$(2,2)$	1	14.82 IV	3	(2)	4
5.14 II	3	(1)	1	9.37 I	1	(1)	2	12.60 III	2	(1)	1	14.83 IV	5	(2)	2
5.15 III	1	(1)	1	9.38 II	1	(1)	2	12.61 III	3	(1)	1	14.84 IV	3	(2)	4
5.16 IV	2	(1)	2	9.39 III	1	(1)	2	12.62 III	7	$(2,2,4)$	1	15.85 I	6	$(2,2)$	2
5.17 IV	3	(2)	2	9.40 IV	1	(1)	2	12.63 IV	6	$(2,2)$	2	15.86 II	6	$(2,2)$	2
6.18 I	3	(1)	1	9.41 IV	1	(2)	2^*	12.64 IV	8	$(2,2)$	2	15.87 III	1	(1)	2
6.19 II	3	(1)	1	10.42 I	15	$(2,2,2)$	1	13.65 I	7	$(2,2)$	2	15.88 III	2	(1)	2
6.20 III	1	(1)	1	10.43 II	15	$(2,2,2)$	1	13.66 II	7	$(2,2)$	2	15.89 III	5	$(2,4)$	2
6.21 IV	3	(1)	2	10.44 III	3	(1)	1	13.67 III	1	(1)	2	15.90 IV	5	$(2,2)$	2
6.22 IV	2	(1)	2	10.45 III	5	(2)	1	13.68 III	3	(2)	2	15.91 IV	4	$(2,2,2)$	2^*
6.23 IV	2	(1)	2	10.46 III	9	$(2,2,2,4)$	1	13.69 III	5	$(2,2,4)$	2				

d: Rank of the band structure group $\{\mathrm{BS}\}$
X_{BS}: Symmetry-based indicators of band topology
ν_{BS}: Set of ν bands are symmetry-forbidden from being isolated by band gaps if $\nu \notin \nu_{\mathrm{BS}}\,\mathbb{Z}$
$*$: Exhibiting exceptional filling pattern; see table S19

Table S15. Characterization of MSGs in the orthorhombic family for spinless electrons.

MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$
16.1 I	13	(1)	1	23.51 III	3	(1)	1	29.101 III	1	(1)	4	33.151 IV	1	(1)	4
16.2 II	13	(1)	1	23.52 IV	5	(2)	2	29.102 III	1	(1)	4	33.152 IV	1	(1)	4
16.3 III	5	(2)	1	24.53 I	4	(1)	2	29.103 III	1	(1)	4	33.153 IV	1	(1)	4
16.4 IV	9	(2)	2	24.54 II	4	(1)	2	29.104 IV	1	(1)	4	33.154 IV	1	(1)	4
16.5 IV	7	(2)	2	24.55 III	2	(1)	2	29.105 IV	1	(2)	4^*	33.155 IV	1	(1)	4
16.6 IV	7	(2)	2	24.56 IV	4	(2)	2	29.106 IV	1	(1)	4	34.156 I	3	(1)	2
17.7 I	5	(1)	2	25.57 I	9	(1)	1	29.107 IV	1	(1)	4	34.157 II	3	(1)	2
17.8 II	5	(1)	2	25.58 II	9	(1)	1	29.108 IV	1	(1)	4	34.158 III	1	(1)	2
17.9 III	1	(1)	2	25.59 III	3	(1)	1	29.109 IV	1	(1)	4	34.159 III	3	(1)	2
17.10 III	3	(2)	2	25.60 III	5	(2)	1	29.110 IV	1	(1)	4	34.160 IV	2	(1)	4
17.11 IV	4	(2)	4	25.61 IV	9	(1)	2	30.111 I	3	(1)	2	34.161 IV	3	(2)	4
17.12 IV	5	(1)	2	25.62 IV	6	(1)	2	30.112 II	3	(1)	2	34.162 IV	2	(1)	4
17.13 IV	4	(1)	2	25.63 IV	5	(1)	2	30.113 III	1	(1)	2	34.163 IV	3	(2)	4
17.14 IV	3	(2)	4	25.64 IV	6	(1)	2	30.114 III	1	(1)	2	34.164 IV	3	(1)	2
17.15 IV	3	(1)	4	25.65 IV	5	(1)	2	30.115 III	3	(1)	2	35.165 I	6	(1)	1
18.16 I	3	(1)	2	26.66 I	3	(1)	2	30.116 IV	2	(1)	4	35.166 II	6	(1)	1
18.17 II	3	(1)	2	26.67 II	3	(1)	2	30.117 IV	3	(2)	4	35.167 III	2	(1)	1
18.18 III	3	(2)	2	26.68 III	1	(1)	2	30.118 IV	3	(2)	4	35.168 III	4	(2)	1
18.19 III	1	(1)	2	26.69 III	3	(1)	2	30.119 IV	3	(1)	2	35.169 IV	6	(1)	2
18.20 IV	2	(1)	4	26.70 III	1	(1)	2	30.120 IV	2	(2)	4	35.170 IV	5	(1)	2
18.21 IV	3	(2)	4	26.71 IV	2	(1)	4	30.121 IV	2	(1)	4	35.171 IV	5	(1)	2
18.22 IV	2	(1)	4	26.72 IV	3	(1)	4	30.122 IV	2	(1)	4	36.172 I	2	(1)	2
18.23 IV	3	(1)	2	26.73 IV	3	(1)	2	31.123 I	2	(1)	2	36.173 II	2	(1)	2
18.24 IV	3	(1)	2	26.74 IV	3	(1)	2	31.124 II	2	(1)	2	36.174 III	1	(1)	2
19.25 I	1	(1)	4	26.75 IV	2	(1)	4	31.125 III	1	(1)	2	36.175 III	2	(1)	2
19.26 II	1	(1)	4	26.76 IV	2	(1)	4	31.126 III	2	(1)	2	36.176 III	1	(1)	2
19.27 III	1	(1)	4	26.77 IV	2	(1)	4	31.127 III	1	(1)	2	36.177 IV	2	(1)	2
19.28 IV	1	(1)	4	27.78 I	5	(2)	2	31.128 IV	2	(1)	4	36.178 IV	2	(1)	4
19.29 IV	1	(1)	4	27.79 II	5	(2)	2	31.129 IV	2	(1)	4	36.179 IV	2	(1)	2
19.30 IV	1	(1)	4	27.80 III	1	(1)	2	31.130 IV	2	(1)	4	37.180 I	4	(2)	2
20.31 I	3	(1)	2	27.81 III	5	(2)	2	31.131 IV	2	(1)	4	37.181 II	4	(2)	2
20.32 II	3	(1)	2	27.82 IV	5	(2)	2	31.132 IV	2	(1)	2	37.182 III	1	(1)	2
20.33 III	1	(1)	2	27.83 IV	3	(2)	4	31.133 IV	2	(1)	4	37.183 III	4	(2)	2
20.34 III	2	(1)	2	27.84 IV	3	(2)	4	31.134 IV	2	(1)	2	37.184 IV	4	(2)	2
20.35 IV	3	(1)	2	27.85 IV	3	(2)	4	32.135 I	3	(1)	2	37.185 IV	3	(2, 2)	2^*
20.36 IV	3	(2)	4	27.86 IV	3	(2)	4	32.136 II	3	(1)	2	37.186 IV	3	(2)	2
20.37 IV	3	(1)	2	28.87 I	4	(1)	2	32.137 III	1	(1)	2	38.187 I	6	(1)	1
21.38 I	8	(1)	1	28.88 II	4	(1)	2	32.138 III	3	(2)	2	38.188 II	6	(1)	1
21.39 II	8	(1)	1	28.89 III	1	(1)	2	32.139 IV	3	(2)	4	38.189 III	2	(1)	1
21.40 III	4	(2)	1	28.90 III	2	(1)	2	32.140 IV	2	(1)	4	38.190 III	3	(1)	1
21.41 III	3	(1)	1	28.91 III	3	(2)	2	32.141 IV	3	(1)	2	38.191 III	3	(1)	1
21.42 IV	6	(2)	2	28.92 IV	4	(1)	2	32.142 IV	2	(1)	4	38.192 IV	4	(1)	2
21.43 IV	7	(2)	2	28.93 IV	3	(1)	4	32.143 IV	3	(1)	4	38.193 IV	6	(1)	2
21.44 IV	5	(2)	2	28.94 IV	4	(1)	4	33.144 I	1	(1)	4	38.194 IV	4	(1)	2
22.45 I	7	(1)	1	28.95 IV	3	(1)	4	33.145 II	1	(1)	4	39.195 I	4	(1)	2
22.46 II	7	(1)	1	28.96 IV	4	(1)	4	33.146 III	1	(1)	4	39.196 II	4	(1)	2
22.47 III	3	(1)	1	28.97 IV	3	(1)	2	33.147 III	1	(1)	4	39.197 III	2	(1)	2
22.48 IV	7	(2)	2	28.98 IV	3	(1)	4	33.148 III	1	(1)	4	39.198 III	1	(1)	2
23.49 I	7	(1)	1	29.99 I	1	(1)	4	33.149 IV	1	(1)	4	39.199 III	3	(2)	2
23.50 II	7	(1)	1	29.100 II	1	(1)	4	33.150 IV	1	(1)	8	39.200 IV	3	(1)	4

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39.201 IV	4	(1)	2	47.251 III	9	(1)	1	51.301 IV	7	(1)	4	54.351 IV	4	(2)	4
39.202 IV	3	(1)	2	47.252 III	15	(2, 2, 2)	1	51.302 IV	9	(1)	2	54.352 IV	4	(2)	8
40.203 I	3	(1)	2	47.253 III	13	(1)	1	51.303 IV	8	(1)	4	55.353 I	9	(1)	2
40.204 II	3	(1)	2	47.254 IV	18	(1)	2	51.304 IV	7	(1)	4	55.354 II	9	(1)	2
40.205 III	1	(1)	2	47.255 IV	15	(1)	2	52.305 I	5	(2)	4	55.355 III	3	(1)	2
40.206 III	2	(1)	2	47.256 IV	14	(1)	2	52.306 II	5	(2)	4	55.356 III	3	(1)	2
40.207 III	2	(1)	2	48.257 I	9	(2)	2	52.307 III	2	(1)	4	55.357 III	9	(2, 2, 2)	2
40.208 IV	3	(1)	2	48.258 II	9	(2)	2	52.308 III	1	(1)	4	55.358 III	5	(2)	2
40.209 IV	3	(1)	4	48.259 III	3	(1)	2	52.309 III	2	(1)	4	55.359 III	3	(1)	2
40.210 IV	3	(1)	2	48.260 III	5	(2)	2	52.310 III	4	(2)	4	55.360 IV	6	(1)	4
41.211 I	2	(1)	2	48.261 III	7	(2)	2	52.311 III	4	(2)	4	55.361 IV	6	(1)	4
41.212 II	2	(1)	2	48.262 IV	6	(2, 2)	4	52.312 III	3	(2)	4	55.362 IV	5	(1)	4
41.213 III	1	(1)	2	48.263 IV	7	(2, 2)	4	52.313 III	3	(1)	4	55.363 IV	9	(1)	2
41.214 III	1	(1)	2	48.264 IV	8	(2)	2	52.314 IV	5	(2, 2)	4	55.364 IV	6	(1)	4
41.215 III	2	(2)	2	49.265 I	14	(2)	2	52.315 IV	4	(2)	4	56.365 I	5	(2)	4
41.216 IV	2	(1)	4	49.266 II	14	(2)	2	52.316 IV	4	(2, 2)	8	56.366 II	5	(2)	4
41.217 IV	2	(2)	2^*	49.267 III	4	(1)	2	52.317 IV	5	(2)	4	56.367 III	1	(1)	4
41.218 IV	2	(1)	2	49.268 III	5	(2)	2	52.318 IV	4	(2)	4	56.368 III	3	(2)	4
42.219 I	5	(1)	1	49.269 III	10	(2, 2)	2	52.319 IV	4	(2, 2)	4	56.369 III	5	(2, 2)	4
42.220 II	5	(1)	1	49.270 III	7	(2, 2)	2	52.320 IV	5	(2)	4	56.370 III	3	(2)	4
42.221 III	2	(1)	1	49.271 III	9	(2)	2	53.321 I	9	(2)	2	56.371 III	3	(2)	4
42.222 III	3	(2)	1	49.272 IV	9	(2)	4	53.322 II	9	(2)	2	56.372 IV	3	(2)	8
42.223 IV	5	(1)	2	49.273 IV	14	(2)	2	53.323 III	3	(1)	2	56.373 IV	4	(2)	4
43.224 I	2	(1)	2	49.274 IV	9	(2)	4	53.324 III	3	(1)	2	56.374 IV	4	(2)	4
43.225 II	2	(1)	2	49.275 IV	8	(2)	4	53.325 III	2	(1)	2	56.375 IV	5	(2)	4
43.226 III	1	(1)	2	49.276 IV	8	(2)	4	53.326 III	5	(2)	2	56.376 IV	4	(2)	4
43.227 III	2	(1)	2	50.277 I	9	(2)	2	53.327 III	8	(2, 2)	2	57.377 I	5	(1)	4
43.228 IV	2	(2)	2^*	50.278 II	9	(2)	2	53.328 III	6	(2, 2)	2	57.378 II	5	(1)	4
44.229 I	5	(1)	1	50.279 III	3	(1)	2	53.329 III	4	(1)	2	57.379 III	3	(1)	4
44.230 II	5	(1)	1	50.280 III	3	(1)	2	53.330 IV	7	(2)	4	57.380 III	2	(1)	4
44.231 III	2	(1)	1	50.281 III	5	(2, 2)	2	53.331 IV	6	(2)	4	57.381 III	1	(1)	4
44.232 III	3	(1)	1	50.282 III	5	(2)	2	53.332 IV	7	(2)	4	57.382 III	4	(2)	4
44.233 IV	5	(1)	2	50.283 III	7	(2)	2	53.333 IV	6	(2)	4	57.383 III	4	(2, 2)	4
44.234 IV	4	(1)	2	50.284 IV	8	(2, 2)	4	53.334 IV	9	(2)	2	57.384 III	3	(2)	4
45.235 I	3	(2)	2	50.285 IV	6	(2, 2)	4	53.335 IV	6	(2)	4	57.385 III	2	(1)	4
45.236 II	3	(1)	2	50.286 IV	5	(2, 2)	4	53.336 IV	6	(2)	4	57.386 IV	4	(1)	8
45.237 III	1	(1)	2	50.287 IV	9	(2)	2	54.337 I	6	(2)	4	57.387 IV	5	(1)	4
45.238 III	3	(2)	2	50.288 IV	6	(2)	4	54.338 II	6	(2)	4	57.388 IV	5	(1)	4
45.239 IV	3	(2)	2	51.289 I	12	(1)	2	54.339 III	1	(1)	4	57.389 IV	5	(1)	4
45.240 IV	2	(2)	2^*	51.290 II	12	(1)	2	54.340 III	2	(1)	4	57.390 IV	4	(1)	4
46.241 I	3	(1)	2	51.291 III	3	(1)	2	54.341 III	3	(2)	4	57.391 IV	4	(1)	4
46.242 II	3	(1)	2	51.292 III	4	(1)	2	54.342 III	5	(2, 2)	4	57.392 IV	4	(1)	4
46.243 III	1	(1)	2	51.293 III	6	(1)	2	54.343 III	3	(2)	4	58.393 I	8	(2)	2
46.244 III	2	(1)	2	51.294 III	7	(2, 2)	2	54.344 III	4	(2, 2)	4	58.394 II	8	(2)	2
46.245 III	2	(1)	2	51.295 III	6	(2)	2	54.345 III	4	(2)	4	58.395 III	2	(1)	2
46.246 IV	3	(1)	2	51.296 III	9	(2, 2, 2)	2	54.346 IV	6	(2)	4	58.396 III	3	(1)	2
46.247 IV	3	(1)	2	51.297 III	5	(1)	2	54.347 IV	4	(2, 2)	4^*	58.397 III	8	(2, 2)	2
46.248 IV	3	(1)	4	51.298 IV	12	(1)	2	54.348 IV	6	(2)	4	58.398 III	5	(2)	2
47.249 I	27	(1)	1	51.299 IV	8	(1)	4	54.349 IV	4	(2)	4	58.399 III	3	(1)	2
47.250 II	27	(1)	1	51.300 IV	9	(1)	4	54.350 IV	6	(2)	4	58.400 IV	5	(2)	4

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58.401	IV	6	(2)	4	62.442	II	4	(1)	4	65.483	III	6	(1)	1	69.524	III	9	(2, 2)	1
58.402	IV	5	(2)	4	62.443	III	2	(1)	4	65.484	III	6	(1)	1	69.525	III	7	(2)	1
58.403	IV	6	(2)	4	62.444	III	1	(1)	4	65.485	III	12	(2, 2, 2)	1	69.526	IV	14	(1)	2
58.404	IV	8	(2)	2	62.445	III	2	(1)	4	65.486	III	10	(2, 2)	1	70.527	I	6	(2)	2
59.405	I	7	(1)	2	62.446	III	3	(2)	4	65.487	III	8	(1)	1	70.528	II	6	(2)	2
59.406	II	7	(1)	2	62.447	III	3	(2)	4	65.488	IV	12	(1)	2	70.529	III	2	(1)	2
59.407	III	2	(1)	2	62.448	III	4	(2)	4	65.489	IV	15	(1)	2	70.530	III	4	(2)	2
59.408	III	5	(1)	2	62.449	III	1	(1)	4	65.490	IV	11	(1)	2	70.531	III	4	(1)	2
59.409	III	5	(2, 2)	2	62.450	IV	3	(1)	4	66.491	I	11	(2)	2	70.532	IV	5	(2, 2)	2*
59.410	III	4	(2)	2	62.451	IV	4	(1)	4	66.492	II	11	(2)	2	71.533	I	15	(1)	1
59.411	III	3	(1)	2	62.452	IV	3	(1)	8	66.493	III	3	(1)	2	71.534	II	15	(1)	1
59.412	IV	6	(1)	4	62.453	IV	3	(1)	4	66.494	III	4	(2)	2	71.535	III	5	(1)	1
59.413	IV	6	(1)	4	62.454	IV	4	(1)	4	66.495	III	9	(2, 2)	2	71.536	III	9	(2, 2)	1
59.414	IV	5	(1)	4	62.455	IV	3	(1)	4	66.496	III	6	(2, 2)	2	71.537	III	7	(1)	1
59.415	IV	7	(1)	2	62.456	IV	4	(1)	4	66.497	III	6	(2)	2	71.538	IV	11	(1)	2
59.416	IV	6	(1)	2	63.457	I	8	(1)	2	66.498	IV	10	(2)	2	72.539	I	9	(1)	2
60.417	I	4	(2)	4	63.458	II	8	(1)	2	66.499	IV	8	(2)	4	72.540	II	9	(1)	2
60.418	II	4	(2)	4	63.459	III	3	(1)	2	66.500	IV	9	(2)	2	72.541	III	3	(1)	2
60.419	III	1	(1)	4	63.460	III	4	(1)	2	67.501	I	13	(1)	2	72.542	III	3	(2)	2
60.420	III	2	(1)	4	63.461	III	2	(1)	2	67.502	II	13	(1)	2	72.543	III	7	(2, 2)	2
60.421	III	1	(1)	4	63.462	III	5	(2)	2	67.503	III	4	(1)	2	72.544	III	5	(2, 2)	2
60.422	III	3	(2)	4	63.463	III	6	(2, 2)	2	67.504	III	5	(1)	2	72.545	III	5	(2)	2
60.423	III	3	(2)	4	63.464	III	5	(2, 2)	2	67.505	III	7	(2, 2)	2	72.546	IV	9	(1)	2
60.424	III	4	(2, 2)	4	63.465	III	3	(1)	2	67.506	III	8	(2, 2)	2	72.547	IV	7	(1)	4
60.425	III	2	(1)	4	63.466	IV	8	(1)	2	67.507	III	7	(2)	2	73.548	I	6	(2)	2*
60.426	IV	3	(2, 2)	4*	63.467	IV	7	(1)	4	67.508	IV	9	(1)	4	73.549	II	6	(1)	4
60.427	IV	3	(2)	8	63.468	IV	7	(1)	2	67.509	IV	13	(1)	2	73.550	III	2	(2)	2*
60.428	IV	4	(2)	4	64.469	I	7	(1)	2	67.510	IV	9	(1)	2	73.551	III	4	(2, 2)	4
60.429	IV	3	(2)	4	64.470	II	7	(1)	2	68.511	I	7	(2)	2	73.552	III	4	(2, 2)	4
60.430	IV	3	(2)	4	64.471	III	2	(1)	2	68.512	II	7	(2)	2	73.553	IV	6	(2)	2*
60.431	IV	4	(2)	4	64.472	III	3	(1)	2	68.513	III	2	(1)	2	74.554	I	10	(1)	2
60.432	IV	4	(2)	4	64.473	III	2	(1)	2	68.514	III	3	(2)	2	74.555	II	10	(1)	2
61.433	I	3	(2)	4	64.474	III	4	(2)	2	68.515	III	5	(2, 2)	2	74.556	III	3	(1)	2
61.434	II	3	(1)	4	64.475	III	6	(2, 2)	2	68.516	III	4	(2, 2)	2	74.557	III	4	(1)	2
61.435	III	1	(1)	4	64.476	III	5	(2, 2)	2	68.517	III	5	(2, 2)	2	74.558	III	6	(2, 2)	2
61.436	III	3	(2)	4	64.477	III	3	(2)	2	68.518	IV	7	(2)	4	74.559	III	7	(2, 2)	2
61.437	III	1	(1)	4	64.478	IV	6	(1)	4	68.519	IV	7	(2, 2)	2*	74.560	III	4	(1)	2
61.438	IV	2	(2)	4*	64.479	IV	6	(1)	4	68.520	IV	7	(2)	2	74.561	IV	8	(1)	4
61.439	IV	3	(2)	4	64.480	IV	7	(1)	2	69.521	I	15	(1)	1	74.562	IV	9	(1)	2
61.440	IV	2	(2)	4*	65.481	I	18	(1)	1	69.522	II	15	(1)	1					
62.441	I	4	(1)	4	65.482	II	18	(1)	1	69.523	III	5	(1)	1					

d: Rank of the band structure group $\{\mathrm{BS}\}$
X_{BS}: Symmetry-based indicators of band topology
ν_{BS}: Set of ν bands are symmetry-forbidden from being isolated by band gaps if $\nu \notin \nu_{\mathrm{BS}}\,\mathbb{Z}$
$*$: Exhibiting exceptional filling pattern; see table S19

Table S16. Characterization of MSGs in the tetragonal family for spinless electrons.

MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$
75.1 I	8	(4)	1	84.51 I	13	(2, 4)	2	90.101 IV	7	(1)	2	97.151 I	8	(1)	1
75.2 II	6	(2)	1	84.52 II	11	(2, 2)	2	90.102 IV	6	(2)	2	97.152 II	8	(1)	1
75.3 III	4	(1)	1	84.53 III	9	(2)	2	91.103 I	4	(1)	4	97.153 III	5	(1)	1
75.4 IV	6	(2)	2	84.54 III	4	(2)	2	91.104 II	4	(1)	4	97.154 III	5	(2)	1
75.5 IV	5	(2)	2	84.55 III	8	(2, 2, 2)	2	91.105 III	3	(1)	4	97.155 III	5	(1)	1
75.6 IV	5	(2)	2	84.56 IV	10	(2)	2	91.106 III	1	(1)	4	97.156 IV	7	(1)	2
76.7 I	1	(1)	4	84.57 IV	8	(2, 2)	4	91.107 III	2	(1)	4	98.157 I	5	(1)	2
76.8 II	1	(1)	4	84.58 IV	9	(2)	2	91.108 IV	4	(1)	4	98.158 II	5	(1)	2
76.9 III	1	(1)	4	85.59 I	11	(2, 4)	2	91.109 IV	3	(2)	4^*	98.159 III	3	(1)	2
76.10 IV	1	(1)	4	85.60 II	8	(2)	2	91.110 IV	3	(1)	4	98.160 III	2	(1)	2
76.11 IV	1	(2)	4^*	85.61 III	5	(1)	2	92.111 I	2	(1)	4	98.161 III	4	(1)	2
76.12 IV	1	(1)	4	85.62 III	5	(2)	2	92.112 II	2	(1)	4	98.162 IV	5	(2)	2^*
77.13 I	4	(2)	2	85.63 III	7	(2, 2, 2)	2	92.113 III	1	(1)	4	99.163 I	9	(1)	1
77.14 II	4	(2)	2	85.64 IV	7	(2)	4	92.114 III	1	(1)	4	99.164 II	9	(1)	1
77.15 III	4	(2)	2	85.65 IV	8	(2)	2	92.115 III	2	(1)	4	99.165 III	5	(1)	1
77.16 IV	4	(1)	2	85.66 IV	7	(2)	2	92.116 IV	2	(1)	4	99.166 III	6	(1)	1
77.17 IV	3	(2, 2)	2^*	86.67 I	9	(2, 2)	2	92.117 IV	2	(2)	4^*	99.167 III	8	(4)	1
77.18 IV	3	(1)	2	86.68 II	7	(2)	2	92.118 IV	2	(1)	4	99.168 IV	9	(1)	2
78.19 I	1	(1)	4	86.69 III	5	(2)	2	93.119 I	10	(1)	2	99.169 IV	6	(1)	2
78.20 II	1	(1)	4	86.70 III	3	(2)	2	93.120 II	10	(1)	2	99.170 IV	6	(1)	2
78.21 III	1	(1)	4	86.71 III	7	(2, 2, 2, 2)	2	93.121 III	8	(2)	2	100.171 I	6	(1)	2
78.22 IV	1	(1)	4	86.72 IV	6	(1)	4	93.122 III	4	(2)	2	100.172 II	5	(1)	2
78.23 IV	1	(2)	4^*	86.73 IV	7	(2, 2)	2^*	93.123 III	6	(2)	2	100.173 III	4	(1)	2
78.24 IV	1	(1)	4	86.74 IV	6	(1)	2	93.124 IV	7	(1)	2	100.174 III	3	(1)	2
79.25 I	5	(2)	1	87.75 I	16	(4, 4)	1	93.125 IV	7	(2)	2^*	100.175 III	5	(4)	2
79.26 II	4	(1)	1	87.76 II	12	(2, 2)	1	93.126 IV	6	(1)	2	100.176 IV	5	(1)	4
79.27 III	3	(1)	1	87.77 III	8	(1)	1	94.127 I	5	(1)	2	100.177 IV	6	(1)	2
79.28 IV	4	(1)	2	87.78 III	4	(2)	1	94.128 II	5	(1)	2	100.178 IV	6	(1)	4
80.29 I	2	(1)	2	87.79 III	7	(2, 2)	1	94.129 III	3	(2)	2	101.179 I	5	(1)	2
80.30 II	2	(1)	2	87.80 IV	11	(2, 2)	2	94.130 III	3	(2)	2	101.180 II	5	(1)	2
80.31 III	2	(1)	2	88.81 I	8	(2, 2)	2	94.131 III	5	(2)	2	101.181 III	5	(1)	2
80.32 IV	2	(2)	2^*	88.82 II	6	(2)	2	94.132 IV	4	(1)	4	101.182 III	4	(2)	2
81.33 I	12	(2, 2, 4)	1	88.83 III	4	(2)	2	94.133 IV	5	(2)	2^*	101.183 III	4	(2)	2
81.34 II	8	(2, 2)	1	88.84 III	2	(1)	2	94.134 IV	4	(1)	2	101.184 IV	5	(1)	2
81.35 III	4	(1)	1	88.85 III	6	(2, 2)	2	95.135 I	4	(1)	4	101.185 IV	4	(1)	4
81.36 IV	8	(2, 2)	2	88.86 IV	5	(2, 2)	2^*	95.136 II	4	(1)	4	101.186 IV	4	(1)	4
81.37 IV	7	(2, 2, 2)	2	89.87 I	12	(1)	1	95.137 III	3	(1)	4	102.187 I	4	(1)	2
81.38 IV	7	(2, 2)	2	89.88 II	12	(1)	1	95.138 III	1	(1)	4	102.188 II	4	(1)	2
82.39 I	11	(2, 2, 2)	1	89.89 III	8	(1)	1	95.139 III	2	(1)	4	102.189 III	4	(1)	2
82.40 II	7	(2)	1	89.90 III	8	(4)	1	95.140 IV	4	(1)	4	102.190 III	3	(2)	2
82.41 III	3	(1)	1	89.91 III	6	(1)	1	95.141 IV	3	(2)	4^*	102.191 III	3	(2)	2
82.42 IV	7	(2, 2)	2	89.92 IV	9	(2)	2	95.142 IV	3	(1)	4	102.192 IV	4	(1)	4
83.43 I	24	(4, 4, 4)	1	89.93 IV	8	(1)	2	96.143 I	2	(1)	4	102.193 IV	4	(1)	4
83.44 II	18	(2, 2, 2)	1	89.94 IV	7	(2)	2	96.144 II	2	(1)	4	102.194 IV	4	(1)	2
83.45 III	12	(1)	1	90.95 I	7	(1)	2	96.145 III	1	(1)	4	103.195 I	6	(2)	2
83.46 III	6	(2)	1	90.96 II	6	(1)	2	96.146 III	1	(1)	4	103.196 II	6	(2)	2
83.47 III	8	(2, 2)	1	90.97 III	3	(1)	2	96.147 III	2	(1)	4	103.197 III	4	(1)	2
83.48 IV	14	(2, 4)	2	90.98 III	5	(4)	2	96.148 IV	2	(1)	4	103.198 III	4	(1)	2
83.49 IV	15	(2, 2, 2)	2	90.99 III	5	(1)	2	96.149 IV	2	(2)	4^*	103.199 III	8	(4)	2
83.50 IV	13	(2, 4)	2	90.100 IV	5	(2)	4	96.150 IV	2	(1)	4	103.200 IV	6	(2)	2

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103.201	IV	4	(1)	4	111.251	I	13	(1)	1	117.301	III	5	(1)	2	124.351	I	17	(4)	2
103.202	IV	4	(2)	4	111.252	II	13	(1)	1	117.302	III	3	(1)	2	124.352	II	15	(2)	2
104.203	I	5	(2)	2	111.253	III	5	(1)	1	117.303	III	7	(2, 2, 4)	2	124.353	III	6	(2)	2
104.204	II	4	(1)	2	111.254	III	8	(1)	1	117.304	IV	6	(2)	4	124.354	III	9	(1)	2
104.205	III	3	(1)	2	111.255	III	12	(2, 2, 4)	1	117.305	IV	9	(2)	2	124.355	III	10	(1)	2
104.206	III	3	(1)	2	111.256	IV	9	(1)	2	117.306	IV	6	(2)	4	124.356	III	8	(1)	2
104.207	III	5	(2)	2	111.257	IV	8	(1)	2	118.307	I	9	(2)	2	124.357	III	16	(4, 4)	2
104.208	IV	4	(2)	4	111.258	IV	8	(1)	2	118.308	II	7	(1)	2	124.358	III	7	(1)	2
104.209	IV	5	(2)	4	112.259	I	12	(2)	2	118.309	III	5	(1)	2	124.359	III	9	(2)	2
104.210	IV	5	(2)	2	112.260	II	10	(1)	2	118.310	III	3	(1)	2	124.360	IV	17	(4)	2
105.211	I	6	(1)	2	112.261	III	4	(2)	2	118.311	III	7	(2, 2, 2)	2	124.361	IV	11	(2)	4
105.212	II	6	(1)	2	112.262	III	8	(1)	2	118.312	IV	6	(2)	4	124.362	IV	11	(4)	4
105.213	III	4	(2)	2	112.263	III	8	(2, 2, 2)	2	118.313	IV	7	(2)	4	125.363	I	13	(1)	2
105.214	III	6	(1)	2	112.264	IV	10	(2)	2	118.314	IV	8	(2)	2	125.364	II	13	(1)	2
105.215	III	4	(2)	2	112.265	IV	7	(2)	2^*	119.315	I	10	(1)	1	125.365	III	6	(1)	2
105.216	IV	6	(1)	2	112.266	IV	7	(2)	2	119.316	II	10	(1)	1	125.366	III	8	(1)	2
105.217	IV	4	(1)	4	113.267	I	8	(1)	2	119.317	III	5	(1)	1	125.367	III	7	(1)	2
105.218	IV	4	(1)	2	113.268	II	6	(1)	2	119.318	III	4	(1)	1	125.368	III	8	(1)	2
106.219	I	3	(2)	2^*	113.269	III	4	(1)	2	119.319	III	11	(2, 2, 2)	1	125.369	III	11	(2, 4)	2
106.220	II	3	(1)	4	113.270	III	3	(1)	2	119.320	IV	8	(1)	2	125.370	III	9	(2)	2
106.221	III	3	(2)	4	113.271	III	7	(2, 2, 4)	2	120.321	I	9	(2)	2	125.371	III	8	(1)	2
106.222	III	3	(2)	2^*	113.272	IV	6	(1)	4	120.322	II	7	(1)	2	125.372	IV	9	(1)	4
106.223	III	3	(2)	4	113.273	IV	8	(1)	2	120.323	III	5	(1)	2	125.373	IV	13	(1)	2
106.224	IV	3	(1)	4	113.274	IV	6	(1)	2	120.324	III	3	(2)	2	125.374	IV	9	(1)	4
106.225	IV	3	(2)	2^*	114.275	I	7	(2)	2	120.325	III	7	(2, 2, 2)	2	126.375	I	10	(2)	2
106.226	IV	3	(1)	4	114.276	II	5	(1)	2	120.326	IV	9	(2)	2	126.376	II	9	(1)	2
107.227	I	6	(1)	1	114.277	III	3	(2)	2	121.327	I	10	(1)	1	126.377	III	5	(2)	2
107.228	II	6	(1)	1	114.278	III	3	(1)	2	121.328	II	9	(1)	1	126.378	III	5	(1)	2
107.229	III	4	(1)	1	114.279	III	7	(2, 2, 2)	2	121.329	III	4	(1)	1	126.379	III	6	(1)	2
107.230	III	4	(1)	1	114.280	IV	5	(2)	4	121.330	III	5	(1)	1	126.380	III	7	(1)	2
107.231	III	5	(2)	1	114.281	IV	5	(2)	2^*	121.331	III	9	(2, 2, 2)	1	126.381	III	8	(2, 2)	2
107.232	IV	6	(1)	2	114.282	IV	7	(2)	2	121.332	IV	8	(1)	2	126.382	III	6	(2)	2
108.233	I	5	(1)	2	115.283	I	12	(1)	1	122.333	I	7	(2)	2	126.383	III	7	(1)	2
108.234	II	5	(1)	2	115.284	II	12	(1)	1	122.334	II	5	(1)	2	126.384	IV	9	(2)	4
108.235	III	4	(1)	2	115.285	III	6	(1)	1	122.335	III	2	(1)	2	126.385	IV	8	(2)	4
108.236	III	3	(1)	2	115.286	III	6	(1)	1	122.336	III	3	(1)	2	126.386	IV	10	(2)	2
108.237	III	5	(4)	2	115.287	III	12	(2, 2, 4)	1	122.337	III	6	(2, 2)	2	127.387	I	18	(1)	2
108.238	IV	5	(1)	2	115.288	IV	9	(1)	2	122.338	IV	5	(2)	2^*	127.388	II	15	(1)	2
109.239	I	3	(1)	2	115.289	IV	8	(1)	2	123.339	I	27	(1)	1	127.389	III	5	(1)	2
109.240	II	3	(1)	2	115.290	IV	7	(1)	2	123.340	II	27	(1)	1	127.390	III	12	(1)	2
109.241	III	2	(1)	2	116.291	I	10	(2)	2	123.341	III	9	(1)	1	127.391	III	9	(1)	2
109.242	III	3	(1)	2	116.292	II	8	(1)	2	123.342	III	15	(1)	1	127.392	III	6	(1)	2
109.243	III	2	(1)	2	116.293	III	6	(1)	2	123.343	III	18	(1)	1	127.393	III	15	(4, 4, 4)	2
109.244	IV	3	(1)	4	116.294	III	4	(2)	2	123.344	III	13	(1)	1	127.394	III	7	(1)	2
110.245	I	2	(2)	2^*	116.295	III	8	(2, 2, 2)	2	123.345	III	24	(4, 4, 4)	1	127.395	III	6	(1)	2
110.246	II	2	(1)	4	116.296	IV	9	(2)	2	123.346	III	12	(1)	1	127.396	IV	11	(1)	4
110.247	III	2	(2)	4	116.297	IV	7	(2)	4	123.347	III	12	(1)	1	127.397	IV	18	(1)	2
110.248	III	2	(2)	2^*	116.298	IV	6	(2)	4	123.348	IV	18	(1)	2	127.398	IV	12	(1)	4
110.249	III	2	(2)	4	117.299	I	9	(2)	2	123.349	IV	18	(1)	2	128.399	I	14	(4)	2
110.250	IV	2	(2)	2^*	117.300	II	7	(1)	2	123.350	IV	15	(1)	2	128.400	II	11	(2)	2

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128.401	III	4	(2)	2	131.444	IV	15	(1)	2	135.487	III	6	(1)	4	138.530	IV	7	(1)	4
128.402	III	8	(1)	2	131.445	IV	11	(1)	4	135.488	III	5	(2)	4	139.531	I	18	(1)	1
128.403	III	7	(1)	2	131.446	IV	12	(1)	2	135.489	III	8	(2, 4)	4	139.532	II	18	(1)	1
128.404	III	5	(1)	2	132.447	I	15	(1)	2	135.490	III	6	(2)	2^*	139.533	III	6	(1)	1
128.405	III	13	(4, 4)	2	132.448	II	15	(1)	2	135.491	III	4	(1)	4	139.534	III	11	(1)	1
128.406	III	6	(1)	2	132.449	III	5	(1)	2	135.492	IV	8	(1)	4	139.535	III	11	(1)	1
128.407	III	5	(1)	2	132.450	III	11	(1)	2	135.493	IV	9	(1)	4	139.536	III	10	(1)	1
128.408	IV	10	(4)	4	132.451	III	10	(2)	2	135.494	IV	9	(1)	4	139.537	III	16	(4, 4)	1
128.409	IV	11	(2)	4	132.452	III	9	(1)	2	136.495	I	12	(1)	2	139.538	III	9	(1)	1
128.410	IV	14	(4)	2	132.453	III	12	(2, 4)	2	136.496	II	11	(1)	2	139.539	III	8	(1)	1
129.411	I	12	(1)	2	132.454	III	10	(2)	2	136.497	III	4	(1)	2	139.540	IV	15	(1)	2
129.412	II	12	(1)	2	132.455	III	8	(1)	2	136.498	III	10	(1)	2	140.541	I	15	(1)	2
129.413	III	6	(1)	2	132.456	IV	14	(1)	2	136.499	III	7	(2)	2	140.542	II	14	(1)	2
129.414	III	8	(1)	2	132.457	IV	11	(1)	4	136.500	III	6	(1)	2	140.543	III	5	(1)	2
129.415	III	6	(1)	2	132.458	IV	10	(1)	4	136.501	III	9	(2, 4)	2	140.544	III	10	(1)	2
129.416	III	8	(1)	2	133.459	I	9	(2)	2^*	136.502	III	7	(2)	2	140.545	III	8	(1)	2
129.417	III	11	(2, 4)	2	133.460	II	8	(1)	4	136.503	III	5	(1)	2	140.546	III	7	(1)	2
129.418	III	8	(1)	2	133.461	III	3	(2)	2^*	136.504	IV	9	(1)	4	140.547	III	13	(4, 4)	2
129.419	III	7	(1)	2	133.462	III	5	(2)	4	136.505	IV	11	(1)	4	140.548	III	8	(1)	2
129.420	IV	9	(1)	4	133.463	III	6	(2)	2^*	136.506	IV	11	(1)	2	140.549	III	7	(2)	2
129.421	IV	12	(1)	2	133.464	III	7	(2)	4	137.507	I	8	(1)	2	140.550	IV	15	(1)	2
129.422	IV	9	(1)	2	133.465	III	6	(2, 2)	4	137.508	II	8	(1)	2	141.551	I	9	(1)	2
130.423	I	8	(2)	4	133.466	III	6	(2, 2)	2^*	137.509	III	4	(1)	2	141.552	II	9	(1)	2
130.424	II	7	(2)	4	133.467	III	6	(1)	4	137.510	III	5	(2)	2	141.553	III	3	(1)	2
130.425	III	4	(2)	4	133.468	IV	8	(1)	4	137.511	III	5	(1)	2	141.554	III	5	(2)	2
130.426	III	5	(1)	4	133.469	IV	9	(2)	2^*	137.512	III	7	(2)	2	141.555	III	6	(1)	2
130.427	III	4	(1)	4	133.470	IV	7	(1)	4	137.513	III	8	(2, 2)	2	141.556	III	7	(2)	2
130.428	III	5	(1)	4	134.471	I	12	(1)	2	137.514	III	7	(1)	2	141.557	III	8	(2, 2)	2
130.429	III	8	(2, 4)	4	134.472	II	12	(1)	2	137.515	III	4	(1)	2	141.558	III	6	(1)	2
130.430	III	6	(1)	4	134.473	III	4	(1)	2	137.516	IV	7	(1)	4	141.559	III	4	(1)	2
130.431	III	6	(2)	4	134.474	III	8	(1)	2	137.517	IV	7	(1)	4	141.560	IV	7	(1)	4
130.432	IV	8	(2)	4	134.475	III	7	(2)	2	137.518	IV	8	(1)	2	142.561	I	7	(2)	2^*
130.433	IV	7	(1)	4	134.476	III	8	(1)	2	138.519	I	10	(1)	4	142.562	II	6	(1)	4
130.434	IV	8	(2)	4	134.477	III	9	(2, 2)	2	138.520	II	9	(1)	4	142.563	III	2	(2)	2^*
131.435	I	18	(1)	2	134.478	III	9	(2, 2)	2	138.521	III	4	(1)	4	142.564	III	4	(2)	4
131.436	II	18	(1)	2	134.479	III	7	(1)	2	138.522	III	8	(1)	4	142.565	III	4	(2)	2^*
131.437	III	6	(1)	2	134.480	IV	8	(1)	4	138.523	III	5	(2)	4	142.566	III	5	(2, 2)	4
131.438	III	10	(2)	2	134.481	IV	10	(1)	4	138.524	III	6	(1)	4	142.567	III	5	(2, 2)	4
131.439	III	15	(1)	2	134.482	IV	10	(1)	2	138.525	III	7	(2, 2)	4	142.568	III	5	(2)	2^*
131.440	III	12	(2)	2	135.483	I	9	(1)	4	138.526	III	7	(2)	4	142.569	III	4	(2)	4
131.441	III	13	(2, 4)	2	135.484	II	8	(1)	4	138.527	III	5	(1)	4	142.570	IV	7	(2)	2^*
131.442	III	9	(1)	2	135.485	III	3	(2)	2^*	138.528	IV	8	(1)	4					
131.443	III	9	(1)	2	135.486	III	7	(2)	4	138.529	IV	10	(1)	4					

d: Rank of the band structure group $\{\mathrm{BS}\}$

X_{BS}: Symmetry-based indicators of band topology

ν_{BS}: Set of ν bands are symmetry-forbidden from being isolated by band gaps if $\nu \notin \nu_{\mathrm{BS}}\,\mathbb{Z}$

$*$: Exhibiting exceptional filling pattern; see table S19

Table S17. Characterization of MSGs in the hexagonal family for spinless electrons.

MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$
143.1 I	7	(3)	1	156.51 III	7	(3)	1	166.101 III	11	(2, 4)	1	177.151 III	5	(1)	1
143.2 II	4	(1)	1	156.52 IV	5	(1)	2	166.102 IV	7	(2)	2	177.152 III	6	(1)	1
143.3 IV	4	(1)	2	157.53 I	5	(1)	1	167.103 I	7	(2)	2	177.153 III	9	(6)	1
144.4 I	1	(1)	3	157.54 II	4	(1)	1	167.104 II	6	(2)	2	177.154 IV	8	(2)	2
144.5 II	1	(1)	3	157.55 III	5	(3)	1	167.105 III	2	(1)	2	178.155 I	3	(1)	6
144.6 IV	1	(1)	6	157.56 IV	4	(1)	2	167.106 III	3	(1)	2	178.156 II	3	(1)	6
145.7 I	1	(1)	3	158.57 I	4	(1)	2	167.107 III	7	(4)	2	178.157 III	2	(1)	6
145.8 II	1	(1)	3	158.58 II	4	(1)	2	167.108 IV	7	(2)	2	178.158 III	2	(1)	6
145.9 IV	1	(1)	6	158.59 III	7	(3)	2	168.109 I	9	(6)	1	178.159 III	1	(1)	6
146.10 I	3	(1)	1	158.60 IV	4	(1)	2	168.110 II	6	(2)	1	178.160 IV	3	(1)	6
146.11 II	2	(1)	1	159.61 I	4	(1)	2	168.111 III	4	(1)	1	179.161 I	3	(1)	6
146.12 IV	2	(1)	2	159.62 II	3	(1)	2	168.112 IV	6	(2)	2	179.162 II	3	(1)	6
147.13 I	13	(2, 12)	1	159.63 III	5	(3)	2	169.113 I	1	(1)	6	179.163 III	2	(1)	6
147.14 II	9	(2, 4)	1	159.64 IV	3	(1)	2	169.114 II	1	(1)	6	179.164 III	2	(1)	6
147.15 III	4	(1)	1	160.65 I	3	(1)	1	169.115 III	1	(1)	6	179.165 III	1	(1)	6
147.16 IV	7	(4)	2	160.66 II	3	(1)	1	169.116 IV	1	(1)	6	179.166 IV	3	(1)	6
148.17 I	11	(2, 4)	1	160.67 III	3	(1)	1	170.117 I	1	(1)	6	180.167 I	7	(1)	3
148.18 II	8	(2, 4)	1	160.68 IV	3	(1)	2	170.118 II	1	(1)	6	180.168 II	7	(1)	3
148.19 III	2	(1)	1	161.69 I	2	(1)	2	170.119 III	1	(1)	6	180.169 III	3	(1)	3
148.20 IV	6	(4)	2	161.70 II	2	(1)	2	170.120 IV	1	(1)	6	180.170 III	3	(1)	3
149.21 I	6	(1)	1	161.71 III	3	(1)	2	171.121 I	3	(2)	3	180.171 III	3	(2)	3
149.22 II	6	(1)	1	161.72 IV	2	(1)	2	171.122 II	3	(2)	3	180.172 IV	5	(2)	6
149.23 III	7	(3)	1	162.73 I	12	(2)	1	171.123 III	1	(1)	3	181.173 I	7	(1)	3
149.24 IV	5	(1)	2	162.74 II	12	(2)	1	171.124 IV	3	(2)	6	181.174 II	7	(1)	3
150.25 I	6	(1)	1	162.75 III	5	(1)	1	172.125 I	3	(2)	3	181.175 III	3	(1)	3
150.26 II	5	(1)	1	162.76 III	5	(1)	1	172.126 II	3	(2)	3	181.176 III	3	(1)	3
150.27 III	5	(3)	1	162.77 III	13	(2, 12)	1	172.127 III	1	(1)	3	181.177 III	3	(2)	3
150.28 IV	4	(1)	2	162.78 IV	8	(2)	2	172.128 IV	3	(2)	6	181.178 IV	5	(2)	6
151.29 I	3	(1)	3	163.79 I	8	(2)	2	173.129 I	5	(3)	2	182.179 I	5	(1)	2
151.30 II	3	(1)	3	163.80 II	7	(2)	2	173.130 II	3	(1)	2	182.180 II	5	(1)	2
151.31 III	1	(1)	3	163.81 III	4	(1)	2	173.131 III	4	(1)	2	182.181 III	4	(1)	2
151.32 IV	2	(1)	6	163.82 III	4	(1)	2	173.132 IV	3	(1)	2	182.182 III	5	(1)	2
152.33 I	3	(1)	3	163.83 III	9	(12)	2	174.133 I	21	(3, 3, 3)	1	182.183 III	5	(3)	2
152.34 II	3	(1)	3	163.84 IV	8	(2)	2	174.134 II	12	(1)	1	182.184 IV	5	(1)	2
152.35 III	1	(1)	3	164.85 I	12	(2)	1	174.135 III	4	(1)	1	183.185 I	8	(1)	1
152.36 IV	2	(1)	6	164.86 II	12	(2)	1	174.136 IV	11	(3)	2	183.186 II	8	(1)	1
153.37 I	3	(1)	3	164.87 III	4	(1)	1	175.137 I	27	(6, 6, 6)	1	183.187 III	5	(1)	1
153.38 II	3	(1)	3	164.88 III	6	(1)	1	175.138 II	18	(2, 2, 2)	1	183.188 III	4	(1)	1
153.39 III	1	(1)	3	164.89 III	13	(2, 12)	1	175.139 III	12	(1)	1	183.189 III	9	(6)	1
153.40 IV	2	(1)	6	164.90 IV	8	(2)	2	175.140 III	6	(2)	1	183.190 IV	8	(1)	2
154.41 I	3	(1)	3	165.91 I	8	(2)	2	175.141 III	9	(2, 4)	1	184.191 I	6	(2)	2
154.42 II	3	(1)	3	165.92 II	7	(2)	2	175.142 IV	15	(2, 6)	2	184.192 II	6	(2)	2
154.43 III	1	(1)	3	165.93 III	3	(1)	2	176.143 I	16	(3, 6)	2	184.193 III	4	(1)	2
154.44 IV	2	(1)	6	165.94 III	5	(1)	2	176.144 II	10	(2)	2	184.194 III	3	(1)	2
155.45 I	4	(1)	1	165.95 III	9	(12)	2	176.145 III	11	(3)	2	184.195 III	9	(6)	2
155.46 II	4	(1)	1	165.96 IV	8	(2)	2	176.146 III	3	(1)	2	184.196 IV	6	(2)	2
155.47 III	3	(1)	1	166.97 I	11	(2)	1	176.147 III	7	(4)	2	185.197 I	4	(1)	2
155.48 IV	3	(1)	2	166.98 II	11	(2)	1	176.148 IV	10	(2)	2	185.198 II	4	(1)	2
156.49 I	5	(1)	1	166.99 III	3	(1)	1	177.149 I	10	(1)	1	185.199 III	5	(1)	2
156.50 II	5	(1)	1	166.100 III	4	(1)	1	177.150 II	10	(1)	1	185.200 III	3	(1)	2

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185.201	III	5	(3)	188.219	III	14	(3, 3)	191.237	III	12	(1)	193.255	III	4	(1)
185.202	IV	4	(1)	188.220	IV	12	(3)	191.238	III	12	(2)	193.256	III	10	(1)
186.203	I	4	(1)	189.221	I	15	(1)	191.239	III	12	(2)	193.257	III	9	(3)
186.204	II	4	(1)	189.222	II	12	(1)	191.240	III	27	(6, 6, 6)	193.258	III	8	(2)
186.205	III	4	(1)	189.223	III	4	(1)	191.241	III	10	(1)	193.259	III	8	(2)
186.206	III	4	(1)	189.224	III	5	(1)	191.242	IV	16	(1)	193.260	III	14	(3, 6)
186.207	III	5	(3)	189.225	III	15	(3, 3, 3)	192.243	I	17	(6)	193.261	III	5	(1)
186.208	IV	4	(1)	189.226	IV	9	(1)	192.244	II	14	(2)	193.262	IV	13	(1)
187.209	I	15	(1)	190.227	I	12	(3)	192.245	III	6	(2)	194.263	I	13	(1)
187.210	II	15	(1)	190.228	II	8	(1)	192.246	III	9	(1)	194.264	II	13	(1)
187.211	III	6	(1)	190.229	III	3	(1)	192.247	III	8	(1)	194.265	III	4	(1)
187.212	III	5	(1)	190.230	III	4	(1)	192.248	III	7	(2)	194.266	III	12	(3)
187.213	III	21	(3, 3, 3)	190.231	III	12	(3, 3)	192.249	III	7	(2)	194.267	III	9	(1)
187.214	IV	10	(1)	190.232	IV	9	(3)	192.250	III	18	(6, 6)	194.268	III	8	(2)
188.215	I	12	(3)	191.233	I	24	(1)	192.251	III	8	(2)	194.269	III	8	(2)
188.216	II	9	(1)	191.234	II	24	(1)	192.252	IV	17	(6)	194.270	III	16	(3, 6)
188.217	III	5	(1)	191.235	III	8	(1)	193.253	I	13	(1)	194.271	III	5	(1)
188.218	III	4	(1)	191.236	III	15	(1)	193.254	II	12	(1)	194.272	IV	12	(1)

185.201	2
185.202	2
186.203	2
186.204	2
186.205	2
186.206	2
186.207	2
186.208	2
187.209	1
187.210	1
187.211	1
187.212	1
187.213	1
187.214	2
188.215	2
188.216	2
188.217	2
188.218	2
188.219	2
188.220	2
189.221	1
189.222	1
189.223	1
189.224	1
189.225	1
189.226	2
190.227	2
190.228	2
190.229	2
190.230	2
190.231	2
190.232	2
191.233	1
191.234	1
191.235	1
191.236	1
191.237	1
191.238	1
191.239	1
191.240	1
191.241	1
191.242	2
192.243	2
192.244	2
192.245	2
192.246	2
192.247	2
192.248	2
192.249	2
192.250	2
192.251	2
192.252	2
193.253	2
193.254	2
193.255	2
193.256	2
193.257	2
193.258	2
193.259	2
193.260	2
193.261	2
193.262	2
194.263	2
194.264	2
194.265	2
194.266	2
194.267	2
194.268	2
194.269	2
194.270	2
194.271	2
194.272	2

d: Rank of the band structure group $\{\mathrm{BS}\}$

X_{BS}: Symmetry-based indicators of band topology

ν_{BS}: Set of ν bands are symmetry-forbidden from being isolated by band gaps if $\nu \notin \nu_{\mathrm{BS}}\,\mathbb{Z}$

Table S18. Characterization of MSGs in the cubic family for spinless electrons.

MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$	MSG	d	$X_{\rm BS}$	$\nu_{\rm BS}$
195.1 I	7	(1)	1	206.39 III	3	(1)	4	216.77 IV	7	(1)	2	224.115 IV	10	(1)	2
195.2 II	6	(1)	1	207.40 I	9	(1)	1	217.78 I	9	(1)	1	225.116 I	17	(1)	1
195.3 IV	4	(2)	2	207.41 II	9	(1)	1	217.79 II	8	(1)	1	225.117 II	17	(1)	1
196.4 I	5	(1)	1	207.42 III	6	(1)	1	217.80 III	5	(1)	1	225.118 III	8	(1)	1
196.5 II	4	(1)	1	207.43 IV	6	(2)	2	218.81 I	9	(2)	2	225.119 III	13	(1)	1
196.6 IV	4	(2)	2	208.44 I	7	(1)	2	218.82 II	7	(1)	2	225.120 III	7	(1)	1
197.7 I	5	(1)	1	208.45 II	7	(1)	2	218.83 III	6	(1)	2	225.121 IV	13	(1)	2
197.8 II	4	(1)	1	208.46 III	6	(2)	2	218.84 IV	6	(2)	2	226.122 I	14	(1)	2
198.9 I	3	(1)	4	208.47 IV	5	(1)	2	219.85 I	8	(2)	2	226.123 II	12	(1)	2
198.10 II	2	(1)	4	209.48 I	7	(1)	1	219.86 II	6	(1)	2	226.124 III	7	(1)	2
198.11 IV	2	(1)	4^*	209.49 II	7	(1)	1	219.87 III	5	(1)	2	226.125 III	10	(1)	2
199.12 I	4	(1)	2^*	209.50 III	5	(1)	1	219.88 IV	8	(2)	2	226.126 III	6	(2)	2
199.13 II	3	(1)	2^*	209.51 IV	6	(1)	2	220.89 I	7	(2)	2^*	226.127 IV	14	(1)	2
200.14 I	17	(1)	1	210.52 I	4	(1)	2	220.90 II	5	(1)	2^*	227.128 I	11	(1)	2
200.15 II	14	(1)	1	210.53 II	4	(1)	2	220.91 III	4	(1)	2^*	227.129 II	11	(1)	2
200.16 III	6	(1)	1	210.54 III	4	(1)	2	221.92 I	22	(1)	1	227.130 III	6	(1)	2
200.17 IV	9	(1)	2	210.55 IV	4	(2)	2^*	221.93 II	22	(1)	1	227.131 III	10	(2)	2
201.18 I	11	(2)	2	211.56 I	7	(1)	1	221.94 III	10	(1)	1	227.132 III	4	(1)	2
201.19 II	8	(2)	2	211.57 II	7	(1)	1	221.95 III	16	(1)	1	227.133 IV	8	(1)	4
201.20 III	4	(2)	2	211.58 III	5	(1)	1	221.96 III	9	(1)	1	228.134 I	9	(2)	2^*
201.21 IV	7	(2)	2	212.59 I	3	(1)	4	221.97 IV	13	(1)	2	228.135 II	7	(1)	4
202.22 I	13	(1)	1	212.60 II	3	(1)	4	222.98 I	11	(2)	2	228.136 III	5	(2)	2^*
202.23 II	10	(1)	1	212.61 III	3	(1)	4	222.99 II	9	(1)	2	228.137 III	7	(2)	4
202.24 III	4	(2)	1	212.62 IV	3	(1)	4^*	222.100 III	6	(1)	2	228.138 III	4	(2)	4
202.25 IV	9	(1)	2	213.63 I	3	(1)	4	222.101 III	8	(1)	2	228.139 IV	9	(2)	2^*
203.26 I	10	(2)	2	213.64 II	3	(1)	4	222.102 III	6	(1)	2	229.140 I	17	(1)	1
203.27 II	7	(2)	2	213.65 III	3	(1)	4	222.103 IV	11	(2)	2	229.141 II	17	(1)	1
203.28 III	3	(1)	2	213.66 IV	3	(1)	4^*	223.104 I	13	(1)	2	229.142 III	9	(1)	1
203.29 IV	6	$(2,2)$	2^*	214.67 I	5	(1)	2^*	223.105 II	12	(1)	2	229.143 III	13	(1)	1
204.30 I	13	(1)	1	214.68 II	5	(1)	2^*	223.106 III	9	(2)	2	229.144 III	7	(1)	1
204.31 II	10	(1)	1	214.69 III	4	(1)	2^*	223.107 III	11	(1)	2	230.145 I	9	(2)	2^*
204.32 III	4	(1)	1	215.70 I	10	(1)	1	223.108 III	6	(1)	2	230.146 II	7	(1)	4^*
205.33 I	9	(2)	4	215.71 II	10	(1)	1	223.109 IV	11	(1)	2	230.147 III	5	(2)	4^*
205.34 II	6	(1)	4	215.72 III	6	(1)	1	224.110 I	13	(1)	2	230.148 III	7	(2)	2^*
205.35 III	2	(1)	4	215.73 IV	7	(1)	2	224.111 II	13	(1)	2	230.149 III	4	(1)	4^*
205.36 IV	5	(2)	4^*	216.74 I	9	(1)	1	224.112 III	7	(1)	2				
206.37 I	10	(2)	2^*	216.75 II	9	(1)	1	224.113 III	11	(2)	2				
206.38 II	7	(1)	4	216.76 III	5	(1)	1	224.114 III	6	(1)	2				

d: Rank of the band structure group $\{\rm BS\}$

$X_{\rm BS}$: Symmetry-based indicators of band topology

$\nu_{\rm BS}$: Set of ν bands are symmetry-forbidden from being isolated by band gaps if $\nu \notin \nu_{\rm BS}\,\mathbb{Z}$

$*$: Exhibiting exceptional filling pattern; see table S19

Table S19. MSGs for which spinless electrons exhibit exceptional filling patterns.

MSG	$\{\nu\}_{\mathrm{AI}}$	$\{\nu\}_{\mathrm{BS}}$	MSG	$\{\nu\}_{\mathrm{AI}}$	$\{\nu\}_{\mathrm{BS}}$	MSG	$\{\nu\}_{\mathrm{AI}}$	$\{\nu\}_{\mathrm{BS}}$	MSG	$\{\nu\}_{\mathrm{AI}}$	$\{\nu\}_{\mathrm{BS}}$
9.41 IV	$4\mathbb{N}$	$2\mathbb{N}$	80.32 IV	$4\mathbb{N}$	$2\mathbb{N}$	133.459 I	$4\mathbb{N}$	$2\mathbb{N}\setminus\{2\}$	212.62 IV	$4\mathbb{N}\setminus\{4\}$	$4\mathbb{N}$
15.91 IV	$4\mathbb{N}$	$2\mathbb{N}$	86.73 IV	$4\mathbb{N}$	$2\mathbb{N}$	133.461 III	$4\mathbb{N}$	$2\mathbb{N}$	213.66 IV	$4\mathbb{N}\setminus\{4\}$	$4\mathbb{N}$
29.105 IV	$8\mathbb{N}$	$4\mathbb{N}$	88.86 IV	$4\mathbb{N}$	$2\mathbb{N}$	133.463 III	$4\mathbb{N}$	$2\mathbb{N}\setminus\{2\}$	214.67 I	$2\mathbb{N}\setminus\{2\}$	–
37.185 IV	$4\mathbb{N}$	$2\mathbb{N}$	91.109 IV	$8\mathbb{N}$	$4\mathbb{N}$	133.466 III	$4\mathbb{N}$	$2\mathbb{N}$	214.68 II	$2\mathbb{N}\setminus\{2\}$	–
41.217 IV	$4\mathbb{N}$	$2\mathbb{N}$	92.117 IV	$8\mathbb{N}$	$4\mathbb{N}$	133.469 IV	$4\mathbb{N}$	$2\mathbb{N}\setminus\{2\}$	214.69 III	$2\mathbb{N}\setminus\{2\}$	–
43.228 IV	$4\mathbb{N}$	$2\mathbb{N}$	93.125 IV	$4\mathbb{N}$	$2\mathbb{N}\setminus\{2\}$	135.485 III	$4\mathbb{N}$	$2\mathbb{N}$	220.89 I	$2\mathbb{N}\setminus\{2,4,10\}$	$2\mathbb{N}\setminus\{2\}$
45.240 IV	$4\mathbb{N}$	$2\mathbb{N}$	94.133 IV	$4\mathbb{N}$	$2\mathbb{N}$	135.490 III	$4\mathbb{N}$	$2\mathbb{N}\setminus\{2\}$	220.90 II	$2\mathbb{N}\setminus\{2,4,10\}$	–
54.347 IV	$8\mathbb{N}$	$4\mathbb{N}$	95.141 IV	$8\mathbb{N}$	$4\mathbb{N}$	142.561 I	$4\mathbb{N}$	$2\mathbb{N}\setminus\{2\}$	220.91 III	$2\mathbb{N}\setminus\{2,4,10\}$	–
60.426 IV	$8\mathbb{N}$	$4\mathbb{N}$	96.149 IV	$8\mathbb{N}$	$4\mathbb{N}$	142.563 III	$4\mathbb{N}$	$2\mathbb{N}$	228.134 I	$4\mathbb{N}$	$2\mathbb{N}\setminus\{2\}$
61.438 IV	$8\mathbb{N}$	$4\mathbb{N}$	98.162 IV	$4\mathbb{N}$	$2\mathbb{N}$	142.565 III	$4\mathbb{N}$	$2\mathbb{N}\setminus\{2\}$	228.136 III	$4\mathbb{N}$	$2\mathbb{N}\setminus\{2\}$
61.440 IV	$8\mathbb{N}$	$4\mathbb{N}$	106.219 I	$4\mathbb{N}$	$2\mathbb{N}$	142.568 III	$4\mathbb{N}$	$2\mathbb{N}\setminus\{2\}$	228.139 IV	$4\mathbb{N}$	$2\mathbb{N}\setminus\{2\}$
68.519 IV	$4\mathbb{N}$	$2\mathbb{N}$	106.222 III	$4\mathbb{N}$	$2\mathbb{N}$	142.570 IV	$4\mathbb{N}$	$2\mathbb{N}\setminus\{2\}$	230.145 I	$4\mathbb{N}\setminus\{4\}$	$2\mathbb{N}\setminus\{2,4\}$
70.532 IV	$4\mathbb{N}$	$2\mathbb{N}$	106.225 IV	$4\mathbb{N}$	$2\mathbb{N}$	198.11 IV	$4\mathbb{N}\setminus\{4\}$	$4\mathbb{N}$	230.146 II	$4\mathbb{N}\setminus\{4\}$	–
73.548 I	$4\mathbb{N}$	$2\mathbb{N}\setminus\{2\}$	110.245 I	$4\mathbb{N}$	$2\mathbb{N}$	199.12 I	$2\mathbb{N}\setminus\{2\}$	–	230.147 III	$4\mathbb{N}\setminus\{4\}$	$4\mathbb{N}$
73.550 III	$4\mathbb{N}$	$2\mathbb{N}$	110.248 III	$4\mathbb{N}$	$2\mathbb{N}$	199.13 II	$2\mathbb{N}\setminus\{2\}$	–	230.148 III	$4\mathbb{N}\setminus\{4\}$	$2\mathbb{N}\setminus\{2,4,10\}$
73.553 IV	$4\mathbb{N}$	$2\mathbb{N}\setminus\{2\}$	110.250 IV	$4\mathbb{N}$	$2\mathbb{N}$	203.29 IV	$4\mathbb{N}$	$2\mathbb{N}$	230.149 III	$4\mathbb{N}\setminus\{4\}$	–
76.11 IV	$8\mathbb{N}$	$4\mathbb{N}$	112.265 IV	$4\mathbb{N}$	$2\mathbb{N}\setminus\{2\}$	205.36 IV	$8\mathbb{N}$	$4\mathbb{N}$			
77.17 IV	$4\mathbb{N}$	$2\mathbb{N}$	114.281 IV	$4\mathbb{N}$	$2\mathbb{N}$	206.37 I	$4\mathbb{N}$	$2\mathbb{N}\setminus\{2\}$			
78.23 IV	$8\mathbb{N}$	$4\mathbb{N}$	122.338 IV	$4\mathbb{N}$	$2\mathbb{N}$	210.55 IV	$4\mathbb{N}$	$2\mathbb{N}$			

$\mathbb{N}$: The set of natural numbers

$\{\nu\}_{\mathrm{AI}}$: Set of fillings for which physical atomic insulators are possible

$\{\nu\}_{\mathrm{BS}}$: Set of fillings for which physical band structure are possible; a dash (–) indicates $\{\nu\}_{\mathrm{BS}} = \{\nu\}_{\mathrm{AI}}$ for that MSG