Supplementary Materials for "Evaluating Classification Performance of Biomarkers in Two-Phase Case-Control Studies"

Lu Wang[1], Ying Huang*[1,2]

[1] *Vaccine and Infectious Disease Division, Fred Hutchinson Cancer Research Center, Seattle, WA 98109, U.S.A.*

[2] *Department of Biostatistics, University of Washington, Seattle, WA 98195, U.S.A.*

* *yhuang@fhcrc.org*

Appendix A. Proof of Theorem 1

Suppose we model the sampling probabilities p_D and $p_{\bar{D}}$ with finite-dimensional parameters θ_D and $\theta_{\bar{D}}$, respectively. Let $\hat{\theta}_D$ and $\hat{\theta}_{\bar{D}}$ be the maximum likelihood estimators and $\hat{p}_D$ and $\hat{p}_{\bar{D}}$ be the corresponding sampling probabilities estimators. To prove Theorem 1, we first show the asymptotic properties of the IPW sensitivity estimator $\widehat{SEN}_{IPW}(c,\hat{p}_D) = \frac{\sum_{i=1}^{N_D} \frac{\delta_{Di}}{\hat{p}_{Di}} I(X_{Di}>c)}{\sum_{i=1}^{N_D} \frac{\delta_{Di}}{\hat{p}_{Di}}}$ and the IPW specificity estimator $\widehat{SPE}_{IPW}(c,\hat{p}_{\bar{D}}) = \frac{\sum_{j=1}^{N_{\bar{D}}} \frac{\delta_{\bar{D}j}}{\hat{p}_{\bar{D}j}} I\left(X_{\bar{D}j}<c\right)}{\sum_{j=1}^{N_{\bar{D}}} \frac{\delta_{\bar{D}j}}{\hat{p}_{\bar{D}j}}}$, where $\hat{p}_D$ and $\hat{p}_{\bar{D}}$ are obtained from the GLM model. The following lemma presents the results.

Lemma A.1 *Assume* $0 < p_D, p_{\bar{D}} \le 1$ *and* $N_D/N \to \lambda$ *as the sample size* $N \to \infty$. *Then as* $N \to \infty$, $\sqrt{N_D}\left[\widehat{SEN}_{IPW}(c,\hat{p}_D) - SEN(c)\right]$ *converges to a normal random variable with mean* 0 *and variance* $\sigma^2_{D,IPW}(c)$, *and* $\sqrt{N_{\bar{D}}}\left[\widehat{SPE}_{IPW}(c,\hat{p}_{\bar{D}}) - SPE(c)\right]$ *converges to a normal random variable with mean* 0 *and variance* $\sigma^2_{\bar{D},IPW}(c)$, *where*

$$\begin{aligned}\sigma^2_{d,IPW}(c) \;&=\; Var(H_d) + E\left[\left(p_d^{-1}-1\right)H_d\right] - SS_d(c)\left\{E\left[\left(p_d^{-1}-1\right)H_d\right] + Cov\left(p_d^{-1}-1, H_d\right)\right\} \\ &\quad - \left[SS_d(c)\times a_d - b_d(c)\right]^T I_d^{-1}\left[SS_d(c)\times a_d - b_d(c)\right], \textit{ for } d = D, \bar{D}\end{aligned}$$

in which $H_D = I(X_D > c)$, $H_{\bar{D}} = I(X_{\bar{D}} < c)$, $SS_D(c) = SEN(c)$, $SS_{\bar{D}}(c) = SPE(c)$, $a_d = E[(1/p_d)(\partial p_d/\partial\theta_d)]$, $b_d(c) = E[H_d(1/p_d)(\partial p_d/\partial\theta_d)]$, $I_d = E\left[\left(\frac{1}{p_d} + \frac{1}{1-p_d}\right)\frac{\partial p_d}{\partial\theta_d}\frac{\partial p_d}{\partial\theta_d}\right]$ *is the information matrix of* θ_d, *for* $d = D, \bar{D}$.

Proof. To begin with, we define $\widetilde{SEN}_{IPW}(c,\hat{p}_D) = \left[\sum_{i=1}^{N_D} \frac{\delta_{Di}}{\hat{p}_{Di}} I(X_{Di} > c)\right]/N_D$ and show its asymptotic normality. In the following proof, we omit the subscript D for simplicity. First, we observe that

$$\begin{aligned}&\sqrt{N_D}\left[\widetilde{SEN}_{IPW}(c,\hat{p}) - SEN(c)\right] \\ =\;& \sqrt{N_D}\left[\widetilde{SEN}_{IPW}(c,\hat{p}) - \widetilde{SEN}_{IPW}(c,p)\right] + \sqrt{N_D}\left[\widetilde{SEN}_{IPW}(c,p) - SEN(c)\right] \\ =\;& \sqrt{N_D}\frac{\partial E\left[\widetilde{SEN}_{IPW}(c,p)\right]}{\partial\theta}\left(\hat{\theta}-\theta\right) + \sqrt{N_D}\left[\frac{1}{N_D}\sum_{i=1}^{N_D}\frac{\delta_i}{p_i} I(X_i > c) - E[I(X > c)]\right] + o_p(1) \\ =\;& U + M + o_p(1),\end{aligned}$$

where

$$U = \sqrt{N_D}\frac{\partial E\left[\widetilde{SEN}_{IPW}(c,p)\right]}{\partial\theta}\left(\hat{\theta}-\theta\right) = -\sqrt{N_D}E\left[I(X>c)\frac{1}{p}\frac{\partial p}{\partial\theta}\right]\left(\hat{\theta}-\theta\right),$$

$$M = \sqrt{N_D}\left[\frac{1}{N_D}\sum_{i=1}^{N_D}\frac{\delta_i}{p_i}I(X_i>c) - E[I(X>c)]\right].$$

Obviously, $E(U)=E(M)=0$. So, by the central limit theorem (CLT), $\sqrt{N_D}\left[\widetilde{SEN}_{IPW}(c,\hat{p}_D)-SEN(c)\right]$ converges to a normal random variable with mean 0 and variance $Var(U+M)$, as $N_D\to\infty$.

Next, we define $\beta=\frac{1}{N_D}\sum_{i=1}^{N_D}(\delta_i/\hat{p}_i)$, then one has

$$\begin{aligned}
&\sqrt{N_D}(\beta-1)\\
=\;& \sqrt{N_D}\left\{\frac{1}{N_D}\sum_{i=1}^{N_D}\frac{\delta_i}{\hat{p}_i}-\frac{1}{N_D}\sum_{i=1}^{N_D}\frac{\delta_i}{p_i}\right\}+\sqrt{N_D}\left(\frac{1}{N_D}\sum_{i=1}^{N_D}\frac{\delta_i}{p_i}-1\right)\\
=\;& -\sqrt{N_D}\frac{1}{N_D}E\left(\sum_{i=1}^{N_D}\frac{\delta_i}{p_i^2}\frac{\partial p}{\partial\theta}\right)\left(\hat{\theta}-\theta\right)+\sqrt{N_D}\left(\frac{1}{N_D}\sum_{i=1}^{N_D}\frac{\delta_i}{p_i}-1\right)+o_p(1)\\
=\;& C^*+E^*+o_p(1),
\end{aligned}$$

where

$$C^* = -\sqrt{N_D}\frac{1}{N_D}E\left(\sum_{i=1}^{N_D}\frac{\delta_i}{p_i^2}\frac{\partial p}{\partial\theta}\right)\left(\hat{\theta}-\theta\right) = -\sqrt{N_D}E\left(\frac{1}{p}\frac{\partial p}{\partial\theta}\right)\left(\hat{\theta}-\theta\right),$$

$$E^* = \sqrt{N_D}\left(\frac{1}{N_D}\sum_{i=1}^{N_D}\frac{\delta_i}{p_i}-1\right).$$

Therefore, by CLT, $\sqrt{N_D}(\beta-1)$ converges to a normal random variable with mean 0 and variance $Var(C^*+E^*)$, as $N_D\to\infty$. Then $\sqrt{N_D}\begin{pmatrix}\widetilde{SEN}_{IPW}(c,\hat{p})-SEN(c)\\ \beta-1\end{pmatrix}$ converges to the bivariate normal random variable with mean $\begin{pmatrix}0\\0\end{pmatrix}$ and covariance matrix $\begin{bmatrix}Var(U+M) & Cov(U+M,C^*+E^*)\\ Cov(U+M,C^*+E^*) & Var(C^*+E^*)\end{bmatrix}$. With the fact that $\widehat{SEN}_{IPW}(c,\hat{p})=\widetilde{SEN}_{IPW}(c,\hat{p})/\beta$, by Delta method, as $N_D\to\infty$, $\sqrt{N_D}\left[\widehat{SEN}_{IPW}(c,\hat{p})-SEN(c)\right]$ converges to a normal random variable with mean 0 and variance

$$\begin{aligned}
&\sigma^2_{D,IPW}(c)\\
=\;&\begin{pmatrix}1 & -SEN(c)\end{pmatrix}\begin{bmatrix}Var(U+M) & Cov(U+M,C^*+E^*)\\ Cov(U+M,C^*+E^*) & Var(C^*+E^*)\end{bmatrix}\begin{pmatrix}1\\ -SEN(c)\end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= Var(U) + Var(M) + 2Cov(U+M) \\
&\quad -2SEN(c) \times [Cov(U,C^*) + Cov(U,E^*) + Cov(M,C^*) + Cov(M,E^*)] \\
&\quad +SEN^2(c) \times [Var(C^*) + Var(E^*) + 2Cov(C^*+E^*)] \\
&= I_1 + I_2, \qquad \text{(A.1)}
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= Var(M) - 2SEN(c) \times Cov(M,E^*) + SEN^2(c) \times Var(E^*), \\
I_2 &= Var(U) + 2Cov(U+M) - 2SEN(c) \times [Cov(U,C^*) + Cov(U,E^*) + Cov(M,C^*)] \\
&\quad +SEN^2(c) \times [Var(C^*) + 2Cov(C^*+E^*)].
\end{aligned}
$$

We notice that

$$
\begin{aligned}
Var(M) &= Var\left[\frac{\delta}{p} I(X>c)\right] = E\left[\left(\frac{1}{p}-1\right) I(X>c)\right] + Var[I(X>c)], \\
Var(E^*) &= Var\left(\frac{\delta}{p}\right) = E\left(\frac{1}{p}-1\right), \\
Cov(M,E^*) &= Cov\left[\frac{\delta}{p} I(X>c), \frac{\delta}{p}\right] = E\left[\left(\frac{1}{p}-1\right) I(X>c)\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_1 &= E\left[\left(\frac{1}{p}-1\right) I(X>c)\right] + Var[I(X>c)] \\
&\quad -SEN(c) \times \left\{2E\left[\left(\frac{1}{p}-1\right) I(X>c)\right] - SEN(c) \times E\left(\frac{1}{p}-1\right)\right\} \\
&= E\left[\left(\frac{1}{p}-1\right) I(X>c)\right] + Var[I(X>c)] \\
&\quad -SEN(c) \times \left\{2E\left[\left(\frac{1}{p}-1\right) I(X>c)\right] - E[I(X>c)]E\left(\frac{1}{p}-1\right)\right\} \\
&= E\left[\left(\frac{1}{p}-1\right) I(X>c)\right] + Var[I(X>c)] \\
&\quad -SEN(c) \times \left\{E\left[\left(\frac{1}{p}-1\right) I(X>c)\right] + Cov\left(\frac{1}{p}-1, I(X>c)\right)\right\}. \qquad \text{(A.2)}
\end{aligned}
$$

Next, let $a_D = E[(1/p)(\partial p/\partial\theta)]$ and $b_D(c) = E[I(X>c)(1/p)(\partial p/\partial\theta)]$, then

$$
\begin{aligned}
Var(U) &= E\left[I(X>c)\frac{1}{p}\frac{\partial p}{\partial\theta}\right]^T I_D^{-1} E\left[I(X>c)\frac{1}{p}\frac{\partial p}{\partial\theta}\right] = b_D^T(c) I_D^{-1} b_D(c), \\
Var(C^*) &= E\left(\frac{1}{p}\frac{\partial p}{\partial\theta}\right)^T I_D^{-1} E\left(\frac{1}{p}\frac{\partial p}{\partial\theta}\right) = a_D^T I_D^{-1} a_D,
\end{aligned}
$$

$$\begin{aligned}
& Cov(U,M) \\
= & -Cov\left(E\left[I(X>c)\frac{1}{p}\frac{\partial p}{\partial\theta}\right]I_D^{-1}\left[\left(\frac{\delta}{p}+\frac{1-\delta}{1-p}\right)\frac{\partial p}{\partial\theta}\right],\frac{\delta}{p}I(X>c)\right) \\
= & -E\left[I(X>c)\frac{1}{p}\frac{\partial p}{\partial\theta}\right]^T I_D^{-1}E\left[I(X>c)\frac{1}{p}\frac{\partial p}{\partial\theta}\right] \\
= & -b_D^T(c)I_D^{-1}b_D(c),
\end{aligned}$$

where I_D is the information matrix for θ_D. Similarly, we can show that $Cov(U,C^*)=b_D^T(c)I_D^{-1}a_D$, $Cov(U,E^*)=Cov(M,C^*)=-b_D^T(c)I_D^{-1}a_D$, $Cov(C^*,E^*)=-a_D^TI_D^{-1}a_D$. So

$$\begin{aligned}
I_2 = & \; b_D^T(c)I_D^{-1}b_D(c)-2b_D^T(c)I_D^{-1}b_D(c)-2SEN(c)\times\left(b_D^T(c)I_D^{-1}a_D-2b_D^T(c)I_D^{-1}a_D\right) \\
& +SEN^2(c)\times\left(a_D^TI_D^{-1}a_D-2a_D^TI_D^{-1}a_D\right) \\
= & -b_D^T(c)I_D^{-1}b_D(c)+2SEN(c)\times b_D^T(c)I_D^{-1}a_D-SEN^2(c)\times a_D^TI_D^{-1}a_D \\
= & -[SEN(c)\times a_D-b_D(c)]^T I_D^{-1}[SEN(c)\times a_D-b_D(c)]. \qquad (A.3)
\end{aligned}$$

Finally, the asymptotic variance of $\widehat{SEN}_{IPW}(c,\hat{p}_D)$ can be obtained from equations (A.1), (A.2) and (A.3). Similar arguments can be used to prove the asymptotic variance of $\widehat{SPE}_{IPW}(c,\hat{p}_{\bar{D}})$. ■

We now show the asymptotic property of $\sqrt{N_D}\left[\widehat{ROC}_{IPW}(t)-ROC(t)\right]$. Observe that

$$\begin{aligned}
& \sqrt{N_D}\left[\widehat{ROC}_{IPW}(t)-ROC(t)\right] \\
= & \sqrt{N_D}\left\{1-\hat{F}_{D,IPW}\left[\hat{F}_{\bar{D},IPW}^{-1}(1-t)\right]-\left[1-F_D\left[F_{\bar{D}}^{-1}(1-t)\right]\right]\right\} \\
= & \sqrt{N_D}\left\{1-\hat{F}_{D,IPW}\left[\hat{F}_{\bar{D},IPW}^{-1}(1-t)\right]-\left[1-F_D\left[\hat{F}_{\bar{D},IPW}^{-1}(1-t)\right]\right]\right\} \\
& +\sqrt{N_D}\left\{1-F_D\left[\hat{F}_{\bar{D},IPW}^{-1}(1-t)\right]-\left[1-F_D\left[F_{\bar{D}}^{-1}(1-t)\right]\right]\right\}. \qquad (A.4)
\end{aligned}$$

Let $c=F_{\bar{D}}^{-1}(1-t)$, then according to Lemme A.1, as $N\to\infty$

$$\sqrt{N_D}\left\{1-\hat{F}_{D,IPW}(c)-\left[1-F_{D,IPW}(c)\right]\right\}\xrightarrow{d}N\left(0,\sigma_{D,IPW}^2(c)\right),$$

and

$$\sqrt{N_{\bar{D}}}\left(\hat{F}_{\bar{D},IPW}(c)-F_{\bar{D}}(c)\right)\xrightarrow{d}N\left(0,\sigma_{\bar{D},IPW}^2(c)\right). \qquad (A.5)$$

Therefore, we have

$$\begin{aligned}
& \sqrt{N_D}\left\{1-\hat{F}_{D,IPW}\left[\hat{F}_{\bar{D},IPW}^{-1}(1-t)\right]-\left[1-F_{D,IPW}\left[\hat{F}_{\bar{D},IPW}^{-1}(1-t)\right]\right]\right\} \\
= & \sqrt{N_D}\left\{1-\hat{F}_{D,IPW}\left[\hat{F}_{\bar{D},IPW}^{-1}(1-t)\right]-\left[1-F_{D,IPW}\left[\hat{F}_{\bar{D},IPW}^{-1}(1-t)\right]\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
& -\sqrt{N_D}\left\{1-\hat{F}_{D,IPW}\left[F^{-1}_{\bar{D},IPW}(1-t)\right]-\left[1-F_{D,IPW}\left[F^{-1}_{\bar{D},IPW}(1-t)\right]\right]\right\} \\
& +\sqrt{N_D}\left\{1-\hat{F}_{D,IPW}\left[F^{-1}_{\bar{D},IPW}(1-t)\right]-\left[1-F_{D,IPW}\left[F^{-1}_{\bar{D},IPW}(1-t)\right]\right]\right\} \\
= \;& \sqrt{N_D}\left\{1-\hat{F}_{D,IPW}(c)-\left[1-F_{D,IPW}(c)\right]\right\}+o_p(1)\xrightarrow{d} N\left(0,\sigma^2_{D,IPW}(c)\right). \qquad \text{(A.6)}
\end{aligned}
$$

Furthermore, by the property of stochastic equicontinuity, the result shown in (A.5) is not affected if $F^{-1}_{\bar{D}}(1-t)$ is replaced by a consistent estimator $\hat{F}^{-1}_{\bar{D},IPW}(1-t)$, thus

$$
\sqrt{N_{\bar{D}}}\left[\hat{F}_{\bar{D},IPW}\left(\hat{F}^{-1}_{\bar{D},IPW}(1-t)\right)-F_{\bar{D}}\left(\hat{F}^{-1}_{\bar{D},IPW}(1-t)\right)\right]\xrightarrow{d} N\left(0,\sigma^2_{\bar{D},IPW}(c)\right),
$$

which implies that

$$
\sqrt{N_{\bar{D}}}\left[F_{\bar{D}}\left(\hat{F}^{-1}_{\bar{D},IPW}(1-t)\right)-(1-t)\right]\xrightarrow{d} N\left(0,\sigma^2_{\bar{D},IPW}(c)\right).
$$

Then using the Delta method via the transformation $F^{-1}_{\bar{D}}$, we have

$$
\sqrt{N_{\bar{D}}}\left[\hat{F}^{-1}_{\bar{D},IPW}(1-t)-F^{-1}_{\bar{D}}(1-t)\right]\xrightarrow{d} N\left(0,\frac{\sigma^2_{\bar{D},IPW}(c)}{f^2_{\bar{D}}(c)}\right).
$$

Next, consider the transformation $g(u)=1-F_D(u)$. We apply Delta method and obtain

$$
\sqrt{N_{\bar{D}}}\left\{1-F_D\left[\hat{F}^{-1}_{\bar{D},IPW}(1-t)\right]-\left[1-F_D\left[F^{-1}_{\bar{D}}(1-t)\right]\right]\right\}\xrightarrow{d} N\left(0,\frac{f^2_D(c)}{f^2_{\bar{D}}(c)}\sigma^2_{\bar{D},IPW}(c)\right).
$$

Therefore, as $N_D/N\rightarrow\lambda$ and $N\rightarrow\infty$,

$$
\sqrt{N_D}\left\{1-F_D\left[\hat{F}^{-1}_{\bar{D},IPW}(1-t)\right]-\left[1-F_D\left[F^{-1}_{\bar{D}}(1-t)\right]\right]\right\}\xrightarrow{d} N\left(0,\frac{\lambda}{1-\lambda}\frac{f^2_D(c)}{f^2_{\bar{D}}(c)}\sigma^2_{\bar{D},IPW}(c)\right). \qquad \text{(A.7)}
$$

Followed by equations (A.4), (A.6) and (A.7), the proof of Theorem 1 is complete.

Appendix B. Improvement on efficiency via including auxiliary variables

We now show that the efficiency of our proposed IPW estimators can be improved by including more auxiliary variables than sampling strata in estimating the sampling probabilities. Here, we take the IPW AUC estimator as an example. Similar arguments can be used for the IPW ROC(t) and partial AUC estimators. Here we consider the following two models in estimating sampling probabilities.

Model A:

$$\log\left(\frac{p_D}{1-p_D}\right) = \theta_{D0} + \theta_{D1}^T V_D,$$
$$\log\left(\frac{p_{\bar{D}}}{1-p_{\bar{D}}}\right) = \theta_{\bar{D}0} + \theta_{\bar{D}1}^T V_{\bar{D}}.$$

Model B:

$$\log\left(\frac{p_D}{1-p_D}\right) = \theta_{D0} + \theta_{D1}^T V_D + \theta_{D2} Z_D,$$
$$\log\left(\frac{p_{\bar{D}}}{1-p_{\bar{D}}}\right) = \theta_{\bar{D}0} + \theta_{\bar{D}1}^T V_{\bar{D}} + \theta_{\bar{D}2} Z_{\bar{D}}.$$

In model A, we only include the dummy variable V for sampling strata to estimate the sampling probabilities. In model B, we add an auxiliary variable Z besides sampling strata. We need to show that

$$\begin{aligned} & \frac{1}{\lambda}\left(AUC \times a_{D_B} - l_{D_B}\right)^T I_{D_B}^{-1}\left(AUC \times a_{D_B} - l_{D_B}\right) + \\ & \frac{1}{1-\lambda}\left(AUC \times a_{\bar{D}_B} - l_{\bar{D}_B}\right)^T I_{\bar{D}_B}^{-1}\left(AUC \times a_{\bar{D}_B} - l_{\bar{D}_B}\right) \\ \geq\; & \frac{1}{\lambda}\left(AUC \times a_{D_A} - l_{D_A}\right)^T I_{D_A}^{-1}\left(AUC \times a_{D_A} - l_{D_A}\right) \\ & + \frac{1}{1-\lambda}\left(AUC \times a_{\bar{D}_A} - l_{\bar{D}_A}\right)^T I_{\bar{D}_A}^{-1}\left(AUC \times a_{\bar{D}_A} - l_{\bar{D}_A}\right), \end{aligned} \tag{A.8}$$

where l_D, $l_{\bar{D}}$, a_D, $a_{\bar{D}}$, I_D and $I_{\bar{D}}$ are defined as aforementioned. We use subscripts A and B to denote the corresponding values obtained from model A and B, respectively. With the fact that $I_{D_B} = \begin{bmatrix} I_{D_A} & e_D \\ e_D^T & g_D \end{bmatrix}$, $a_{D_B} = \begin{bmatrix} a_{D_A} \\ m_D \end{bmatrix}$, and $l_{D_B} = \begin{bmatrix} l_{D_A} \\ u_D \end{bmatrix}$, where $e_D = E\left[\left(\frac{1}{p_D} + \frac{1}{1-p_D}\right)\frac{\partial p_D}{\partial \theta_{D(-2)}}\frac{\partial p_D}{\partial \theta_{D2}}\right]$, $g_D = E\left[\left(\frac{1}{p_D} + \frac{1}{1-p_D}\right)\frac{\partial p_D}{\partial \theta_{D2}}\frac{\partial p_D}{\partial \theta_{D2}}\right]$, $m_D = E[(1/p_D)(\partial p_D/\partial \theta_{D2})]$, $u_D = E[I(X_D > X_{\bar{D}})(1/p_D)(\partial p_D/\partial \theta_{D2})]$, in which $\theta_{D(-2)} = \left(\theta_{D0}, \theta_{D1}^T\right)^T$. Then we have

$$\begin{aligned} & \left(AUC \times a_{D_B} - l_{D_B}\right)^T I_{D_B}^{-1}\left(AUC \times a_{D_B} - l_{D_B}\right) \\ = & \begin{bmatrix} \left(AUC \times a_{D_A} - l_{D_A}\right)^T & AUC \times m_D - u_D \end{bmatrix} I_{D_B}^{-1} \begin{bmatrix} \left(AUC \times a_{D_A} - l_{D_A}\right)^T \\ AUC \times m_D - u_D \end{bmatrix} \\ = & \left(AUC \times a_{D_A} - l_{D_A}\right)^T I_{D_A}^{-1}\left(AUC \times a_{D_A} - l_{D_A}\right) + \begin{bmatrix} \left(AUC \times a_{D_A} - l_{D_A}\right)^T & AUC \times m_D - u_D \end{bmatrix} \\ & \times \begin{bmatrix} I_{D_A}^{-1} e_D \left(g_D - e_D^T I_{D_A}^{-1} e_D\right)^{-1} e_D^T I_{D_A}^{-1} & -\left(g_D - e_D^T I_{D_A}^{-1} e_D\right)^{-1} I_{D_A}^{-1} e_D \\ \left(g_D - e_D^T I_{D_A}^{-1} e_D\right)^{-1} e_D^T I_{D_A}^{-1} & \left(g_D - e_D^T I_{D_A}^{-1} e_D\right)^{-1} \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
& \times \left[\begin{array}{c} \left(AUC \times a_{D_A} - l_{D_A}\right)^T \\ AUC \times m_D - u_D \end{array} \right] \\
= & \left(AUC \times a_{D_A} - l_{D_A}\right)^T I_{D_A}^{-1} \left(AUC \times a_{D_A} - l_{D_A}\right) \\
& + \left[\begin{array}{cc} \left(AUC \times a_{D_A} - l_{D_A}\right)^T & AUC \times m_D - u_D \end{array} \right] \times \left[\begin{array}{c} I_{D_A}^{-1} e_D \left(g_D - e_D^T I_{D_A}^{-1} e_D\right)^{-1/2} \\ -\left(g_D - e_D^T I_{D_A}^{-1} e_D\right)^{-1/2} \end{array} \right] \\
& \times \left[\begin{array}{cc} \left(g_D - e_D^T I_{D_A}^{-1} e_D\right)^{-1/2} e_D^T I_{D_A}^{-1} & -\left(g_D - e_D^T I_{D_A}^{-1} e_D\right)^{-1/2} \end{array} \right] \times \left[\begin{array}{c} \left(AUC \times a_{D_A} - l_{D_A}\right)^T \\ AUC \times m_D - u_D \end{array} \right] \\
= & \left(AUC \times a_{D_A} - l_{D_A}\right)^T I_{D_A}^{-1} \left(AUC \times a_{D_A} - l_{D_A}\right) + \\
& \left[\left(AUC \times a_{D_A} - l_{D_A}\right)^T I_{D_A}^{-1} e_D \left(g_D - e_D^T I_{D_A}^{-1} e_D\right)^{-1/2} - \left(AUC \times m_D - u_D\right) \left(g_D - e_D^T I_{D_A}^{-1} e_D\right)^{-1/2} \right]^2 \\
\geq & \left(AUC \times a_{D_A} - l_{D_A}\right)^T I_{D_A}^{-1} \left(AUC \times a_{D_A} - l_{D_A}\right).
\end{aligned}
$$

Similarly, for the controls, one has

$$
\left(AUC \times a_{\bar{D}_B} - l_{\bar{D}_B}\right)^T I_{\bar{D}_B}^{-1} \left(AUC \times a_{\bar{D}_B} - l_{\bar{D}_B}\right) \geq \left(AUC \times a_{\bar{D}_A} - l_{\bar{D}_A}\right)^T I_{\bar{D}_A}^{-1} \left(AUC \times a_{\bar{D}_A} - l_{\bar{D}_A}\right).
$$

Therefore, the inequation (A.8) is hold. Note the equality holds when

$$
\left(AUC \times a_{D_A} - l_{D_A}\right)^T I_{D_A}^{-1} e_D = \left(AUC \times m_D - u_D\right) \text{ and } \left(AUC \times a_{\bar{D}_A} - l_{\bar{D}_A}\right)^T I_{\bar{D}_A}^{-1} e_{\bar{D}} = \left(AUC \times m_{\bar{D}} - u_{\bar{D}}\right), \tag{A.9}
$$

otherwise the efficiency would always be improved when the auxiliary variables are added in the model.

Appendix C. Proof of Theorem 3

For simplicity, we represent $\widehat{pAUC}_{IPW}(t_0, t_1)$ and $pAUC(t_0, t_1)$ as $\widehat{pAUC}(\hat{p}, \hat{q})$ and $pAUC$, respectively. We note that

$$
\begin{aligned}
& \sqrt{N}\left[\widehat{pAUC}(\hat{p}, \hat{q}) - pAUC\right] \\
= & \sqrt{N}\left[\widehat{pAUC}(\hat{p}, \hat{q}) - \widehat{pAUC}(\hat{p}, q)\right] + \sqrt{N}\left[\widehat{pAUC}(\hat{p}, q) - pAUC\right] \\
= & I + II.
\end{aligned}
\tag{A.10}
$$

Since $\sqrt{N}\left[\hat{F}_D(u) - F_D(u)\right]$ and $\sqrt{N}\left[\hat{F}_{\bar{D}}(u) - F_{\bar{D}}(u)\right]$ converges to a Gaussian process, using Clivenko-Cantelli theorem, $\sup_x |\hat{F}_D(u) - F_D(u)| \to 0$ and $\sup_x |\hat{F}_{\bar{D}}(u) - F_{\bar{D}}(u)| \to 0$ a.s. For the first term I in

(A.10), we have

$$
\begin{aligned}
I &= \sqrt{N}\left[\widehat{pAUC}(\hat{p},\hat{q})-\widehat{pAUC}(\hat{p},q)\right] \\
&= \sqrt{N}\left[\int_{\hat{q}_1}^{\hat{q}_0}\left(1-\hat{F}_D(u)\right)d\hat{F}_{\bar{D}}(u)-\int_{q_1}^{q_0}\left(1-\hat{F}_D(u)\right)d\hat{F}_{\bar{D}}(u)\right] \\
&= \sqrt{N}\left[\int_{\hat{q}_1}^{\hat{q}_0}(1-F_D(u))\,d\hat{F}_{\bar{D}}(u)-\int_{q_1}^{q_0}(1-F_D(u))\,d\hat{F}_{\bar{D}}(u)\right]+o_p(1) \\
&= \sqrt{N}\left[\int_{\hat{q}_1}^{\hat{q}_0}(1-F_D(u))\,d(F_{\bar{D}}(u))-\int_{q_1}^{q_0}(1-F_D(u))\,d(F_{\bar{D}}(u))\right] \\
&\quad+\sqrt{N}\left[\int_{\hat{q}_1}^{\hat{q}_0}(1-F_D(u))\,d\left(\hat{F}_{\bar{D}}(u)-F_{\bar{D}}(u)\right)-\int_{q_1}^{q_0}(1-F_D(u))\,d\left(\hat{F}_{\bar{D}}(u)-F_{\bar{D}}(u)\right)\right] \\
&= \sqrt{N}\left[\int_{\hat{q}_1}^{\hat{q}_0}(1-F_D(u))\,d(F_{\bar{D}}(u))-\int_{q_1}^{q_0}(1-F_D(u))\,d(F_{\bar{D}}(u))\right]+o_p(1) \\
&= \sqrt{N}\left[\int_{\hat{q}_1}^{\hat{q}_0}(1-F_D(u))\,f_{\bar{D}}(u)\,d(u)-\int_{q_1}^{\hat{q}_0}(1-F_D(u))\,f_{\bar{D}}(u)\,d(u)\right] \\
&\quad+\sqrt{N}\left[\int_{q_1}^{\hat{q}_0}(1-F_D(u))\,f_{\bar{D}}(u)\,d(u)-\int_{q_1}^{q_0}(1-F_D(u))\,f_{\bar{D}}(u)\,d(u)\right]+o_p(1) \\
&= \sqrt{N}\int_{\hat{q}_1}^{q_1}(1-F_D(u))\,f_{\bar{D}}(u)\,d(u)+\sqrt{N}\int_{q_0}^{\hat{q}_0}(1-F_D(u))\,f_{\bar{D}}(u)\,d(u)+o_p(1) \\
&= -\sqrt{N}(1-F_D(q_1))\,f_{\bar{D}}(q_1)(\hat{q}_1-q_1)+\sqrt{N}(1-F_D(q_0))\,f_{\bar{D}}(q_0)(\hat{q}_0-q_0)+o_p(1).
\end{aligned}
$$

Using the Bahadur representation,

$$
\hat{q}_k-q_k=-\frac{1}{f_{\bar{D}}(q_k)}\left[\hat{F}_{IPW,\bar{D}}(q_k)-(1-t_k)\right]+o_p\left(1/\sqrt{N_{\bar{D}}}\right).
$$

Therefore,

$$
\begin{aligned}
I &\approx (1-F_D(q_1))\sqrt{N}\left[\hat{F}_{IPW,\bar{D}}(q_1)-(1-t_1)\right]-(1-F_D(q_0))\sqrt{N}\left[\hat{F}_{IPW,\bar{D}}(q_0)-(1-t_0)\right] \\
&= (1-F_D(q_1))\sqrt{N}\left[\hat{F}_{IPW,\bar{D}}(q_1)-F_{\bar{D}}(q_1)\right]-(1-F_D(q_0))\sqrt{N}\left[\hat{F}_{IPW,\bar{D}}(q_0)-F_{\bar{D}}(q_0)\right]
\end{aligned}
$$

In addition, according to the proof of Lemma A.1, we have

$$
\sqrt{N_{\bar{D}}}\left[\tilde{F}_{IPW,\bar{D}}(c)-F_{\bar{D}}(c)\right]=\tilde{U}(c)+\tilde{M}(c)+o_p(1),
$$

$$
\sqrt{N_{\bar{D}}}\,[\beta-1]=\sqrt{N_{\bar{D}}}\left[\sum_{j=1}^{N_D}\left(\delta_{Dj}/\hat{p}_{Dj}\right)/N_{\bar{D}}-1\right]=\tilde{E}^*+\tilde{C}^*+o_p(1)
$$

where

$$\begin{aligned}
\tilde{U}(c) &= -\sqrt{N_{\bar{D}}}E\left[I(X_{\bar{D}}<c)\frac{1}{p_{\bar{D}}}\frac{\partial p_{\bar{D}}}{\partial \theta_{\bar{D}}}\right]\left(\hat{\theta}_{\bar{D}}-\theta_{\bar{D}}\right),\\
\tilde{M}(c) &= \sqrt{N_{\bar{D}}}\left[\frac{1}{N_{\bar{D}}}\sum_{j=1}^{N_{\bar{D}}}\frac{\delta_{\bar{D}j}}{p_{\bar{D}j}}I\left(X_{\bar{D}j}<c\right)-E\left[I(X_{\bar{D}}<c)\right]\right],\\
\tilde{E}^{*} &= \sqrt{N_{\bar{D}}}\left(\sum_{j=1}^{N_{\bar{D}}}\frac{\delta_{\bar{D}j}}{p_{\bar{D}j}N_{\bar{D}}}-1\right),\\
\tilde{C}^{*} &= -\sqrt{N_{\bar{D}}}E\left(\frac{1}{p_{\bar{D}}}\frac{\partial p_{\bar{D}}}{\partial \theta_{\bar{D}}}\right)\left(\hat{\theta}_{\bar{D}}-\theta_{\bar{D}}\right).
\end{aligned}$$

Then

$$\begin{aligned}
&\sqrt{N_{\bar{D}}}\left[\hat{F}_{IPW,\bar{D}}(c)-F_{\bar{D}}(c)\right]=\sqrt{N_{\bar{D}}}\left[\tilde{F}_{IPW,\bar{D}}(c)/\beta-F_{\bar{D}}(c)\right]\\
=\;&\sqrt{N_{\bar{D}}}\left[\tilde{F}_{IPW,\bar{D}}(c)-F_{\bar{D}}(c)\right]+\sqrt{N_{\bar{D}}}\left[\tilde{F}_{IPW,\bar{D}}(c)/\beta-\tilde{F}_{IPW,\bar{D}}(c)\right]\\
=\;&\sqrt{N_{\bar{D}}}\left[\tilde{F}_{IPW,\bar{D}}(c)-F_{\bar{D}}(c)\right]+\sqrt{N_{\bar{D}}}\left[F_{\bar{D}}(c)/\beta-F_{\bar{D}}(c)\right]+o_p(1)\\
=\;&\sqrt{N_{\bar{D}}}\left[\tilde{F}_{IPW,\bar{D}}(c)-F_{\bar{D}}(c)\right]-F_{\bar{D}}(c)\sqrt{N_{\bar{D}}}(\beta-1)+o_p(1)\\
=\;&\tilde{U}(c)+\tilde{M}(c)-F_{\bar{D}}(c)\left(\tilde{E}^{*}+\tilde{C}^{*}\right)+o_p(1).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
I &\approx (1-F_D(q_1))\frac{\sqrt{N}}{\sqrt{N_{\bar{D}}}}\left[\tilde{U}(q_1)+\tilde{M}(q_1)-F_{\bar{D}}(q_1)\left(\tilde{E}^{*}+\tilde{C}^{*}\right)\right]\\
&\quad-(1-F_D(q_0))\frac{\sqrt{N}}{\sqrt{N_{\bar{D}}}}\left[\tilde{U}(q_0)+\tilde{M}(q_0)-F_{\bar{D}}(q_0)\left(\tilde{E}^{*}+\tilde{C}^{*}\right)\right]\\
&= I_{(1)}+I_{(2)}. \qquad\qquad \text{(A.11)}
\end{aligned}$$

We then look at the second term II in (A.10). Define

$$\widetilde{pAUC}(\hat{p},q)=\left[\sum_{i=1}^{N_D}\sum_{j=1}^{N_{\bar{D}}}\frac{\delta_{Di}}{\hat{p}_{Di}}\frac{\delta_{\bar{D}j}}{\hat{p}_{\bar{D}j}}I\left(X_{Di}>X_{\bar{D}j},X_{\bar{D}j}\in(q_1,q_0)\right)\right]/(N_D N_{\bar{D}}),$$

then we have

$$\begin{aligned}
&\sqrt{N}\left[\widetilde{pAUC}(\hat{p},q)-pAUC\right]\\
=\;&\sqrt{N}\left[\widetilde{pAUC}(\hat{p},q)-\widetilde{pAUC}(p,q)\right]+\sqrt{N}\left[\widetilde{pAUC}(p,q)-pAUC\right]\\
=\;&\sqrt{N}\frac{\partial E\left[\widetilde{pAUC}(p,q)\right]}{\partial \theta_D}\left(\hat{\theta}_D-\theta_D\right)+\sqrt{N}\frac{\partial E\left[\widetilde{pAUC}(p,q)\right]}{\partial \theta_{\bar{D}}}\left(\hat{\theta}_{\bar{D}}-\theta_{\bar{D}}\right)
\end{aligned}$$

$$
\begin{aligned}
& +\sqrt{N}\left\{\sum_{i=1}^{N_D}\frac{\delta_{Di}}{p_{Di}N_D}P\left(X_{\bar{D}}<X_{Di},X_{\bar{D}}\in(q_1,q_0)\right)-E\left[P\left(X_D>X_{\bar{D}},X_{\bar{D}}\in(q_1,q_0)\right)\right]\right\} \\
& +\sqrt{N}\left\{\sum_{j=1}^{N_{\bar{D}}}\frac{\delta_{\bar{D}j}}{p_{\bar{D}j}N_D}P\left(X_D>X_{\bar{D}j},X_{\bar{D}j}\in(q_1,q_0)\right)-E\left[P\left(X_D>X_{\bar{D}},X_{\bar{D}}\in(q_1,q_0)\right)\right]\right\} \\
& +o_p(1) \\
= \; & B+\tilde{B}+A+\tilde{A}+o_p(1),
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \sqrt{N}\left\{\sum_{i=1}^{N_D}\frac{\delta_{Di}}{p_{Di}N_D}P\left(X_{\bar{D}}<X_{Di},X_{\bar{D}}\in(q_1,q_0)\right)-E\left[P\left(X_D>X_{\bar{D}},X_{\bar{D}}\in(q_1,q_0)\right)\right]\right\}, \\
\tilde{A} &= \sqrt{N}\left\{\sum_{j=1}^{N_{\bar{D}}}\frac{\delta_{\bar{D}j}}{p_{\bar{D}j}N_D}P\left(X_D>X_{\bar{D}j},X_{\bar{D}j}\in(q_1,q_0)\right)-E\left[P\left(X_D>X_{\bar{D}},X_{\bar{D}}\in(q_1,q_0)\right)\right]\right\}, \\
B &= \sqrt{N}\frac{\partial E\left[\widetilde{pAUC}(p)\right]}{\partial\theta_D}\left(\hat{\theta}_D-\theta_D\right)=-\sqrt{N}E\left[I\left(X_D>X_{\bar{D}},X_{\bar{D}}\in(q_1,q_0)\right)\frac{1}{p_D}\frac{\partial p_D}{\partial\theta_D}\right]\left(\hat{\theta}_D-\theta_D\right), \\
\tilde{B} &= \sqrt{N}\frac{\partial E\left[\widetilde{pAUC}(p)\right]}{\partial\theta_{\bar{D}}}\left(\hat{\theta}_{\bar{D}}-\theta_{\bar{D}}\right) \\
&= -\sqrt{N}E\left[I\left(X_D>X_{\bar{D}},X_{\bar{D}}\in(q_1,q_0)\right)\frac{1}{p_{\bar{D}}}\frac{\partial p_{\bar{D}}}{\partial\theta_{\bar{D}}}\right]\left(\hat{\theta}_{\bar{D}}-\theta_{\bar{D}}\right).
\end{aligned}
$$

let $\eta=\frac{1}{N_DN_{\bar{D}}}\sum_{i=1}^{N_D}\sum_{j=1}^{N_{\bar{D}}}\frac{\delta_{Di}}{\hat{p}_{Di}}\frac{\delta_{\bar{D}j}}{\hat{p}_{\bar{D}j}}$ and note that

$$
\begin{aligned}
& \sqrt{N}(\eta-1) \\
= \; & \sqrt{N}\left[\frac{1}{N_DN_{\bar{D}}}\sum_{i=1}^{N_D}\sum_{j=1}^{N_{\bar{D}}}\frac{\delta_{Di}\delta_{\bar{D}j}}{\hat{p}_{Di}\hat{p}_{\bar{D}j}}-\frac{1}{N_DN_{\bar{D}}}\sum_{i=1}^{N_D}\sum_{j=1}^{N_{\bar{D}}}\frac{\delta_{Di}\delta_{\bar{D}j}}{p_{Di}p_{\bar{D}j}}\right]+\sqrt{N}\left(\frac{1}{N_DN_{\bar{D}}}\sum_{i=1}^{N_D}\sum_{j=1}^{N_{\bar{D}}}\frac{\delta_{Di}\delta_{\bar{D}j}}{p_{Di}p_{\bar{D}j}}-1\right) \\
= \; & -\sqrt{N}\frac{1}{N_DN_{\bar{D}}}E\left(\sum_{i=1}^{N_D}\sum_{j=1}^{N_{\bar{D}}}\frac{\delta_{Di}\delta_{\bar{D}j}}{p_{Di}^2p_{\bar{D}j}}\frac{\partial p_D}{\partial\theta_D}\right)\left(\hat{\theta}_D-\theta_D\right)-\sqrt{N}\frac{1}{N_DN_{\bar{D}}}E\left(\sum_{i=1}^{N_D}\sum_{j=1}^{N_{\bar{D}}}\frac{\delta_{Di}\delta_{\bar{D}j}}{p_{Di}p_{\bar{D}j}^2}\frac{\partial p_{\bar{D}}}{\partial\theta_{\bar{D}}}\right)\left(\hat{\theta}_{\bar{D}}-\theta_{\bar{D}}\right) \\
& +\sqrt{N}\left[\left(\sum_{i=1}^{N_D}\frac{\delta_{Di}}{p_{Di}N_D}-1\right)+\left(\sum_{j=1}^{N_{\bar{D}}}\frac{\delta_{\bar{D}j}}{p_{\bar{D}j}N_{\bar{D}}}-1\right)\right]+o_p(1) \\
= \; & C+\tilde{C}+E+\tilde{E}+o_p(1),
\end{aligned}
$$

where

$$
\begin{aligned}
C &= -\frac{\sqrt{N}}{N_DN_{\bar{D}}}E\left(\sum_{i=1}^{N_D}\sum_{j=1}^{N_{\bar{D}}}\frac{\delta_{Di}\delta_{\bar{D}j}}{p_{Di}^2p_{\bar{D}j}}\frac{\partial p_D}{\partial\theta_D}\right)\left(\hat{\theta}_D-\theta_D\right)=-\sqrt{N}E\left(\frac{1}{p_D}\frac{\partial p_D}{\partial\theta_D}\right)\left(\hat{\theta}_D-\theta_D\right), \\
\tilde{C} &= -\frac{\sqrt{N}}{N_DN_{\bar{D}}}E\left(\sum_{i=1}^{N_D}\sum_{j=1}^{N_{\bar{D}}}\frac{\delta_{Di}\delta_{\bar{D}j}}{p_{Di}p_{\bar{D}j}^2}\frac{\partial p_{\bar{D}}}{\partial\theta_{\bar{D}}}\right)\left(\hat{\theta}_{\bar{D}}-\theta_{\bar{D}}\right)=-\sqrt{N}E\left(\frac{1}{p_{\bar{D}}}\frac{\partial p_{\bar{D}}}{\partial\theta_{\bar{D}}}\right)\left(\hat{\theta}_{\bar{D}}-\theta_{\bar{D}}\right), \\
E &= \sqrt{N}\left(\sum_{i=1}^{N_D}\frac{\delta_{Di}}{p_{Di}N_D}-1\right),\text{ and }\tilde{E}=\sqrt{N}\left(\sum_{j=1}^{N_{\bar{D}}}\frac{\delta_{\bar{D}j}}{p_{\bar{D}j}N_{\bar{D}}}-1\right).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
& II = \sqrt{N}\left[\widehat{pAUC}(\hat{p},q) - pAUC\right] \\
= & \sqrt{N}\left[\widetilde{pAUC}(\hat{p},q)/\eta - pAUC\right] \\
= & \sqrt{N}\left[\widetilde{pAUC}(\hat{p},q) - pAUC\right] + \sqrt{N}\left[\widetilde{pAUC}(\hat{p},q)/\eta - \widetilde{pAUC}(\hat{p},q)\right] \\
= & \sqrt{N}\left[\widetilde{pAUC}(\hat{p},q) - pAUC\right] - pAUC \times \sqrt{N}(\eta - 1) + o_p(1) \\
= & A + \tilde{A} + B + \tilde{B} - pAUC \times \left(C + \tilde{C} + E + \tilde{E}\right) + o_p(1). \qquad \text{(A.12)}
\end{aligned}
$$

It follows from (A.10), (A.11) and (A.12) that

$$
\begin{aligned}
& \sqrt{N}\left[\widehat{pAUC}(\hat{p},\hat{q}) - pAUC\right] \\
\approx & I_{(1)} + I_{(2)} + II \\
= & (1 - F_D(q_1))\frac{\sqrt{N}}{\sqrt{N_{\bar{D}}}}\left[\tilde{U}(q_1) + \tilde{M}(q_1) - F_{\bar{D}}(q_1)\left(\tilde{E}^* + \tilde{C}^*\right)\right] \\
& -(1 - F_D(q_0))\frac{\sqrt{N}}{\sqrt{N_{\bar{D}}}}\left[\tilde{U}(q_0) + \tilde{M}(q_0) - F_{\bar{D}}(q_0)\left(\tilde{E}^* + \tilde{C}^*\right)\right] \\
& + A + \tilde{A} + B + \tilde{B} - pAUC \times \left(C + \tilde{C} + E + \tilde{E}\right)
\end{aligned}
$$

Therefore, $\sqrt{N}\left[\widehat{pAUC}(\hat{p},\hat{q}) - pAUC\right]$ converges to a normal random variable with mean 0 and variance

$$
\begin{aligned}
\sigma^2_{pAUC,IPW} &= Var\left(I_{(1)} + I_{(2)} + II\right) \\
&= VarI_{(1)} + VarI_{(2)} + VarII \\
&\quad + 2\left[Cov\left(I_{(1)}, I_{(2)}\right) + Cov\left(I_{(1)}, II\right) + Cov\left(I_{(2)}, II\right)\right] \qquad \text{(A.13)}
\end{aligned}
$$

as $N \to \infty$.

According to Lemma A.1,

$$
\begin{aligned}
VarI_{(1)} &= \frac{1}{1-\lambda}\left[1 - F_D(q_1)\right]^2 \sigma^2_{\bar{D},IPW}(q_1), \\
VarI_{(2)} &= \frac{1}{1-\lambda}\left[1 - F_D(q_0)\right]^2 \sigma^2_{\bar{D},IPW}(q_0), \qquad \text{(A.14)}
\end{aligned}
$$

where $\sigma^2_{\bar{D},IPW}(\cdot)$ is given in Lemma A.1. For the term II,

$$
Var(II) = Var\left(A + \tilde{A} + B + \tilde{B}\right)
$$

$$-2pAUC \times Cov\left(A+\tilde{A}+B+\tilde{B}, C+\tilde{C}+E+\tilde{E}\right) + pAUC^2 \times Var\left(C+\tilde{C}+E+\tilde{E}\right)$$
$$= II_1 + II_2 + II_3, \tag{A.15}$$

where

$$\begin{aligned}
II_1 &= Var(A) + Var\left(\tilde{A}\right) - 2pAUC \times \left[Cov(A,E) + Cov\left(\tilde{A},\tilde{E}\right)\right] + pAUC^2 \times \left[Var(E) + Var\left(\tilde{E}\right)\right], \\
II_2 &= Var(B) + 2Cov(A,B) - 2pAUC \times [Cov(A,C) + Cov(B,E) + Cov(B,C)] \\
&\quad + pAUC^2 \times [Var(C) + 2Cov(E,C)], \\
II_3 &= Var\left(\tilde{B}\right) + 2Cov\left(\tilde{A},\tilde{B}\right) - 2pAUC \times \left[Cov\left(\tilde{A},\tilde{C}\right) + Cov\left(\tilde{B},\tilde{E}\right) + Cov\left(\tilde{B},\tilde{C}\right)\right] \\
&\quad + pAUC^2 \times \left[Var\left(\tilde{C}\right) + 2Cov\left(\tilde{E},\tilde{C}\right)\right].
\end{aligned}$$

As $N_D/N \to \lambda$, we have

$$\begin{aligned}
Var(A) &= \frac{N}{N_D} Var\left[\frac{\delta_D}{p_D} P\left(X_{\bar{D}} < X_D, X_{\bar{D}} \in (q_1,q_0)\right)\right] \\
&= \frac{1}{\lambda}\left\{Var\left[P\left(X_{\bar{D}} < X_D, X_{\bar{D}} \in (q_1,q_0)\right)\right] + E\left[\left(\frac{1}{p_D} - 1\right) P^2\left(X_{\bar{D}} < X_D, X_{\bar{D}} \in (q_1,q_0)\right)\right]\right\}, \\
Var\left(\tilde{A}\right) &= \frac{N}{N_{\bar{D}}} Var\left[\frac{\delta_{\bar{D}}}{p_{\bar{D}}} P\left(X_{\bar{D}} < X_D, X_{\bar{D}} \in (q_1,q_0)\right)\right] \\
&= \frac{1}{1-\lambda}\left\{Var\left[P\left(X_D > X_{\bar{D}}, X_{\bar{D}} \in (q_1,q_0)\right)\right] + E\left[\left(\frac{1}{p_{\bar{D}}} - 1\right) P^2\left(X_D > X_{\bar{D}}, X_{\bar{D}} \in (q_1,q_0)\right)\right]\right\}, \\
Cov(A,E) &= \frac{N}{N_D} Cov\left[\frac{\delta_D}{p_D} P\left(X_{\bar{D}} < X_D, X_{\bar{D}} \in (q_1,q_0)\right), \frac{\delta_D}{p_D}\right] \\
&= \frac{1}{\lambda} E\left[\left(\frac{1}{p_D} - 1\right) P\left(X_{\bar{D}} < X_D, X_{\bar{D}} \in (q_1,q_0)\right)\right], \\
Cov\left(\tilde{A},\tilde{E}\right) &= \frac{N}{N_{\bar{D}}} Cov\left[\frac{\delta_{\bar{D}}}{p_{\bar{D}}} P\left(X_{\bar{D}} < X_D, X_{\bar{D}} \in (q_1,q_0)\right), \frac{\delta_{\bar{D}}}{p_{\bar{D}}}\right] \\
&= \frac{1}{1-\lambda} E\left[\left(\frac{1}{p_{\bar{D}}} - 1\right) P\left(X_D > X_{\bar{D}}, X_{\bar{D}} \in (q_1,q_0)\right)\right], \\
Var(E) &= \frac{N}{N_D} Var\left(\frac{\delta_D}{p_D}\right) = \frac{1}{\lambda} E\left(\frac{1}{p_D} - 1\right), \\
Var\left(\tilde{E}\right) &= \frac{N}{N_{\bar{D}}} Var\left(\frac{\delta_{\bar{D}}}{p_{\bar{D}}}\right) = \frac{1}{1-\lambda} E\left(\frac{1}{p_{\bar{D}}} - 1\right).
\end{aligned}$$

So

$$\begin{aligned}
& II_1 \\
= \; & \frac{1}{\lambda}\left\{Var\left[P\left(X_{\bar{D}} < X_D, X_{\bar{D}} \in (q_1,q_0)\right)\right] + E\left[\left(\frac{1}{p_D} - 1\right) P^2\left(X_{\bar{D}} < X_D, X_{\bar{D}} \in (q_1,q_0)\right)\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{1-\lambda}\left\{Var\left[P\left(X_D > X_{\bar{D}}, X_{\bar{D}} \in (q_1,q_0)\right)\right] + E\left[\left(\frac{1}{p_{\bar{D}}}-1\right)P^2\left(X_D > X_{\bar{D}}, X_{\bar{D}} \in (q_1,q_0)\right)\right]\right\} \\
&-\frac{pAUC}{\lambda}\left\{2E\left[\left(\frac{1}{p_D}-1\right)P\left(X_{\bar{D}} < X_D, X_{\bar{D}} \in (q_1,q_0)\right)\right] - pAUCE\left(\frac{1}{p_D}-1\right)\right\} \\
&-\frac{pAUC}{1-\lambda}\left\{2E\left[\left(\frac{1}{p_{\bar{D}}}-1\right)P\left(X_D > X_{\bar{D}}, X_{\bar{D}} \in (q_1,q_0)\right)\right] - pAUCE\left(\frac{1}{p_{\bar{D}}}-1\right)\right\} \\
= &\ \frac{1}{\lambda}\left\{Var\left[P\left(X_{\bar{D}} < X_D, X_{\bar{D}} \in (q_1,q_0)\right)\right] + E\left[\left(\frac{1}{p_D}-1\right)P^2\left(X_{\bar{D}} < X_D, X_{\bar{D}} \in (q_1,q_0)\right)\right]\right\} \\
&-\frac{1}{1-\lambda}\left\{Var\left[P\left(X_D > X_{\bar{D}}, X_{\bar{D}} \in (q_1,q_0)\right)\right] + E\left[\left(\frac{1}{p_{\bar{D}}}-1\right)P^2\left(X_D > X_{\bar{D}}, X_{\bar{D}} \in (q_1,q_0)\right)\right]\right\} \\
&-\frac{pAUC}{\lambda}\left\{E\left[\left(\frac{1}{p_D}-1\right)P\left(X_{\bar{D}} < X_D, X_{\bar{D}} \in (q_1,q_0)\right)\right] + Cov\left[\frac{1}{p_D}-1, P\left(X_{\bar{D}} < X_D, X_{\bar{D}} \in (q_1,q_0)\right)\right]\right\} \\
&-\frac{pAUC}{1-\lambda}\left\{E\left[\left(\frac{1}{p_{\bar{D}}}-1\right)P\left(X_D > X_{\bar{D}}, X_{\bar{D}} \in (q_1,q_0)\right)\right] + Cov\left[\frac{1}{p_{\bar{D}}}-1, P\left(X_D > X_{\bar{D}}, X_{\bar{D}} \in (q_1,q_0)\right)\right]\right\}.
\end{aligned}
$$

Similar, we can show that

$$
\begin{aligned}
II_2 &= -\frac{1}{\lambda}\left[pAUC \times a_D - s_D\right]^T I_D^{-1}\left[pAUC \times a_D - s_D\right], \\
II_3 &= -\frac{1}{1-\lambda}\left[pAUC \times a_{\bar{D}} - s_{\bar{D}}\right]^T I_{\bar{D}}^{-1}\left[pAUC \times a_{\bar{D}} - s_{\bar{D}}\right],
\end{aligned}
$$

where $s_d = E\left[I\left(X_D > X_{\bar{D}}, X_{\bar{D}} \in (q_1,q_0)\right)(1/p_d)(\partial p_d/\partial \theta_d)\right]$, a_d are defined in Lemma A.1, for $d = D, \bar{D}$.

Then, $Cov\left(I_{(1)}, I_{(2)}\right), Cov\left(I_{(1)}, II\right)$ and $Cov\left(I_{(2)}, II\right)$ can be derived similarly, and we get

$$
\begin{aligned}
Cov\left(I_{(1)}, I_{(2)}\right) = &-\frac{1}{1-\lambda}\left[1-F_D(q_0)\right]\left[1-F_D(q_1)\right] \times \\
&\left\{Cov\left[\left(\frac{1}{p_{\bar{D}}}-1\right)I\left(X_{\bar{D}} < q_1\right), I\left(X_{\bar{D}} < q_0\right)\right]\right. \\
&+Cov\left[I\left(X_{\bar{D}} < q_0\right), I\left(X_{\bar{D}} < q_1\right)\right] - F_{\bar{D}}(q_1) \times Cov\left[\frac{1}{p_{\bar{D}}}-1, I\left(X_{\bar{D}} < q_0\right)\right] \\
&\left.-\left[SPE(q_0) \times a_{\bar{D}} - b_{\bar{D}}(q_0)\right]^T I_{\bar{D}}^{-1}\left[SPE(q_1) \times a_{\bar{D}} - b_{\bar{D}}(q_1)\right]\right\},
\end{aligned}
\tag{A.16}
$$

$$
\begin{aligned}
Cov\left(I_{(1)}, II\right) = &\ \frac{1}{1-\lambda}\left[1-F_D(q_1)\right] \times \\
&\left\{Cov\left[\left(\frac{1}{p_{\bar{D}}}-1\right)P\left(X_{\bar{D}} < X_D, X_{\bar{D}} \in (q_1,q_0)\right), I\left(X_{\bar{D}} < q_1\right)\right]\right. \\
&+Cov\left[P\left(X_{\bar{D}} < X_D, X_{\bar{D}} \in (q_1,q_0)\right), I\left(X_{\bar{D}} < q_1\right)\right] \\
&-pAUC \times Cov\left[\left(\frac{1}{p_{\bar{D}}}-1\right), I\left(X_{\bar{D}} < q_1\right)\right]
\end{aligned}
$$

$$-\left[pAUC\left(t_0,t_1\right)\times a_{\bar{D}}-s_{\bar{D}}\right]^T I_{\bar{D}}^{-1}\left[SPE\left(q_1\right)\times a_{\bar{D}}-b_{\bar{D}}\left(q_1\right)\right]\Bigg\}, \quad (A.17)$$

and

$$\begin{aligned} Cov\left(I_{(2)},II\right) &= -\frac{1}{1-\lambda}\left[1-F_D\left(q_0\right)\right]\times \\ &\Bigg\{Cov\left[\left(\frac{1}{p_{\bar{D}}}-1\right)P\left(X_{\bar{D}}<X_D,X_{\bar{D}}\in\left(q_1,q_0\right)\right),I\left(X_{\bar{D}}<q_0\right)\right] \\ &+Cov\left[P\left(X_{\bar{D}}<X_D,X_{\bar{D}}\in\left(q_1,q_0\right)\right),I\left(X_{\bar{D}}<q_0\right)\right] \\ &-pAUC\times Cov\left[\left(\frac{1}{p_{\bar{D}}}-1\right),I\left(X_{\bar{D}}<q_0\right)\right] \\ &-\left[pAUC\left(t_0,t_1\right)\times a_{\bar{D}}-s_{\bar{D}}\right]^T I_{\bar{D}}^{-1}\left[SPE\left(q_1\right)\times a_{\bar{D}}-b_{\bar{D}}\left(q_0\right)\right]\Bigg\}, \quad (A.18) \end{aligned}$$

Finally, followed by equations (A.13), (A.14), (A.15), (A.16), (A.17) and (A.18), the proof is complete.

Appendix D. Results of simulation studies for Bernoulli sampling design

Table 1: Estimate, bias, variance, median of estimated variance ($Med(\widehat{Var})$) and coverage of 95% confidence interval (CI) of $ROC\,(t)$ estimator using the empirical method, IPW method with the estimated sampling probabilities $\hat{p}^{Str}$ and $\hat{p}^{Aux}$, for scenarios where $\rho_{XV^*} = 0.5$, $\rho_{XW} = 0.5$, $\rho_{WV^*} = 0.1$, $\mu_{X_{\bar{D}}} = 0$, and $\mu_{X_D} = 1$ in **Bernoulli sampling**.

Method	$n_D = n_{\bar{D}}$	Estimate	Bias ×100	Var×N	$Med(\widehat{Var}) \times N$	Coverage of 95% CI
			$ROC(0.1) = 0.3891$			
$\widehat{ROC}_{em}$	100	0.3414	-4.7713	11.677	11.742	93.5%
	250	0.3399	-4.9206	11.815	11.827	85.1%
	500	0.3396	-4.9533	11.760	11.991	73.2%
$\widehat{ROC}_{IPW}^{Str}$	100	0.3905	0.1341	11.813	11.235	95.6%
	250	0.3887	-0.0473	12.192	11.753	95.0%
	500	0.3896	0.0431	12.194	11.892	95.4%
$\widehat{ROC}_{IPW}^{Aux}$	100	0.3907	0.1603	10.604	9.858	95.0%
	250	0.3890	-0.0113	10.606	10.312	95.1%
	500	0.3892	0.0072	10.779	10.434	95.0%
			$ROC(0.2) = 0.5629$			
$\widehat{ROC}_{em}$	100	0.5060	-5.6969	10.640	11.073	89.3%
	250	0.5054	-5.7500	10.934	11.234	77.7%
	500	0.5058	-5.7166	10.888	11.186	60.7%
$\widehat{ROC}_{IPW}^{Str}$	100	0.5611	-0.1841	10.400	10.285	95.5%
	250	0.5612	-0.1696	10.624	10.465	95.3%
	500	0.5623	-0.0610	10.241	10.540	95.3%
$\widehat{ROC}_{IPW}^{Aux}$	100	0.5611	-0.1795	8.958	8.827	95.4%
	250	0.5613	-0.1626	9.124	8.941	95.3%
	500	0.5622	-0.0709	8.743	8.998	95.5%
			$ROC(0.5) = 0.8413$			
$\widehat{ROC}_{em}$	100	0.7983	-4.3073	5.548	5.800	85.0%
	250	0.7976	-4.3713	5.471	5.681	70.2%
	500	0.7989	-4.2402	5.479	5.653	52.2%
$\widehat{ROC}_{IPW}^{Str}$	100	0.8389	-0.2423	4.784	4.550	96.0%
	250	0.8388	-0.2592	4.870	4.620	94.6%
	500	0.8399	-0.1423	4.520	4.621	95.2%
$\widehat{ROC}_{IPW}^{Aux}$	2000	0.8392	-0.2159	4.206	3.952	95.5%
	250	0.8390	-0.2348	4.270	4.039	95.0%
	500	0.8399	-0.1477	3.974	4.029	95.0%

Table 2: Estimate, bias, variance, median of estimated variance ($Med(\widehat{Var})$) and coverage of 95% confidence interval (CI) of AUC and $pAUC(t_0, t_1)$ estimators using the empirical method, IPW method with the estimated sampling probabilities $\hat{p}^{Str}$ and $\hat{p}^{Aux}$, for scenarios where $\rho_{XV^*} = 0.5$, $\rho_{XW} = 0.5$, $\rho_{WV^*} = 0.1$, $\mu_{X_{\bar{D}}} = 0$, and $\mu_{X_D} = 1$ in **Bernoulli sampling**.

Method	$n_D = n_{\bar{D}}$	Estimate	Bias×100	Var×N	$Med(\widehat{Var}) \times N$	Coverage of 95% CI
			$AUC = 0.7602$			
$\widehat{AUC}_{em}$	100	0.7273	-3.2975	2.4736	2.4664	83.9%
	250	0.7267	-3.3503	2.3962	2.4739	65.7%
	500	0.7269	-3.3387	2.5145	2.4814	41.1%
$\widehat{AUC}_{IPW}^{Str}$	100	0.7611	0.0842	2.1561	2.0903	94.7%
	5000	0.7606	0.0310	2.1307	2.0985	94.8%
	10000	0.7606	0.0373	2.2022	2.0965	94.3%
$\widehat{AUC}_{IPW}^{Aux}$	100	0.7607	0.0443	1.7517	1.6180	94.3%
	250	0.7606	0.0392	1.6479	1.6144	95.0%
	500	0.7605	0.0285	1.7007	1.6103	94.3%
			$pAUC(0, 0.1) = 0.0244$			
$\widehat{pAUC}_{em}$	100	0.0200	-0.4393	0.0687	0.0679	94.6%
	250	0.0203	-0.4042	0.0685	0.0699	87.7%
	500	0.0205	-0.3868	0.0690	0.0709	76.5%
$\widehat{pAUC}_{IPW}^{Str}$	100	0.0230	-0.1315	0.0724	0.0645	94.2%
	250	0.0237	-0.0697	0.0720	0.0682	94.6%
	500	0.0239	-0.0422	0.0724	0.0698	94.4%
$\widehat{pAUC}_{IPW}^{Aux}$	100	0.0230	-0.1331	0.0658	0.0618	95.0%
	250	0.0237	-0.0698	0.0640	0.0646	95.7%
	500	0.0239	-0.0444	0.0646	0.0658	95.5%
			$pAUC(0, 0.2) = 0.0726$			
$\widehat{pAUC}_{em}$	100	0.0620	-1.0550	0.2928	0.2972	91.3%
	250	0.0627	-0.9944	0.2961	0.2985	80.5%
	500	0.0630	-0.9611	0.2953	0.3027	64.1%
$\widehat{pAUC}_{IPW}^{Str}$	100	0.0700	-0.2636	0.2895	0.2692	95.0%
	250	0.0711	-0.1505	0.2989	0.2810	94.5%
	500	0.0717	-0.0912	0.2946	0.2852	95.1%
$\widehat{pAUC}_{IPW}^{Aux}$	100	0.0699	-0.2716	0.2489	0.2535	95.9%
	250	0.0711	-0.1485	0.2510	0.2616	95.9%
	500	0.0716	-0.0957	0.2495	0.2646	96.3%

Table 3: Efficiency comparison of the IPW $ROC(t)$, AUC and $pAUC(t_0, t_1)$ estimators with three different types of estimated sampling weights, for scenarios with varying μ_{X_D} (the population mean of the biomarker X among cases), where $n_D = n_{\bar{D}} = 500$, $\rho_{XV^*} = 0.3$, $\rho_{WV^*} = 0$, and $\mu_{X_{\bar{D}}} = 0$.

			Bernoulli Sampling				Finite-population stratified sampling			
			ρ_{XW}				ρ_{XW}			
μ_{X_D}		Parameter	0.1	0.3	0.5	0.7	0.1	0.3	0.5	0.7
0	$\hat{p}^{Aux}$ vs. $\hat{p}^{Str}$	$ROC(0.1)$	1.002	1.029	1.080	1.135	1.014	1.024	1.085	1.141
		$ROC(0.2)$	1.017	1.049	1.101	1.232	1.018	1.045	1.120	1.245
		AUC	1.034	1.104	1.241	1.513	1.042	1.106	1.248	1.549
		$pAUC(0, 0.1)$	1.008	1.027	1.084	1.120	1.019	1.022	1.087	1.133
		$pAUC(0, 0.2)$	1.023	1.059	1.101	1.220	1.026	1.043	1.112	1.231
0.6	$\hat{p}^{Aux}$ vs. $\hat{p}^{Str}$	$ROC(0.1)$	1.006	1.045	1.104	1.182	1.007	1.025	1.102	1.166
		$ROC(0.2)$	1.023	1.065	1.123	1.261	1.017	1.061	1.134	1.258
		AUC	1.035	1.105	1.242	1.511	1.039	1.104	1.241	1.537
		$pAUC(0, 0.1)$	1.009	1.037	1.100	1.170	1.021	1.028	1.107	1.181
		$pAUC(0, 0.2)$	1.029	1.065	1.131	1.272	1.033	1.046	1.134	1.272
1.5	$\hat{p}^{Aux}$ vs. $\hat{p}^{Str}$	$ROC(0.1)$	1.004	1.043	1.106	1.181	1.003	1.031	1.109	1.186
		$ROC(0.2)$	1.012	1.054	1.121	1.245	1.017	1.048	1.128	1.252
		AUC	1.028	1.094	1.204	1.411	1.028	1.088	1.192	1.424
		$pAUC(0, 0.1)$	1.001	1.041	1.112	1.167	1.021	1.025	1.111	1.170
		$pAUC(0, 0.2)$	1.024	1.055	1.141	1.253	1.036	1.039	1.127	1.254

Appendix E. Results for prostate cancer study example

Table 4: Prostate cancer study example: Estimate, variance, 95% confidence interval (CI) and corresponding length of 95% CI for $ROC(t)$, AUC, and $pAUC(t_0, t_1)$ estimators using the empirical method and IPW methods with the estimated sampling probabilities $\hat{p}^{Str}$ and $\hat{p}^{Aux}$.

	Method	Est	Var×N	95% CI	Length of 95% CI
$ROC(0.1)$	Empirical	0.250	1,859	(0.134, 0.366)	0.232
	IPW with $\hat{p}^{Str}$	0.317	1.704	(0.206, 0.428)	0.222
	IPW with $\hat{p}^{Aux}$	0.315	1.664	(0.206, 0.425)	0.219
$ROC(0.2)$	Empirical	0.367	2.371	(0.236, 0.497)	0.262
	IPW with $\hat{p}^{Str}$	0.425	1.935	(0.307, 0.543)	0.236
	IPW with $\hat{p}^{Aux}$	0.423	1.853	(0.307, 0.539)	0.231
AUC	Empirical	0.647	0.645	(0.579, 0.715)	0.136
	IPW with $\hat{p}^{Str}$	0.672	0.570	(0.608, 0.736)	0.128
	IPW with $\hat{p}^{Aux}$	0.673	0.541	(0.611, 0.736)	0.125
$pAUC(0, 0.1)$	Empirical	0.010	0.011	(0.001, 0.019)	0.018
	IPW with $\hat{p}^{Str}$	0.018	0.013	(0.009, 0.028)	0.019
	IPW with $\hat{p}^{Aux}$	0.017	0.013	(0.008, 0.027)	0.019
$pAUC(0, 0.2)$	Empirical	0.047	0.045	(0.029, 0.065)	0.036
	IPW with $\hat{p}^{Str}$	0.060	0.044	(0.042, 0.078)	0.036
	IPW with $\hat{p}^{Aux}$	0.060	0.044	(0.042, 0.078)	0.036

Appendix F. Results for using the estimated sampling probabilities based on the model including additional variables uncorrelated with the biomarker

We here adopt the same simulation setting for the finite-population stratified sampling design as described in Section 3 of the main text. In addition, we further independently generate five more variables $W_1^*, \ldots, W_5^*$ from the standard normal distribution; each of them is independent of disease outcome D and also uncorrelated with the biomarker X, auxiliary variables V^* and W.

For each generated data set, we estimate the points on the ROC curve, the AUC, and the partial AUC using the proposed IPW method with three different types of estimated sampling probabilities for cases/controls: $\hat{p}_D^{Str}/\hat{p}_{\bar{D}}^{Str}$ (conditional on V only), $\hat{p}_D^{Aux}/\hat{p}_{\bar{D}}^{Aux}$ (based on a logistic regression of sampling probability conditional on V, V^* and W), and $\hat{p}_D^{Aux^*}/\hat{p}_{\bar{D}}^{Aux^*}$ (based on a logistic regression of sampling probability conditional on V, V^*, W and $W_1^*, \ldots, W_5^*$). The simulation results are presented in the following Table 5.

Table 5: Estimate, bias, variance, and coverage of 95% confidence interval (CI) of $ROC(t)$, AUC and $pAUC\,(t_0, t_1)$ estimators using the IPW method with the estimated sampling probabilities $\hat{p}^{Str}$, $\hat{p}^{Aux}$, and $\hat{p}^{Aux^*}$ for scenarios where $\rho_{XV^*} = 0.5$, $\rho_{XW} = 0.5$, and $\rho_{WV^*} = 0.1$ in finite-population stratified sampling.

Method	$n_D = n_{\bar{D}}$	Estimate	Bias×100	Var×N	Coverage of 95% CI
		$ROC(0.1) = 0.3891$			
$\widehat{ROC}_{IPW}^{Str}$	100	0.3901	0.0968	12.183	94.5%
	250	0.3882	-0.0961	12.064	95.0%
	500	0.3876	-0.1550	11.999	95.3%
$\widehat{ROC}_{IPW}^{Aux}$	100	0.3906	0.1475	11.006	94.5%
	250	0.3883	-0.0842	10.555	94.9%
	500	0.3873	-0.1842	10.415	95.2%
$\widehat{ROC}_{IPW}^{Aux^*}$	100	0.3907	0.1582	11.657	93.4%
	250	0.3883	-0.0872	10.784	94.4%
	500	0.3873	-0.1867	10.547	95.0%
		$ROC(0.2) = 0.5629$			
$\widehat{ROC}_{IPW}^{Str}$	100	0.5611	-0.1819	10.472	95.2%
	250	0.5607	-0.2263	10.409	95.0%
	500	0.5615	-0.1375	10.499	95.3%
$\widehat{ROC}_{IPW}^{Aux}$	100	0.5622	-0.0735	8.977	95.2%
	250	0.5608	-0.2110	8.904	95.3%
	500	0.5613	-0.1581	9.033	94.9%
$\widehat{ROC}_{IPW}^{Aux^*}$	100	0.5620	-0.0924	9.478	94.5%
	250	0.5607	-0.2223	9.035	95.1%
	500	0.5613	-0.1595	9.099	95.0%
		$AUC = 0.7602$			
$\widehat{AUC}_{IPW}^{Str}$	100	0.7596	-0.0639	2.180	94.4%
	250	0.7603	0.0085	2.035	95.3%
	500	0.7604	0.0103	2.116	94.9%
$\widehat{AUC}_{IPW}^{Aux}$	100	0.7601	-0.0110	1.705	94.2%
	250	0.7604	0.0175	1.560	95.4%

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Table 5 – *Continued from previous page*

Method	$n_D = n_{\bar{D}}$	Estimate	Bias×100	Var×N	Coverage of 95% CI
	500	0.7602	-0.0009	1.642	94.9%
$\widehat{AUC}_{IPW}^{Aux^*}$	100	0.7601	-0.0129	1.796	93.6%
	250	0.7604	0.0158	1.597	95.0%
	500	0.7602	-0.0012	1.659	94.8%
		$pAUC(0,0.1) = 0.0244$			
$\widehat{pAUC}_{IPW}^{Str}$	100	0.0231	-0.1307	0.069	94.5%
	250	0.0236	-0.0717	0.070	95.1%
	500	0.0238	-0.0597	0.072	94.8%
$\widehat{pAUC}_{IPW}^{Aux}$	100	0.0231	-0.1256	0.063	94.8%
	250	0.0236	-0.0720	0.061	95.3%
	500	0.0238	-0.0603	0.063	95.4%
$\widehat{pAUC}_{IPW}^{Aux^*}$	100	0.0231	-0.1287	0.066	94.5%
	250	0.0236	-0.0729	0.063	95.7%
	500	0.0238	-0.0597	0.064	95.2%
		$pAUC(0,0.2) = 0.0726$			
$\widehat{pAUC}_{IPW}^{Str}$	100	0.0699	-0.2680	0.289	94.6%
	250	0.0711	-0.1526	0.287	94.9%
	500	0.0714	-0.1186	0.295	94.3%
$\widehat{pAUC}_{IPW}^{Aux}$	100	0.0700	-0.2584	0.253	95.8%
	250	0.0711	-0.1500	0.244	96.1%
	500	0.0714	-0.1241	0.245	95.7%
$\widehat{pAUC}_{IPW}^{Aux^*}$	100	0.0699	-0.2681	0.262	95.4%
	250	0.0711	-0.1535	0.248	96.1%
	500	0.0713	-0.1264	0.248	95.6%

Appendix G. Results for using the estimated sampling probabilities based on misspecified models

Consider a binary disease outcome D with prevalence $P(D=1) = \lambda = 0.1$. Let X be a single biomarker. We consider two auxiliary variables V^* and W. Among the controls, (X, V^*, W) follows a multivariate normal distribution with zero mean and covariance Σ; among the cases, (X, V^*, W) has a multivariate normal distribution with mean $(1, 0.5, 0.5)^T$ and covariance Σ, where

$\Sigma = \begin{pmatrix} 1 & \rho_{XV^*} & \rho_{XW} \\ \rho_{XV^*} & 1 & \rho_{WV^*} \\ \rho_{XW} & \rho_{WV^*} & 1 \end{pmatrix}$ in which ρ_{XV^*}, ρ_{XW} and ρ_{WV^*} represent the correlations between X and V^*, X and W, W and V^*, respectively. Subjects are stratified into two strata based on the value of V^*. Here we consider the following two scenarios: In scenario (1), we set $\rho_{XV^*} = 0.5$, $\rho_{XW} = 0.5$, $\rho_{WV^*} = 0.1$, and the discrete stratum variable V_1 equals $V_1 = 1$ if $V^* \leq \Phi^{-1}(0.5)$, and $V_1 = 2$ if $V^* \geq \Phi^{-1}(0.5)$, where Φ is the CDF of the standard normal distribution; in scenario (2), we set $\rho_{XV^*} = 0.8$, $\rho_{XW} = 0.5$, $\rho_{WV^*} = 0.1$, and the discrete stratum variable V_2 equals $V_2 = 1$ if $\Phi^{-1}(0.1) \leq V^* \leq \Phi^{-1}(0.75)$, and $V_2 = 2$ otherwise.

In phase one, we randomly draw $N = 2000$, 5000, or 10000 subjects from the population whose disease status D and auxiliary variables V^* and W are measured. In phase two, subsamples are randomly sampled from the phase-one cohort and the biomarker X is measured. We here adopt the finite-population stratified sampling scheme. Firstly, $n_D = 0.5\lambda N$ cases are randomly sampled without replacement. Then, according to the sampling strata defined by V_1 or V_2, in each stratum equal numbers of controls as cases are randomly selected without replacement. With the phase-one sample size $N = 2000$, 5000, and 10000, the corresponding numbers of cases and controls sampled equal $n_D = n_{\bar{D}} = 100$, 250, and 500, respectively.

For estimation of the points on the ROC curve, the AUC, and the partial AUC using the proposed IPW method, we consider the estimated sampling probabilities for cases/controls based on two different misspecified models: $\hat{p}_D^{Mis1}/\hat{p}_{\bar{D}}^{Mis1}$ (based on a logistic regression of sampling probability conditional on V^* and W), and $\hat{p}_D^{Mis2}/\hat{p}_{\bar{D}}^{Mis2}$ (based on a logistic regression of sampling probability conditional on W only). The simulation results for scenario (1) and (2) are shown in Table 6 and Table 7, respectively.

Table 6: Estimate, bias, variance, and coverage of 95% confidence interval (CI) of $ROC(t)$, AUC and $pAUC\,(t_0, t_1)$ estimators using the IPW method with the estimated sampling probabilities $\hat{p}^{Mis1}$ and $\hat{p}^{Mis2}$, for scenario (1) with the discrete stratum variable V_1, $\rho_{XV^*} = 0.5$, $\rho_{XW} = 0.5$, and $\rho_{WV^*} = 0.1$ in finite-population stratified sampling.

Method	$n_D = n_{\bar{D}}$	Estimate	Bias×100	Var×N	Coverage of 95% CI
		$ROC(0.1) = 0.3891$			

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Table 6 – *Continued from previous page*

Method	$n_D = n_{\bar{D}}$	Estimate	Bias×100	Var×N	Coverage of 95% CI
$\widehat{ROC}_{IPW}^{Mis1}$	100	0.3885	-0.0634	10.183	95.2%
	250	0.3879	-0.1263	9.865	95.9%
	500	0.3877	-0.1394	10.394	95.1%
$\widehat{ROC}_{IPW}^{Mis2}$	100	0.3453	-4.3878	10.385	95.0%
	250	0.3445	-4.4679	10.131	88.3%
	500	0.3434	-4.5783	10.762	74.9%
		$ROC(0.2) = 0.5629$			
$\widehat{ROC}_{IPW}^{Mis1}$	100	0.5573	-0.5572	8.650	95.8%
	250	0.5578	-0.5083	8.754	95.4%
	500	0.5580	-0.4904	8.734	94.7%
$\widehat{ROC}_{IPW}^{Mis2}$	100	0.5094	-5.3558	9.132	90.1%
	250	0.5105	-5.2434	9.259	80.4%
	500	0.5105	-5.2408	9.303	63.2%
		$AUC = 0.7602$			
$\widehat{AUC}_{IPW}^{Mis1}$	100	0.7579	-0.2367	1.582	95.1%
	250	0.7585	-0.1775	1.566	95.1%
	500	0.7584	-0.1847	1.671	94.4%
$\widehat{AUC}_{IPW}^{Mis2}$	100	0.7295	-3.0726	1.786	84.1%
	250	0.7301	-3.0165	1.738	67.1%
	500	0.7299	-3.0311	1.904	41.4%
		$pAUC(0, 0.1) = 0.0244$			
$\widehat{pAUC}_{IPW}^{Mis1}$	100	0.0231	-0.1213	0.059	95.8%
	250	0.0238	-0.0559	0.058	96.4%
	500	0.0240	-0.0351	0.064	95.5%
$\widehat{pAUC}_{IPW}^{Mis2}$	100	0.0201	-0.4307	0.060	94.5%
	250	0.0206	-0.3767	0.057	88.9%
	500	0.0208	-0.3585	0.063	77.3%
		$pAUC(0, 0.2) = 0.0726$			
$\widehat{pAUC}_{IPW}^{Mis1}$	100	0.0698	-0.2764	0.238	95.9%
	250	0.0711	-0.1515	0.230	96.3%
	500	0.0715	-0.1093	0.247	95.5%
$\widehat{pAUC}_{IPW}^{Mis2}$	100	0.0622	-1.0398	0.245	91.9%
	250	0.0633	-0.9278	0.233	82.2%

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Table 6 – *Continued from previous page*

Method	$n_D = n_{\bar{D}}$	Estimate	Bias×100	Var×N	Coverage of 95% CI
	500	0.0636	-0.8955	0.257	65.2%

Table 7: Estimate, bias, variance, and coverage of 95% confidence interval (CI) of $ROC(t)$, AUC and $pAUC\,(t_0, t_1)$ estimators using the IPW method with the estimated sampling probabilities $\hat{p}^{Mis1}$ and $\hat{p}^{Mis2}$, for scenario (2) with the discrete stratum variable V_2, $\rho_{XV^*} = 0.8$, $\rho_{XW} = 0.5$, and $\rho_{WV^*} = 0.1$ in finite-population stratified sampling.

Method	$n_D = n_{\bar{D}}$	Estimate	Bias ×100	Var×N	Coverage of 95% CI
		$ROC(0.1) = 0.3891$			
$\widehat{ROC}_{IPW}^{Mis1}$	100	0.3620	-2.7139	8.321	96.2%
	250	0.3604	-2.8792	8.384	92.6%
	500	0.3601	-2.9040	8.473	86.7%
$\widehat{ROC}_{IPW}^{Mis2}$	100	0.3415	-4.7685	9.325	97.0%
	250	0.3392	-4.9963	9.357	88.2%
	500	0.3392	-4.9971	9.448	73.7%
		$ROC(0.2) = 0.5629$			
$\widehat{ROC}_{IPW}^{Mis1}$	100	0.5340	-2.8896	7.187	93.2%
	250	0.5334	-2.9477	6.920	88.9%
	500	0.5333	-2.9578	7.291	80.3%
$\widehat{ROC}_{IPW}^{Mis2}$	100	0.5100	-5.2968	8.442	92.6%
	250	0.5089	-5.4004	8.481	81.6%
	500	0.5087	-5.4241	8.504	62.7%
		$AUC = 0.7602$			
$\widehat{AUC}_{IPW}^{Mis1}$	100	0.7531	-0.7148	1.020	90.6%
	250	0.7532	-0.7091	0.998	89.3%
	500	0.7531	-0.7161	1.054	85.0%
$\widehat{AUC}_{IPW}^{Mis2}$	100	0.7384	-2.1838	1.618	91.0%
	250	0.7385	-2.1738	1.648	82.3%
	500	0.7385	-2.1709	1.704	66.2%
		$pAUC(0, 0.1) = 0.0244$			
$\widehat{pAUC}_{IPW}^{Mis1}$	100	0.0212	-0.3140	0.049	96.4%
	250	0.0217	-0.2617	0.050	93.0%
	500	0.0219	-0.2412	0.052	87.7%
$\widehat{pAUC}_{IPW}^{Mis2}$	100	0.0197	-0.4663	0.053	94.7%

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Table 7 – *Continued from previous page*

Method	$n_D = n_{\bar{D}}$	Estimate	Bias×100	Var×N	Coverage of 95% CI
	250	0.0202	-0.4143	0.055	87.4%
	500	0.0205	-0.3905	0.057	74.5%
		$pAUC(0, 0.2) = 0.0726$			
$\widehat{pAUC}_{IPW}^{Mis1}$	100	0.0654	-0.7199	0.181	95.1%
	250	0.0664	-0.6201	0.180	90.8%
	500	0.0668	-0.5820	0.188	82.0%
$\widehat{pAUC}_{IPW}^{Mis2}$	100	0.0617	-1.0927	0.214	93.4%
	250	0.0625	-1.0068	0.217	81.5%
	500	0.0629	-0.9657	0.221	61.0%