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Persistence of False Paradigms in Low-Power Sciences: Supporting Information

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Appendix A. Proof of Proposition 1

Equation [5] is a logistic differential equation with threshold σ^*. This type of differential equation is well known: it is commonly used to model population dynamics—for instance, in ecology or in population genetics. Its properties are easy to derive.

We define a polynomial P by

$$P(\sigma) = \lambda\sigma\,(1-\sigma)\,J(\sigma),$$

where $\lambda > 0$ and $J(\sigma)$ is the Youden index:

$$J(\sigma) = 1 - (1+\epsilon)\,(\alpha+\beta) + 2\epsilon\,(\alpha+\beta)\,\sigma.$$

Equation [5] can be written $\dot{\sigma}(t) = P(\sigma(t))$, so the time path of $\sigma(t)$ is determined by the properties of P.

We first consider the case $\epsilon = 0$. Then $J(\sigma) = 1 - \beta - \alpha > 0$, so the polynomial P has two roots, 0 and 1, and $P(\sigma) > 0$ for all $\sigma \in (0,1)$. Hence, the differential equation [5] has two critical points: $\sigma = 0$ is a source, and $\sigma = 1$ is a sink. Consequently, for any initial condition $\sigma(0) \in (0,1)$, $\sigma(t)$ converges toward 1.

Next, we turn to the case $\epsilon > 0$ and

$$1 - \beta \geq \alpha + \frac{\epsilon}{1+\epsilon}.$$

Then $J(\sigma) > 0$ for all $\sigma \in (0,1)$, so the polynomial P has two roots on $[0,1]$, 0 and 1, and $P(\sigma) > 0$ for all $\sigma \in (0,1)$. Hence, again, the differential equation [5] has two critical points on $[0,1]$: $\sigma = 0$ is a source, and $\sigma = 1$ is a sink. Consequently, for any $\sigma(0) \in (0,1)$, $\sigma(t)$ converges toward 1.

Finally, we turn to the case $\epsilon > 0$ and

$$1 - \beta < \alpha + \frac{\epsilon}{1+\epsilon}.$$

Then $J(\sigma) < 0$ for all $\sigma < \sigma^*$, $J(\sigma^*) = 0$, and $J(\sigma) > 0$ for all $\sigma > \sigma^*$, where $\sigma^* \in (0, 1/2)$ is given by equation [3]. Hence, the polynomial P has three roots, 0, σ^*, and 1, and for any $\sigma \in (0,1)$, $P(\sigma) > 0$ if $\sigma > \sigma^*$ and $P(\sigma) < 0$ if $\sigma < \sigma^*$. Accordingly, the differential equation [5] has three critical points: $\sigma = 0$ is a sink, σ^* is a source, and $\sigma = 1$ is a sink. Thus, $\sigma(t)$ converges to 1 from any $\sigma(0) \in (\sigma^*, 1)$, and $\sigma(t)$ converges to 0 from any $\sigma(0) \in (0, \sigma^*)$.

Appendix B. Extensions of the Model

We analyze three extensions of our model: tenure committees are composed of several tenured scientists; Better and Worse scientists have different homophilous biases; and Better and Worse scientists train advisees at different rates. We find that the results generalize to all three cases.

Tenure Committees with Several Members

In universities, tenure cases are reviewed by tenure committees composed of several tenured faculty members. Furthermore, tenure committees request outside letters from a large number of experts in the field. Hence, we now assume that each untenured scientist is evaluated by a tenure committee composed of $2n-1$ tenured scientists, with $n \geq 1$. The committee members are selected at random from the population of tenured scientists. The committee makes a decision by a majority vote: a candidate is granted tenure when at least n committee members vote favorably.

Let's first consider a Better tenure candidate. As in the baseline model, a member of the tenure committee is a Better scientist with probability σ and a Worse scientist with probability $1-\sigma$. A Better committee member votes to deny tenure to the Better candidate with probability $(1-\epsilon)\,\alpha$, whereas a Worse committee member votes to deny tenure to the Better candidate with probability $(1+\epsilon)\,\alpha$. Accordingly, the probability that a committee member votes to deny tenure to the Better candidate is

$$A(\sigma) = \sigma\,(1-\epsilon)\,\alpha + (1-\sigma)\,(1+\epsilon)\,\alpha = (1+\epsilon)\,\alpha - 2\epsilon\alpha\sigma.$$

($A(\sigma)$ is exactly the probability that a Better candidate is denied tenure in the baseline model.) Hence, each committee member votes to deny tenure with probability $A(\sigma)$ and to grant tenure with probability $1-A(\sigma)$. Since all committee members vote independently, the random variable counting the number of unfavorable tenure votes is a binomial random variable with parameters $A(\sigma)$ (probability that one vote is unfavorable) and $2n-1$ (number of votes). The distribution function for this binomial random variable is

$$\mathcal{B}(x; A(\sigma), 2n-1) = \sum_{i=0}^{x} \binom{2n-1}{i} A(\sigma)^i \,\left[1 - A(\sigma)\right]^{2n-1-i},$$

where x is an integer between 0 and $2n-1$. Thus, the probability to grant tenure to a Better candidate, which is the probability that at most $n-1$ committee members vote unfavorably, is

$$1 - \hat{\alpha}(\sigma) = \mathcal{B}(n-1; A(\sigma), 2n-1).$$

Following the same logic, the probability that a committee member votes to grant tenure to the Worse candidate is

$$B(\sigma) = \sigma\,(1-\epsilon)\,\beta + (1-\sigma)\,(1+\epsilon)\,\beta = (1+\epsilon)\,\beta - 2\epsilon\beta\sigma.$$

($B(\sigma)$ is exactly the probability that a Worse candidate is granted tenure in the baseline model.) Conversely, a committee member votes to deny tenure to the Worse candidate with probability $1 - B(\sigma)$. Once again, the random variable counting the number of unfavorable tenure votes is a binomial random variable with parameters $1 - B(\sigma)$ (probability that one vote is unfavorable) and $2n-1$ (number of votes). The distribution function for this binomial random variable is

$$\mathcal{B}(x; 1-B(\sigma), 2n-1) = \sum_{i=0}^{x} \binom{2n-1}{i} \,[1-B(\sigma)]^i \; B(\sigma)^{2n-1-i}.$$

Thus, the probability to grant tenure to a Better candidate, which is the probability that at most $n-1$ committee members vote unfavorably, is

$$\hat{\beta}(\sigma) = \mathcal{B}(n-1; 1-B(\sigma), 2n-1).$$

Accordingly, with a tenure committee of $2n-1$ members, population dynamics continue to be governed by the differential equation [5], where the Youden index is given by

$$J(\sigma) = 1 - \hat{\alpha}(\sigma) - \hat{\beta}(\sigma) = \mathcal{B}(n-1; A(\sigma), 2n-1) - \mathcal{B}(n-1; 1-B(\sigma), 2n-1).$$

Although the Youden index admits a more complicated expression than in the baseline model, its main properties are unchanged, which will imply that population dynamics remain unchanged.

First, the Youden index is strictly increasing in $\sigma \in [0,1]$. To see this, note that for any integers m and x such that $m \geq 1$ and $0 \leq x \leq m-1$, a binomial distribution function $\mathcal{B}(x; p, m)$ is strictly decreasing in $p \in (0,1)$. The intuition is simple: as the probability of success p of each of m independent Bernoulli trials increases, the probability that less than x of the Bernoulli trials are successful goes down—because each trial is more likely to be successful.[1] Second, note that $A(\sigma)$ and $B(\sigma)$ are strictly decreasing in σ. Combining these two results, we infer that $\mathcal{B}(n-1; A(\sigma), 2n-1)$ is strictly increasing in σ and $\mathcal{B}(n-1; 1-B(\sigma), 2n-1)$ is strictly decreasing in σ. Accordingly, the Youden index $J(\sigma)$ is strictly increasing in $\sigma \in [0,1]$.

[1]Formally, the binomial distribution function is given by

$$\mathcal{B}(x; p, m) = \sum_{i=0}^{x} \binom{m}{i} p^i \,(1-p)^{m-i}.$$

Second, when

$$1 - \beta \geq \alpha + \frac{\epsilon}{1+\epsilon},$$

the Youden index is positive for all $\sigma \in (0,1)$. Indeed, under this condition,

$$1 - B(0) = 1 - (1+\epsilon)\,\beta \geq (1+\epsilon)\,\alpha = A(0),$$

which implies that

$$\mathcal{B}(n-1; A(0), 2n-1) \geq \mathcal{B}(n-1; 1-B(0), 2n-1)$$

and thus $J(0) \geq 0$. In addition, $J(\sigma)$ is strictly increasing in $\sigma \in [0,1]$, so $J(\sigma) > 0$ for all $\sigma \in (0,1)$.

Third, when

$$1 - \beta < \alpha + \frac{\epsilon}{1+\epsilon},$$

the Youden index is negative for $\sigma < \sigma^*$, zero at $\sigma = \sigma^*$, and positive for $\sigma > \sigma^*$, where $\sigma^* \in (0, 1/2)$ is defined by equation [3]. Indeed, by definition, $1 - B(\sigma^*) = A(\sigma^*)$, which implies

$$\mathcal{B}(n-1; A(\sigma^*), 2n-1) = \mathcal{B}(n-1; 1-B(\sigma^*), 2n-1)$$

and thus $J(\sigma^*) = 0$. Moreover, $J(\sigma)$ is strictly increasing in $\sigma \in [0,1]$, so $J(\sigma) < 0$ for $\sigma < \sigma^*$, $J(\sigma) = 0$ at $\sigma = \sigma^*$, and $J(\sigma) > 0$ for $\sigma > \sigma^*$.

As the proof of proposition 1 solely relies on the three previous properties of the Youden index, we conclude that proposition 1 holds when tenure committees comprise $2n - 1 \geq 1$ scientists, exactly as when tenure is decided by a single scientist. Hence, the dynamics of science

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Consider first the case $x < mp$. Taking the derivative of the distribution function with respect to p yields

$$\frac{\partial \mathcal{B}}{\partial p} = \sum_{i=0}^{x} \binom{m}{i} p^{i-1}\,(1-p)^{m-i-1}\,(i - mp).$$

Since $x < mp$, $i - mp < 0$ for all $i \leq x$, so $\partial \mathcal{B}/\partial p < 0$. Consider next the case $x \geq mp$. The binomial distribution function can be written as

$$\mathcal{B}(x; p, m) = 1 - \sum_{i=x+1}^{m} \binom{m}{i} p^{i}\,(1-p)^{m-i}.$$

Taking the derivative of the distribution function with respect to p yields

$$\frac{\partial \mathcal{B}}{\partial p} = -\sum_{i=x+1}^{m} \binom{m}{i} p^{i-1}\,(1-p)^{m-i-1}\,(i - mp).$$

Since $x \geq mp$, $i - mp > 0$ for all $i \geq x+1$, so once again we find $\partial \mathcal{B}/\partial p < 0$. We conclude that the binomial distribution function $\mathcal{B}(x; p, m)$ is strictly decreasing in p.

are independent of the size of tenure committees. In particular, the size of tenure committees does not influence the power threshold under which a science is at risk of converging to inferior paradigms (the threshold is $\alpha + \epsilon/(1+\epsilon)$, which is independent of n). And the size of tenure committees does not influence the range of initial conditions for which a science gravitates toward inferior paradigms (since σ^* is independent of n).

Overall, the analysis suggests that relying on a larger number of scientists to make tenure decisions (either by increasing the size of tenure committees in universities or by requesting more outside letters) will not change the nature of scientific progress: a scientific field will be neither more likely nor less likely to converge to the truth. The only aspect of scientific progress modified by the number of scientists involved in the tenure decision is the speed at which a field converges to the dominant paradigm, because the shape of the Youden index $J(\sigma)$ changes when n changes, which affects $\dot{\sigma}$ through equation [5]. Of course, there might be other reasons for multiple-member tenure committees, such as the interchange of knowledge between the evaluators and fairness to the individual researcher. Finally, tenure committees narrowly defined are only one of the many evaluators who determine grant of tenure. Those evaluators also include those who have previously evaluated the candidate's prospects and record—previous hiring committees, grant committees, referees of publications, journal editors, and so on. Thus the property that our analysis remains substantially unchanged with compound evaluations is important to the robustness of our analysis.

Heterogeneous Homophilous Biases

Given that Better and Worse scientists have different views of the world, it is natural to allow them to have different homophilous biases. Hence we introduce two biases: $\epsilon^B \in [0,1]$ for Better tenured scientists, and $\epsilon^W \in [0,1]$ for Worse tenured scientists. Then, a Better evaluator denies tenure to a Better scientist with lowered probability $(1-\epsilon^B)\,\alpha$ and grants tenure to a Worse scientist with lowered probability $(1-\epsilon^B)\,\beta$. And, a Worse evaluator denies tenure to a Better scientist with increased probability $(1+\epsilon^W)\,\alpha$ and grants tenure to a Worse scientist with increased probability $(1+\epsilon^W)\,\beta$. (To ensure that all probabilities remain in $[0,1]$, we add the restrictions that $\alpha \leq 1/(1+\epsilon^W)$ and $\beta \leq 1/(1+\epsilon^W)$.)

As a result, Better scientists are denied tenure with probability

$$\hat{\alpha}(\sigma) = \sigma\left(1-\epsilon^B\right)\alpha + (1-\sigma)\left(1+\epsilon^W\right)\alpha = \left(1+\epsilon^W\right)\alpha - \left(\epsilon^B+\epsilon^W\right)\alpha\sigma.$$

Similarly, Worse scientists are granted tenure with probability

$$\hat{\beta}(\sigma) = \sigma\left(1 - \epsilon^B\right)\beta + (1-\sigma)\left(1 + \epsilon^W\right)\beta = \left(1 + \epsilon^W\right)\beta - \left(\epsilon^B + \epsilon^W\right)\beta\sigma.$$

Given these tenure probabilities, the Youden index becomes

$$J(\sigma) = 1 - \hat{\alpha}(\sigma) - \hat{\beta}(\sigma) = 1 - \left(1 + \epsilon^W\right)(\alpha + \beta) + \left(\epsilon^B + \epsilon^W\right)(\alpha + \beta)\,\sigma.$$

The Youden index is linearly increasing in $\sigma \in [0, 1]$, from

$$J(0) = 1 - \left(1 + \epsilon^W\right)(\alpha + \beta)$$

to

$$J(1) = 1 - \left(1 - \epsilon^B\right)(\alpha + \beta).$$

Therefore, two cases arise. If

$$1 - \beta \geq \alpha + \frac{\epsilon^W}{1 + \epsilon^W},$$

the Youden index is positive for all $\sigma \in (0, 1)$. And if

$$1 - \beta < \alpha + \frac{\epsilon^W}{1 + \epsilon^W},$$

the Youden index is negative for $\sigma < \sigma^*_\epsilon$, zero at $\sigma = \sigma^*_\epsilon$, and positive for $\sigma > \sigma^*_\epsilon$, where the threshold $\sigma^*_\epsilon \in (0, \epsilon^W/(\epsilon^B + \epsilon^W))$ is given by

$$\sigma^*_\epsilon = \frac{\epsilon^W}{\epsilon^B + \epsilon^W}\left[1 - \frac{1 - (\alpha + \beta)}{\alpha + \beta} \cdot \frac{1}{\epsilon^W}\right].$$

Population dynamics continue to be governed by the differential equation [5]. And since the Youden index retains similar properties as in the baseline model, we obtain a proposition similar to proposition 1:

PROPOSITION S1: *With heterogeneous homophilous biases, population dynamics depend on power. With high power ($1 - \beta \geq \alpha + \epsilon^W/(1 + \epsilon^W)$), the Better paradigm eventually prevails ($\lim_{t\to\infty} \sigma(t) = 1$), irrespective of initial conditions. With low power ($1 - \beta < \alpha + \epsilon^W/(1 + \epsilon^W)$), initial conditions matter: if the initial fraction of Better scientists is high ($\sigma(0) > \sigma^*_\epsilon$), the Better paradigm eventually prevails ($\lim_{t\to\infty} \sigma(t) = 1$); but if the initial fraction of Better scientists is low ($\sigma(0) < \sigma^*_\epsilon$), the Worse paradigm eventually prevails ($\lim_{t\to\infty} \sigma(t) = 0$).*

Here again, branches of science with homophilous bias ($\epsilon^W > 0$) and sufficiently low power

$(1-\beta < \alpha + \epsilon^W/(1+\epsilon^W))$ are at risk of converging to inferior paradigms. Critically, the bias of Better scientists (ϵ^B) does not influence the possibility of convergence to the Worse paradigm; what matters is the bias of Worse scientists (ϵ^W). This is because for σ close to 0 there are almost no Better scientists, so their bias is irrelevant to determine whether the critical point 0 is a source or a sink, which in turn decides whether convergence to the Worse paradigm happens or not. Accordingly, even if Better scientists have stronger homophilous bias than Worse scientists, Friedman's conjecture is refuted: a lack of power does not just slow down convergence to the Better paradigm; it makes convergence to the Worse paradigm possible.

While the bias of Better scientists does not influence the possibility of convergence to the inferior paradigm, it does affect science dynamics in other ways. For instance, an increase in the bias of Better scientists decreases the threshold σ^*_ϵ and thus reduces the range of initial conditions for which science gravitates toward the Worse paradigm.

Heterogeneous Advising Rates

The race toward having a larger number of advisees suggests that advising is an important aspect of knowledge creation. Maybe some paradigms lend themselves to having more advising; for instance, because they are more applicable, or because they can be extended more easily. Hence we introduce different advising rates across paradigms: Better tenured scientists train advisees at rate $\lambda^B > 0$, and Worse tenured scientists train advisees at rate $\lambda^W > 0$.

The growth rates of the populations of Better and Worse tenured scientists are now given by

$$g^B(t) = \lambda^B(1-\hat{\alpha}(\sigma(t))) - \delta$$
$$g^W(t) = \lambda^W\hat{\beta}(\sigma(t)) - \delta,$$

where, as in the baseline model,

$$\hat{\alpha}(\sigma) = (1+\epsilon)\alpha - 2\epsilon\alpha\sigma$$
$$\hat{\beta}(\sigma) = (1+\epsilon)\beta - 2\epsilon\beta\sigma.$$

We infer that

$$g^B(t) - g^W(t) = \lambda^B J_\lambda(\sigma(t)),$$

where $J_\lambda(\sigma(t))$ is a new Youden index, tailored to a situation with heterogeneous advising rates:

$$J_\lambda(\sigma) = 1 - \hat{\alpha}(\sigma) - \frac{\lambda^W}{\lambda^B}\hat{\beta}(\sigma) = 1 - (1+\epsilon)\left(\alpha + \frac{\lambda^W}{\lambda^B}\beta\right) + 2\epsilon\left(\alpha + \frac{\lambda^W}{\lambda^B}\beta\right)\sigma.$$

The Youden index is linearly increasing in $\sigma \in [0,1]$, from

$$J_\lambda(0) = 1 - (1+\epsilon)\left(\alpha + \frac{\lambda^W}{\lambda^B}\beta\right)$$

to

$$J_\lambda(1) = 1 - (1-\epsilon)\left(\alpha + \frac{\lambda^W}{\lambda^B}\beta\right).$$

Here there are three cases to consider. First, if

$$1 - \frac{\lambda^W}{\lambda^B}\beta \geq \alpha + \frac{\epsilon}{1+\epsilon},$$

the Youden index is positive for all $\sigma \in (0,1)$. Second, if

$$1 - \frac{\lambda^W}{\lambda^B}\beta \leq \alpha - \frac{\epsilon}{1-\epsilon},$$

the Youden index is negative for all $\sigma \in (0,1)$. Third, if

$$1 - \frac{\lambda^W}{\lambda^B}\beta \in \left(\alpha - \frac{\epsilon}{1-\epsilon}, \alpha + \frac{\epsilon}{1+\epsilon}\right),$$

the Youden index changes sign on $(0,1)$: it is negative for $\sigma < \sigma^*_\lambda$, zero at $\sigma = \sigma^*_\lambda$, and positive for $\sigma > \sigma^*_\lambda$, where the threshold $\sigma^*_\lambda \in (0,1)$ is given by

$$\sigma^*_\lambda = \frac{1}{2}\left[1 - \frac{1 - \alpha - \left(\lambda^W/\lambda^B\right)\beta}{\alpha + \left(\lambda^W/\lambda^B\right)\beta} \cdot \frac{1}{\epsilon}\right].$$

The dynamics of $\sigma(t)$ continue to satisfy the differential equation [4]. This equation can be rewritten as

$$\dot{\sigma}(t) = \lambda^B \sigma(t)\,[1-\sigma(t)]\,J_\lambda(\sigma(t)),$$

Exploiting the properties of the Youden index, we obtain the following proposition:

PROPOSITION S2: *With heterogeneous advising rates, population dynamics depend on power and advising rates. When*

$$1 - \frac{\lambda^W}{\lambda^B}\beta \geq \alpha + \frac{\epsilon}{1+\epsilon},$$

the Better paradigm eventually prevails ($\lim_{t\to\infty}\sigma(t) = 1$)*, irrespective of initial conditions. When*

$$1 - \frac{\lambda^W}{\lambda^B}\beta \leq \alpha - \frac{\epsilon}{1-\epsilon},$$

the Worse paradigm eventually prevails ($\lim_{t\to\infty}\sigma(t)=0$*), irrespective of initial conditions. And when*

$$1-\frac{\lambda^W}{\lambda^B}\beta\in\left(\alpha-\frac{\epsilon}{1-\epsilon},\,\alpha+\frac{\epsilon}{1+\epsilon}\right),$$

initial conditions matter: if the initial fraction of Better scientists is high ($\sigma(0)>\sigma^*_\lambda$*), the Better paradigm eventually prevails (*$\lim_{t\to\infty}\sigma(t)=1$*); but if the initial fraction of Better scientists is low (*$\sigma(0)<\sigma^*_\lambda$*), the Worse paradigm eventually prevails (*$\lim_{t\to\infty}\sigma(t)=0$*).*

Here again, science is at risk of converging to inferior paradigms. This risk is present even if Better scientists train advisees at a higher rate than Worse scientists ($\lambda^B>\lambda^W$). Indeed, assume that the low-power condition of proposition 1 holds: $1-\beta<\alpha+\epsilon/(1+\epsilon)$. Then $1-(\lambda^W/\lambda^B)\beta<\alpha+\epsilon/(1+\epsilon)$ even when $\lambda^B>\lambda^W$, as long as λ^B/λ^W is not too large:

$$\frac{\lambda^B}{\lambda^W}\in\left(1,\frac{\beta}{1-\alpha-\epsilon/(1+\epsilon)}\right).$$

In that case, even though Better scientists train advisees at a higher rate than Worse advisees, Friedman's conjecture does not hold: there are initial conditions for which science never converges to the Better paradigm.

Friedman's conjecture holds for some advising rates, of course. For any level of power, if

$$\frac{\lambda^B}{\lambda^W}>\frac{\beta}{1-\alpha-\epsilon/(1+\epsilon)},$$

then $1-(\lambda^W/\lambda^B)\beta>\alpha+\epsilon/(1+\epsilon)$: science is guaranteed to converge to the truth. So if Better scientists train sufficiently more advisees than Worse scientists, the Worse paradigm will always be weeded out—even if power is low.

Finally, in the same way that advising by Better scientists helps science converge to the truth, advising by Worse scientists pushes science toward falsehood. And when Worse scientists advise sufficiently more students than Better scientists, scientific inquiry necessarily leads to falsehood. This happens when

$$\frac{\lambda^W}{\lambda^B}>\frac{1-\alpha+\epsilon/(1-\epsilon)}{\beta},$$

which ensures that $1-(\lambda^W/\lambda^B)\beta\le\alpha-\epsilon/(1-\epsilon)$.