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SUPPLEMENT TO "STATISTICAL INFERENCE FOR THE MEAN OUTCOME UNDER A POSSIBLY NON-UNIQUE OPTIMAL TREATMENT STRATEGY"

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CONTENTS

SUPPLEMENTARY APPENDIX A: PROOFS

A.1. Proofs of results from Section 3.

PROOF OF THEOREM 1. Let $d'(P)$ represent the function $w \mapsto I(\bar{Q}_b(P)(w) > 0)$. For any P, let $\Psi(P) \triangleq E_P E_P[Y|A = d(P)(W), W]$. Note that

$$\begin{aligned}\Psi(P) - E_P E_P[Y|A = 0, W] &= E_P\left[d^*(P)(W)\bar{Q}_b(P)(W)\right] \\ &= E_P\left[d'(P)(W)\bar{Q}_b(P)(W)\right],\end{aligned}$$

where we used the fact that $d^*(P)(w) = d'(P)(w)$ on the set where $\bar{Q}_b(P)(w) \neq 0$. Let the fluctuation submodel $\{P_\epsilon : \epsilon\}$ through P_0 be as defined in Section 3 of the main text, where we note that $P_0 = P_{\epsilon=0}$. Telescoping shows that, for fixed ϵ,

$$\begin{aligned}\Psi(P_\epsilon) - \Psi(P_0) =& E_{P_\epsilon}\left[\left(I(\bar{Q}_{b,\epsilon} > 0) - I(\bar{Q}_{b,0} > 0)\right)\bar{Q}_{b,\epsilon}\right] \\ &+ \Psi_{d'_0}(P_\epsilon) - \Psi_{d'_0}(P_0).\end{aligned} \tag{A.1}$$

It is well known that $\Psi_d(P) \triangleq E_P E_P[Y|A = d(W), W]$ is pathwise differentiable for fixed d. Thus dividing the second line above by ϵ and taking the limit as $\epsilon \to 0$

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yields the pathwise derivative that treats the rule d_0' as known. For a given S_Y, the fluctuated $\bar{Q}_{b,0}$ at $w \in \mathcal{W}$ is given by

$$
\begin{aligned}
\bar{Q}_{b,\epsilon}(w) &\triangleq \int y\left(dQ_{Y,\epsilon}(y|A=1,W=w) - dQ_{Y,\epsilon}(y|A=0,W=w)\right) \\
&= \bar{Q}_{b,0}(w) + \epsilon\Big(E_0\left[YS_Y(Y|1,W)|A=1,W=w\right] \\
&\quad - E_0\left[YS_Y(Y|0,W)|A=0,W=w\right]\Big) \\
&\triangleq \bar{Q}_{b,0}(w) + \epsilon h(w), \qquad \text{(A.2)}
\end{aligned}
$$

where we note that $\sup_w |h(w)| < \infty$ because Y and S_Y are uniformly bounded.

Pathwise differentiable if (3).
Suppose (3). Let $B_1 \triangleq \{w : \bar{Q}_{b,0}(w) = 0\}$ and $B_2 \triangleq \{w : \bar{Q}_{b,0}(w) = 0, \max_a \sigma_0(a,w) = 0\}$. Noting that $B_2 \subseteq B_1$ shows

$$
\begin{aligned}
&E_{P_\epsilon}\left[\left(I(\bar{Q}_{b,\epsilon} > 0) - I(\bar{Q}_{b,0} > 0)\right)\bar{Q}_{b,\epsilon}\right] \\
&\quad = \int_{\mathcal{W}\backslash B_1}\left(I(\bar{Q}_{b,\epsilon} > 0) - I(\bar{Q}_{b,0} > 0)\right)\bar{Q}_{b,\epsilon} dQ_{W,\epsilon} \\
&\qquad + \int_{B_1\backslash B_2}\left(I(\bar{Q}_{b,\epsilon} > 0) - I(\bar{Q}_{b,0} > 0)\right)\bar{Q}_{b,\epsilon} dQ_{W,\epsilon} \\
&\qquad + \int_{B_2}\left(I(\bar{Q}_{b,\epsilon} > 0) - I(\bar{Q}_{b,0} > 0)\right)\bar{Q}_{b,\epsilon} dQ_{W,\epsilon}. \qquad \text{(A.3)}
\end{aligned}
$$

Because $\bar{Q}_{b,0} \neq 0$ on $\mathcal{W}\backslash B_2$, the first term above is $o(|\epsilon|)$ by a slight generalization of Lemma 2 in van der Laan and Luedtke [2014a] to finite measures (since $\mathrm{Pr}_0(\mathcal{W}\backslash B_2)$ may be less than 1). The second term is zero because $\mathrm{Pr}_0(B_1\backslash B_2) = 0$ by (3). Let $f(a,w) \triangleq E_0\left[YS_Y(Y|1,W)|A=1,W=w\right]$. For the third term, note that, for $(a,w) \in \{0,1\} \times B_2$,

$$
\begin{aligned}
&\int_{B_2}\left(I(\bar{Q}_{b,\epsilon} > 0) - I(\bar{Q}_{b,0} > 0)\right)\bar{Q}_{b,\epsilon} dQ_{W,\epsilon} \\
&\quad = \epsilon\int_{B_2}\left(I(\bar{Q}_{b,\epsilon} > 0) - I(\bar{Q}_{b,0} > 0)\right)\left(f(1,w) - f(0,w)\right) dQ_{W,\epsilon}
\end{aligned}
$$

Note that $f(a,w) = \mathrm{Cov}_{P_0}(Y, S_Y(Y|A,W)|A=a,W=w)$ for $a = 0,1$ because $E[S_Y|A,W] = 0$, and thus $f(a,w) = 0$ for $(a,w) \in \{0,1\} \times B_2$ since Y has conditional variance 0 given $A = a$ and $W = w$. This shows that the third term in (A.3) is exactly zero. Hence,

$$
\lim_{\epsilon \to 0} \frac{1}{\epsilon} E_{P_\epsilon}\left[\left(I(\bar{Q}_{b,\epsilon} > 0) - I(\bar{Q}_{b,0} > 0)\right)\bar{Q}_{b,\epsilon}\right] = 0.
$$

Thus Ψ has canonical gradient $D(d_0', P_0)$, i.e. the same canonical gradient as the parameter $\Psi_{d_0'}$. Recall that

$$D(d,P)(O) = \frac{I(A = d(W))}{g(A|W)}(Y - \bar{Q}(A,W)) + \bar{Q}(d(W),W) - \Psi_d(P).$$

If (3) holds, then either i) $Y = \bar{Q}(A,W)$ or ii) $d_0^* = d_0'$ with P_0 probability 1. Thus $D(d_0^*, P_0) = D(d_0', P_0)$ almost surely if (3) holds. It follows that Ψ has canonical gradient $D(d_0^*, P_0)$.

Not pathwise differentiable if not (3).
We wish to construct a submodel so that (4) holds. Let $S_W(w) = 0$ for all w. Without loss of generality, suppose that

$$P_0\left(\bar{Q}_{b,0}(W) = 0, \sigma_0(1,W) > 0\right) > 0. \tag{A.4}$$

Let

$$R(w) \triangleq \frac{\Pr_0(Y \le \bar{Q}_0(1,W)|A = 1, W = w)}{\Pr_0(Y > \bar{Q}_0(1,W)|A = 1, W = w)},$$

where we let $R(w) = \infty$ when $\Pr_0(Y > \bar{Q}_0(1,W)|A = 1, W = w) = 0$. Define S_Y as follows:

$$S_Y(y|a,w) \triangleq \begin{cases} \min\{1, R(w)\}, & \text{if } a = 1 \text{ and } y > \bar{Q}_0(1,w) \\ -\min\{1, 1/R(w)\}, & \text{if } a = 1 \text{ and } y \le \bar{Q}_0(1,w) \\ 0, & \text{if } a = 0. \end{cases}$$

Above we let $\min\{1, 1/R(W)\} = 0$ when $R(W) = \infty$ and $\min\{1, 1/R(W)\} = 1$ when $R(W) = 0$. Note that $\sup_{w,a,y} |S_Y(y|a,w)| \le 1$ and $E[S_Y|A = a, W = w] = 0$ for all a, w. We define B_+ and B_- as follows:

$$\begin{aligned} B_+ &\triangleq B_0 \cap \{w : h(w) > 0\} \\ B_- &\triangleq B_0 \cap \{w : h(w) < 0\}, \end{aligned}$$

where h is defined in (A.2). By (A.4), $\Pr_0(\bar{Q}_{b,0}(W) = 0, 0 < R(W) < \infty) > 0$, and hence $\Pr_0(B_+) > 0$ and $\Pr_0(B_-) > 0$. Let

$$m(w) \triangleq (I(\bar{Q}_{b,\epsilon}(w) > 0) - I(\bar{Q}_{b,0}(w) > 0))\bar{Q}_{b,\epsilon}(w).$$

The first term in (A.1) yields the following limit from above:

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_{\mathcal{W}} m(w) dQ_{W,0}(w)$$

$$= \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_{B_+} m(w) dQ_{W,0}(w)$$
$$+ \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_{B_-} m(w) dQ_{W,0}(w) + \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_{\mathcal{W} \setminus (B_+ \cup B_-)} m(w) dQ_{W,0}(w)$$
$$= \lim_{\epsilon \downarrow 0} \int_{B_+} I(\epsilon h(w) > 0) h(w) dQ_{W,0}(w) + \lim_{\epsilon \downarrow 0} \int_{B_-} I(\epsilon h(w) > 0) h(w) dQ_{W,0}(w)$$
$$+ \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_{\mathcal{W} \setminus (B_+ \cup B_-)} m(w) dQ_{W,0}(w)$$
(A.5)
$$= \int_{B_+} h(w) dQ_{W,0}(w) > 0,$$

where the integral over B_- is equal to zero because the indicator in m is 0 for all $\epsilon > 0$ and the integral over $\mathcal{W} \setminus (B_+ \cup B_-)$ is $o(|\epsilon|)$ because

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_{\mathcal{W} \setminus (B_+ \cup B_-)} m(w) dQ_{W,0}(w)$$
$$= \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_{\mathcal{W} \setminus B_0} m(w) dQ_{W,0}(w) + \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_{\{w : h(w) = 0\} \cap B_0} m(w) dQ_{W,0}(w) = 0,$$

where we used that the first term is 0 by a slight generalization of Lemma 2 in van der Laan and Luedtke [2014a] to finite measures and the second term is 0 because $\bar{Q}_{b,\epsilon} = 0$ on $\{w : h(w) = 0\} \cap B_0$. The inequality in (A.5) is strict because $\Pr_0(B_+) > 0$ and $h > 0$ on B_+. Similarly,

$$\lim_{\epsilon \uparrow 0} \frac{1}{\epsilon} \int m(w) dQ_{W,0}(w) = \int_{B_-} h(w) dQ_{W,0}(w) < 0.$$

Contrasting the above with (A.5) shows that there exists a path about P_0 which results in a fluctuation h for which the limit of the first term in (A.1) divided by ϵ does not exist as $\epsilon \to 0$. But then Ψ cannot be pathwise differentiable: one of the limits in the sum on the right-hand side of (A.1) exists, so the limit on the left-hand side cannot exist. Specifically, suppose c_n has a limit as $n \to \infty$ and $a_n = b_n + c_n$. If b_n does not have a limit, then a_n does not have a limit, since a_n having a limit implies that $b_n = a_n - c_n$ has a limit, contradiction. □

A.2. Proofs of results from Section 5.

PROOF OF THEOREM 2. We have that

$$\Gamma_n \left(\hat{\Psi}(P_n) - \Psi(P_0) \right)$$

(A.6)
$$= \frac{1}{n-\ell_n} \sum_{j=\ell_n+1}^{n} \tilde{\sigma}_{n,j}^{-1} \left(\tilde{D}_{n,j}(O_j) - \Psi(P_0) \right)$$
(A.7)
$$= \frac{1}{n-\ell_n} \sum_{j=\ell_n+1}^{n} \tilde{\sigma}_{n,j}^{-1} \left(\left[\tilde{D}_{n,j}(O_j) - \Psi_{d_{n,j}}(P_0) \right] + \left[\Psi_{d_{n,j}}(P_0) - \Psi(P_0) \right] \right)$$
(A.8)
$$= \frac{1}{n-\ell_n} \sum_{j=\ell_n+1}^{n} \tilde{\sigma}_{n,j}^{-1} \left(\tilde{D}_{n,j}(O_j) - \Psi_{d_{n,j}}(P_0) \right) + o_{P_0}(n^{-1/2})$$
(A.9)
$$= \frac{1}{n-\ell_n} \sum_{j=\ell_n+1}^{n} \tilde{\sigma}_{n,j}^{-1} \left(\tilde{D}_{n,j}(O_j) - E_0 \left[\tilde{D}_{n,j}(O_j) | O_1, ..., O_{j-1} \right] \right) + R_{1n} + o_{P_0}(n^{-1/2})$$
(A.10)
$$= \frac{1}{n-\ell_n} \sum_{j=\ell_n+1}^{n} \tilde{\sigma}_{n,j}^{-1} \left(\tilde{D}_{n,j}(O_j) - E_0 \left[\tilde{D}_{n,j}(O_j) | O_1, ..., O_{j-1} \right] \right) + o_{P_0}(n^{-1/2}).$$

Above (A.6) is a result of moving the $\Psi(P_0)$ into the summation in the defintion of Γ_n, (A.7) adds zero to the line above, (A.8) follows by C5), (A.9) is a consequence of the fact that $\Psi_d(P_0) = P_0\tilde{D}(\bar{Q}, g, d) - E_0[(1 - \frac{g_0(d(W)|W)}{g(d(W)|W)})(\bar{Q}(d(W), W) - \bar{Q}_0(d(W), W))]$ for any fixed $\bar{Q}$, g, and d, and (A.10) follows by C4).

For $j = 1, ..., n - \ell_n$, let

$$M_{n,j} \triangleq \frac{1}{\sqrt{n-\ell_n}} \frac{\left(\tilde{D}(d_{n,j+\ell_n})(O_{j+\ell_n}) - E_0 \left[\tilde{D}(d_{n,j+\ell_n})(O_{j+\ell_n}) | O_1, ..., O_{j+\ell_n} \right] \right)}{\tilde{\sigma}_{n,j+\ell_n}}.$$

Note that, for each n, $\{M_{n,j} : j = 1, ..., n - \ell_n\}$ is a discrete-time martingale with respect to the filtration $\mathcal{F}_j$, where each $\mathcal{F}_j$ is the sigma-field generated by $O_1, ..., O_{j+\ell_n}$. In particular, we have that, for all $j \geq 1$, $E_0[M_{n,j}|\mathcal{F}_{j-1}] = 0$. We also have that $\sum_{j=1}^{n-\ell_n} E_0[M_{n,j}^2|\mathcal{F}_{j-1}] = \frac{1}{n-\ell_n} \sum_{j=1}^{n-\ell_n} \frac{\tilde{\sigma}_{0,n,j+\ell_n}^2}{\tilde{\sigma}_{n,j+\ell_n}^2} \to 1$ by C3). Because the conditional Lindeberg condition in C2) holds, the martingale CLT for triangular arrays [see, e.g., Theorem 2 in Gaenssler et al., 1978] shows that

(A.11)
$$\sum_{j=1}^{n-\ell_n} M_{n,j} \rightsquigarrow N(0, 1).$$

Plugging this into (A.10) gives that

$$\Gamma_n \sqrt{n-\ell_n}\left(\hat{\Psi}(P_n) - \Psi(P_0)\right) \rightsquigarrow N(0,1).$$

The asymptotically valid $1-\alpha$ CI is now constructed in the usual way. □

PROOF OF COROLLARY 3. In this proof we use "$\lesssim$" to denote less than or equal to up to a positive multiplicative constant. Let $\mathcal{F}_j$ represent the sigma-field generated by $O_1, ..., O_j$. Let $\tilde{D}_0 \triangleq \tilde{D}(d_0, \bar{Q}_0, g_0)$ and $s_0^2 \triangleq \mathrm{Var}_{P_0}(\tilde{D}(d_0, \bar{Q}_0, g_0)(O))$. The proof can be broken into four parts, which show that: (1) $\tilde{D}_{n,j}$ approximates $\tilde{D}_0$ in mean-square; (2) $\Gamma_n^{-1} \to s_0$ in probability; (3) $\Gamma_n(\hat{\Psi}(P_n) - \Psi(P_0))$ behaves like an empirical mean of the normalized efficient influence curve; (4) $\hat{\Psi}(P_n)$ is RAL and efficient.

Part 1: $\tilde{D}_{n,j}$ approximates $\tilde{D}_0$. Note that

$$\begin{aligned}
&\frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n} E_0\left[\left(\tilde{D}_{n,j} - \tilde{D}_0\right)^2 \middle| \mathcal{F}_{j-1}\right] \\
&\quad\le \frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n} E_0\left[\left(\tilde{D}(d_{n,j}, \bar{Q}_{n,j}, g_{n,j}) - \tilde{D}(d_0, \bar{Q}_{n,j}, g_{n,j})\right)^2 \middle| \mathcal{F}_{j-1}\right] \\
&\qquad + \frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n} E_0\left[\left(\tilde{D}(d_0, \bar{Q}_{n,j}, g_{n,j}) - \tilde{D}(d_0, \bar{Q}_{n,j}, g_0)\right)^2 \middle| \mathcal{F}_{j-1}\right] \\
&\qquad + \frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n} E_0\left[\left(\tilde{D}(d_0, \bar{Q}_{n,j}, g_0) - \tilde{D}(d_0, \bar{Q}_0, g_0)\right)^2 \middle| \mathcal{F}_{j-1}\right] \\
&\quad\lesssim \frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n} E_0\left[\left(d_{n,j}(W) - d_0(W)\right)^2 \middle| \mathcal{F}_{j-1}\right] \\
&\qquad + \frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n} E_0\left[\left(g_{n,j}(d(W)|W) - g_0(d(W)|W)\right)^2 \middle| \mathcal{F}_{j-1}\right] \\
&\qquad + \frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n} E_0\left[\left(\bar{Q}_{n,j}(d_0(W), W) - \bar{Q}_0(d_0(W), W)\right)^2 \middle| \mathcal{F}_{j-1}\right]
\end{aligned}$$

(A.12)

$$= o_{P_0}(1)$$

where the constant in the second inequality relies on the bounds on Y, $\bar{Q}_{n,j}$, g_0, and $g_{n,j}$.

Part 2: $\Gamma_n^{-1} \to s_0$ in probability. We have that

$$
\begin{aligned}
(\Gamma_n - s_0^{-1})^2 &\leq \left(\frac{1}{n-\ell_n} \sum_{j=\ell_n+1}^{n} \tilde{\sigma}_{n,j}^{-1} s_0^{-1} |\tilde{\sigma}_{n,j} - s_0| \right)^2 \\
&\lesssim \left(\frac{1}{n-\ell_n} \sum_{j=\ell_n+1}^{n} |\tilde{\sigma}_{n,j} - s_0| \right)^2 \\
&\lesssim \frac{1}{n-\ell_n} \sum_{j=\ell_n+1}^{n} (\tilde{\sigma}_{n,j} - s_0)^2, \qquad \text{(A.13)}
\end{aligned}
$$

where the second inequality on the first line holds by the assumed bounds on $\tilde{\sigma}_{n,j}$ and the final inequality holds by Cauchy-Schwarz. Note that, for any positive real numbers x_1, x_2,

$$
(x_1 - x_2)^2 \leq 2|x_1^2 - x_2^2|. \qquad \text{(A.14)}
$$

By the above and Condition C3') , we have that

$$
\begin{aligned}
\frac{1}{n-\ell_n} \sum_{j=\ell_n+1}^{n} (\tilde{\sigma}_{n,j} - \tilde{\sigma}_{0,n,j})^2 &\lesssim \frac{1}{n-\ell_n} \sum_{j=\ell_n+1}^{n} \left| \tilde{\sigma}_{n,j}^2 - \tilde{\sigma}_{0,n,j}^2 \right| \\
&\lesssim \frac{1}{n-\ell_n} \sum_{j=\ell_n+1}^{n} \left| \frac{\tilde{\sigma}_{0,n,j}^2}{\tilde{\sigma}_{n,j}^2} - 1 \right| = o_{P_0}(1).
\end{aligned}
$$

We also have that

$$
\begin{aligned}
\frac{1}{n-\ell_n} \sum_{j=1}^{n} (\tilde{\sigma}_{0,n,j} - s_0)^2 &\leq \frac{2}{n-\ell_n} \sum_{j=1}^{n} |\tilde{\sigma}_{0,n,j}^2 - s_0^2| \\
&= \frac{2}{n-\ell_n} \sum_{j=\ell_n+1}^{n} \left| E_0\left[\tilde{D}_{n,j}^2 - \tilde{D}_0^2 | \mathcal{F}_{j-1} \right] + E_0\left[\tilde{D}_{n,j} | \mathcal{F}_{j-1} \right]^2 - E_0\left[\tilde{D}_0 | \mathcal{F}_{j-1} \right]^2 \right| \\
&\lesssim \frac{1}{n-\ell_n} \sum_{j=\ell_n+1}^{n} E_0\left[\left| \tilde{D}_{n,j} - \tilde{D}_0 \right| | \mathcal{F}_{j-1} \right] \\
&\lesssim \sqrt{ \frac{1}{n-\ell_n} \sum_{j=\ell_n+1}^{n} E_0\left[\left(\tilde{D}_{n,j} - \tilde{D}_0 \right)^2 | \mathcal{F}_{j-1} \right] },
\end{aligned}
$$

where: the first inequality holds by (A.14); the equality holds by the definition of conditional variance; the second inequality holds by twice using that $x_1^2 - x_2^2 =$

$(x_1 + x_2)(x_1 - x_2)$, the strong positivity assumption, and the bounds on Y and $\bar{Q}_{n,j}$; and the final inequality holds by the Cauchy-Schwarz inequality applied to the expectations and the concavity of $x \mapsto \sqrt{x}$. By (A.12), the upper bound above is $o_{P_0}(1)$. By the triangle inequality and the previous two indented equations,

(A.15)
$$\frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n}(\tilde{\sigma}_{n,j}-s_0)^2 \le \frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n}\left[(\tilde{\sigma}_{n,j}-\tilde{\sigma}_{0,n,j})^2+(\tilde{\sigma}_{0,n,j}-s_0)^2\right] = o_{P_0}(1).$$

Plugging this into (A.13) shows that $\Gamma_n = s_0^{-1} + o_{P_0}(1)$. By the continuous mapping theorem, $\Gamma_n^{-1} = s_0 + o_{P_0}(1)$.

Part 3: $\Gamma_n(\hat{\Psi}(P_n) - \Psi(P_0))$ **behaves like an empirical mean.** For each $n > 1$ and $j = \ell_n + 1, ..., n$, define

$$M'_{n,j} \triangleq \frac{\tilde{D}_{n,j}(O_j) - E_0\left[\tilde{D}_{n,j}(O)|\mathcal{F}_{j-1}\right]}{\tilde{\sigma}_{n,j}} - \frac{\tilde{D}_0(O_j) - E_0\left[\tilde{D}_0(O) \mid \mathcal{F}_{j-1}\right]}{s_0}.$$

We first show that $\frac{1}{\sqrt{n-\ell_n}}\sum_{j=\ell_n+1}^{n} M'_{n,j} \to 0$ in probability. Note that

$$\begin{aligned}
V'_{n,j} \triangleq \mathrm{Var}_{P_0}\left(M'_{n,j}\middle|\mathcal{F}_{j-1}\right) &= E_0\left[\left(\frac{\tilde{D}_{n,j}(O_j)}{\tilde{\sigma}_{n,j}} - \frac{\tilde{D}_0(O_j)}{s_0}\right)^2\middle|\mathcal{F}_{j-1}\right] \\
&\le E_0\left[\left(\frac{\tilde{D}_{n,j}(O_j)}{\tilde{\sigma}_{n,j}} - \frac{\tilde{D}_0(O_j)}{\tilde{\sigma}_{n,j}}\right)^2\middle|\mathcal{F}_{j-1}\right] + E_0\left[\left(\frac{\tilde{D}_0(O_j)}{\tilde{\sigma}_{n,j}} - \frac{\tilde{D}_0(O_j)}{s_0}\right)^2\middle|\mathcal{F}_{j-1}\right] \\
&\lesssim E_0\left[\left(\tilde{D}_{n,j}(O_j) - \tilde{D}_0(O_j)\right)^2\middle|\mathcal{F}_{j-1}\right] + E_0\left[(\tilde{\sigma}_{n,j}-s_0)^2\middle|\mathcal{F}_{j-1}\right]
\end{aligned}$$

where the constants in the second inequality rely on the bounds on $g_{n,j}$, g_0, $\bar{Q}_{n,j}$, Y, $\tilde{\sigma}_{0,n,j}$, and s_0. By (A.12) and (A.15),

$$\text{(A.16)} \qquad \frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n} V'_{n,j} = o_{P_0}(1).$$

Fix $\epsilon, \delta > 0$ and let $v_{\epsilon,\delta} \triangleq \frac{\epsilon^2}{\log(4/\delta)}$. We will show that there exists some N such that

$$\text{(A.17)} \qquad \mathrm{Pr}_0\left(\frac{1}{\sqrt{n-\ell_n}}\sum_{j=\ell_n+1}^{n} M'_{n,j} \ge \epsilon\right) < \delta \text{ for all } n \ge N.$$

Note that

$$\begin{aligned}
&\Pr_0\left(\frac{1}{\sqrt{n-\ell_n}}\sum_{j=\ell_n+1}^{n} M'_{n,j} \geq \epsilon\right) \\
&= \Pr_0\left(\frac{1}{\sqrt{n-\ell_n}}\sum_{j=\ell_n+1}^{n} M'_{n,j} \geq \epsilon, \frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n} V'_{n,j} \leq v_{\epsilon,\delta}\right) \\
&\quad + \Pr_0\left(\frac{1}{\sqrt{n-\ell_n}}\sum_{j=\ell_n+1}^{n} M'_{n,j} \geq \epsilon, \frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n} V'_{n,j} > v_{\epsilon,\delta}\right).
\end{aligned}$$

We will bound the terms on the right separately. By our bounding assumptions, there exists some $m^* \in (0,\infty)$ such that $\Pr_0(\sup_{j\leq n}|M_{n,j}| < m^*) = 1$. By Bernstein's inequality for martingale difference sequences with bounded increments [see, e.g, Steiger, 1969; Theorem 1.6 of Freedman, 1975], we have that

$$\begin{aligned}
&\Pr_0\left(\frac{1}{\sqrt{n-\ell_n}}\sum_{j=\ell_n+1}^{n} M'_{n,j} \geq \epsilon, \frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n} V'_{n,j} \leq v_{\epsilon,\delta}\right) \\
&\leq \Pr_0\left(\frac{1}{\sqrt{n-\ell_n}}\sum_{j=\ell_n+1}^{\tilde{n}} M'_{n,j} \geq \epsilon, \frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{\tilde{n}} V'_{n,j} \leq v_{\epsilon,\delta} \text{ for some } \tilde{n} \in \{\ell_n+1,\dots,n\}\right) \\
&\leq \Pr_0\left(\sum_{j=\ell_n+1}^{\tilde{n}} \frac{M'_{n,j}}{m^*} \geq \frac{\epsilon\sqrt{n-\ell_n}}{m^*}, \sum_{j=\ell_n+1}^{\tilde{n}} \frac{V'_{n,j}}{(m^*)^2} \leq \frac{v_{\epsilon,\delta}(n-\ell_n)}{(m^*)^2} \text{ for some } \tilde{n} \in \{\ell_n+1,\dots,n\}\right) \\
&\leq \exp\left(-\frac{\epsilon^2\sqrt{n-\ell_n}}{2(m^*\epsilon + v_{\epsilon,\delta}\sqrt{n-\ell_n})}\right) \overset{n\to\infty}{\longrightarrow} \delta/4.
\end{aligned}$$

It follows that there exists some N_1 such that the upper bound above is less than or equal to $\delta/2$ for all $n \geq N_1$. We also have that

$$\begin{aligned}
&\Pr_0\left(\frac{1}{\sqrt{n-\ell_n}}\sum_{j=\ell_n+1}^{n} M'_{n,j} \geq \epsilon, \frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n} V'_{n,j} > v_{\epsilon,\delta}\right) \\
&\leq \Pr_0\left(\frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n} V'_{n,j} \geq v_{\epsilon,\delta}\right).
\end{aligned}$$

By (A.16), there exists some N_2 so that the upper bound above is no greater than $\delta/4$ for all $n \geq N_2$. Combining the previous two sets of inequalities shows that (A.17) is satisfied for $N \triangleq \max\{N_1, N_2\}$. Thus $\frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^n M'_{n,j} = o_{P_0}(\sqrt{n-\ell_n})$. Because $\ell_n = o(n)$, $\frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^n M'_{n,j} = o_{P_0}(n^{-1/2})$. Combining this with (A.10) shows that

$$\begin{aligned}
&\Gamma_n\left(\hat{\Psi}(P_n) - \Psi(P_0)\right) \\
&= \frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^n \frac{\tilde{D}_{n,j}(O_j) - E_0[\tilde{D}_{n,j}(O)|\mathcal{F}_{j-1}]}{\tilde{\sigma}_{n,j}} + o_{P_0}(n^{-1/2}) \\
&= s_0^{-1}\frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^n \left(\tilde{D}_0(O_j) - E_0\left[\tilde{D}_0(O)\right]\right) \\
&\quad + \frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^n M'_{n,j} + o_{P_0}(n^{-1/2}) \\
&= s_0^{-1}\frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^n \left(\tilde{D}_0(O_j) - E_0\left[\tilde{D}_0(O)\right]\right) + o_{P_0}(n^{-1/2}) \\
&= s_0^{-1}\frac{1}{n}\sum_{j=1}^n \left(\tilde{D}_0(O_j) - E_0\left[\tilde{D}_0(O)\right]\right) + o_{P_0}(n^{-1/2}),
\end{aligned}$$

where the final equality uses that $\ell_n = o(n)$ and that $\tilde{D}_0$ is bounded.

Part 4: $\hat{\Psi}(P_n)$ is RAL and efficient. Combining Parts 2 and 3 shows that

$$\begin{aligned}
\hat{\Psi}(P_n) - \Psi(P_0) &= \Gamma_n^{-1}\Gamma_n\left(\hat{\Psi}(P_n) - \Psi(P_0)\right) \\
&= (s_0 + o_{P_0}(1))\Gamma_n\left(\hat{\Psi}(P_n) - \Psi(P_0)\right) \\
&= \frac{1}{n}\sum_{j=1}^n \left(\tilde{D}_0(O_j) - E_0\left[\tilde{D}_0(O)\right]\right) + o_{P_0}(n^{-1/2}).
\end{aligned}$$

Thus $\hat{\Psi}(P_n)$ is an asymptotically linear estimator of $\Psi(P_0)$ with influence curve $D(d_0, P_0) = \tilde{D}_0(O_j) - E_0\left[\tilde{D}_0(O)\right]$. If P_0 satisfies (3) so that $D(d_0, P_0) = D(d_0^*, P_0)$ almost surely, then Theorem 1 shows that $D(d_0^*, P_0)$ is the efficient influence curve of Ψ. By Proposition 1 of Section 3.3 in Bickel et al. [1993], it follows that (3) holds if and only if $\hat{\Psi}(P_n)$ is a RAL estimator and is asymptotically efficient among all RAL estimators. □

PROOF OF THEOREM 4. The below is an abbreviated version of (A.6) through (A.10) and (A.11), with an added inequality which holds because $R_{2n} \leq 0$:

$$\sqrt{n-\ell_n}\Gamma_n\left(\hat{\Psi}(P_n) - \Psi(P_0)\right)$$

$$= \frac{1}{\sqrt{n-\ell_n}} \sum_{j=\ell_n+1}^{n} \tilde{\sigma}_{n,j}^{-1} \left(\left[\tilde{D}_{n,j}(O_j) - \Psi_{d_{n,j}}(P_0) \right] + \left[\Psi_{d_{n,j}}(P_0) - \Psi(P_0) \right] \right)$$
$$\leq \frac{1}{\sqrt{n-\ell_n}} \sum_{j=\ell_n+1}^{n} \tilde{\sigma}_{n,j}^{-1} \left(\tilde{D}_{n,j}(O_j) - \Psi_{d_{n,j}}(P_0) \right)$$
$$= \frac{1}{\sqrt{n-\ell_n}} \sum_{j=\ell_n+1}^{n} \tilde{\sigma}_{n,j}^{-1} \left(\tilde{D}_{n,j}(O_j) - E_0 \left[\tilde{D}_{n,j}(O_j) | O_1, ..., O_{j-1} \right] \right) + o_{P_0}(1)$$
$$\rightsquigarrow N(0,1).$$

Thus,

$$\liminf_{n\to\infty} \Pr_0 \left(\sqrt{n-\ell_n} \Gamma_n \left(\hat{\Psi}(P_n) - \Psi(P_0) \right) \leq z_{1-\alpha} \right) \geq 1-\alpha.$$

The first result follows by rearranging terms in the probability statement. The second result is an immediate corollary of Theorem 2. □

PROOF OF THEOREM 5. Note that

$$\frac{1}{n-\ell_n} \sum_{j=\ell_n+1}^{n} \tilde{\sigma}_{n,j}^{-1} \left(\Psi_{d_{n,j}}(P_0) - \Psi_{d_n}(P_0) \right)$$
$$= \frac{1}{n-\ell_n} \sum_{j=\ell_n+1}^{n} \tilde{\sigma}_{n,j}^{-1} \left(\Psi_{d_{n,j}}(P_0) - \psi_1 \right) - \Gamma_n \left(\Psi_{d_n}(P_0) - \psi_1 \right) = o_{P_0}(n^{-1/2}).$$

by Conditions C6) and C7). Following the proof of Theorem 2, we have

$$\Gamma_n \left(\hat{\Psi}(P_n) - \Psi_{d_n}(P_0) \right)$$
$$= \frac{1}{n-\ell_n} \sum_{j=\ell_n+1}^{n} \tilde{\sigma}_{n,j}^{-1} \left(\left[\tilde{D}_{n,j}(O_j) - \Psi_{d_{n,j}}(P_0) \right] + \left[\Psi_{d_{n,j}}(P_0) - \Psi_{d_n}(P_0) \right] \right)$$
$$= \frac{1}{n-\ell_n} \sum_{j=\ell_n+1}^{n} \tilde{\sigma}_{n,j}^{-1} \left(\left[\tilde{D}_{n,j}(O_j) - \Psi_{d_{n,j}}(P_0) \right] \right) + o_{P_0}(n^{-1/2}).$$

The remainder of the proof is identical to that of Theorem 2. □

A.3. Proofs of results from Section 7.

PROOF OF LEMMA 6. By the almost sure representation theorem [see, e.g., Theorem 1.10.3 in Billingsley, 1999], there exists a probability space $(\Omega', \mathcal{F}', P')$ and a sequence of random variables $R'_n : \Omega' \to \mathbb{R}$ such that $n^\beta R'_n \stackrel{d}{=} n^\beta R_n$ and

$n^\beta R_n'(\omega') \to 0$ for all $\omega' \in \Omega'$. Fix $\epsilon > 0$ and $\omega' \in \Omega'$. There exists some N that, for all $n \geq N$, $n^\beta |R_n'(\omega')| < \frac{(1-\beta)\epsilon}{2}$. Also note that

$$\frac{1}{n^{1-\beta}} \sum_{j=1}^{n} j^{-\beta} \leq \frac{1}{n^{1-\beta}} \int_1^n (j-1)^{-\beta} dj = \frac{1}{1-\beta}.$$

Hence, for all $n \geq N$,

$$\begin{aligned}
\frac{1}{n^{1-\beta}} \sum_{j=1}^{n} |R_j'(\omega')| &= \frac{1}{n^{1-\beta}} \sum_{j=1}^{N-1} |R_j'(\omega')| + \frac{1}{n^{1-\beta}} \sum_{j=N}^{n} \frac{1}{j^\beta} j^\beta |R_j'(\omega')| \\
&< \frac{1}{n^{1-\beta}} \sum_{j=1}^{N-1} |R_j'(\omega')| + \frac{(1-\beta)\epsilon}{2n^{1-\beta}} \sum_{j=N}^{n} \frac{1}{j^\beta} \\
&\leq \frac{1}{n^{1-\beta}} \sum_{j=1}^{N-1} |R_j'(\omega')| + \frac{\epsilon}{2}.
\end{aligned}$$

It follows that $\frac{1}{n^{1-\beta}} \sum_{j=1}^{n} |R_j'(\omega')| < \epsilon$ for all n large enough, and thus that $\lim_{n\to\infty} \frac{1}{n^{1-\beta}} \sum_{j=1}^{n} R_j'(\omega') = 0$. Noting that $\frac{1}{n^{1-\beta}} \sum_{j=1}^{n} R_j \stackrel{d}{=} \frac{1}{n^{1-\beta}} \sum_{j=1}^{n} R_j'(\omega')$ for all n, we have that $\frac{1}{n} \sum_{j=1}^{n} R_j = o_{P_0}(n^{-\beta})$. □

PROOF OF THEOREM 7. Let $\tilde{\mathcal{D}}_1 \triangleq \{\tilde{D}(d, \bar{Q}, g) : d, \bar{Q}, g\}$, $\tilde{\mathcal{D}}_2 \triangleq \{\tilde{D}^2(d, \bar{Q}, g) : d, \bar{Q}, g\}$, and $j^* \triangleq \min\{j : \delta_j \leq \delta_0\}$. The class $\tilde{\mathcal{D}}_1$ is P_0 Glivenko-Cantelli (GC) by assumption, and $\tilde{\mathcal{D}}_2$ is GC by Theorem 2 of van der Vaart and Wellner [2000]. For all $j \geq j^*$, we have that

$$\begin{aligned}
\left|\tilde{\sigma}_j^2 - \tilde{\sigma}_{0,j}^2\right| \leq& \left| \frac{1}{j-1} \sum_{i=1}^{j-1} \tilde{D}_j^2(O_i) - E_0\left[\tilde{D}_j^2(O) \middle| O_1, ..., O_{j-1} \right] \right| \\
&+ \left| \left(\frac{1}{j-1} \sum_{k=1}^{j-1} \tilde{D}_j(O_k) \right)^2 - E_0\left[\tilde{D}_j(O) \middle| O_1, ..., O_{j-1} \right]^2 \right|. \qquad \text{(A.18)}
\end{aligned}$$

The first term on the right converges to 0 in probability because $\tilde{\mathcal{D}}_2$ is GC. For the second term, the mean value theorem shows that

$$\begin{aligned}
&\left(\frac{1}{j-1} \sum_{k=1}^{j-1} \tilde{D}_j(O_k) \right)^2 - E_0\left[\tilde{D}_j(O) \middle| O_1, ..., O_{j-1} \right]^2 \\
&= 2m_j \underbrace{\left(\frac{1}{j-1} \sum_{k=1}^{j-1} \tilde{D}_j(O_k) - E_0\left[\tilde{D}_j(O) \middle| O_1, ..., O_{j-1} \right] \right)}_{\triangleq \|P_j - P_0\|_{\tilde{\mathcal{D}}_1}},
\end{aligned}$$

where m_j is an intermediate value between the two squared values on the first line. Using that $\tilde{\mathcal{D}}_1$ is a GC class, we have that m_j converges to $E_0[\tilde{D}_j(O)|O_1,...,O_{j-1}]$ in probability and $\|P_j - P_0\|_{\tilde{\mathcal{D}}_1} = o_{P_0}(1)$. Thus the above is $o_{P_0}(1)$, and plugging this into (A.18) shows that $|\tilde{\sigma}_j^2 - \tilde{\sigma}_{0,j}^2| = o_{P_0}(1)$. The continuous mapping theorem shows that (13) is also satisfied. Combining this with Lemma 6 with $\beta = 0$ shows that Condition C3) is satisfied. □

PROOF OF THEOREM 8. In this proof we will omit the dependence of d_0^*, d_n, $\bar{Q}_{b,0}$, and $\bar{Q}_{b,n}$ on W in the notation. Suppose that $\left\|\bar{Q}_{b,n} - \bar{Q}_{b,0}\right\|_{2,P_0} = o_{P_0}(1)$. This part of the proof mimics the proof of Lemma 5.2 in Audibert and Tsybakov [2007]. For any $t > 0$,

$$\begin{aligned}
&|\Psi_{d_n}(P_0) - \Psi_{d_0^*}(P_0)| \\
&\quad= E_0[|\bar{Q}_{b,0}|I(d_0^* \neq d_n)] \\
&\quad= E_0[|\bar{Q}_{b,0}|I(d_0^* \neq d_n)I(0 < |\bar{Q}_{b,0}| \leq t)] \\
&\qquad + E_0[|\bar{Q}_{b,0}|I(d_0^* \neq d_n)I(|\bar{Q}_{b,0}| > t)] \\
&\quad\leq E_0[|\bar{Q}_{b,n} - \bar{Q}_{b,0}|I(0 < |\bar{Q}_{b,0}| \leq t)] \\
&\qquad + E_0[|\bar{Q}_{b,n} - \bar{Q}_{b,0}|I(|\bar{Q}_{b,n} - \bar{Q}_{b,0}| > t)] \\
&\quad\leq \left\|\bar{Q}_{b,n} - \bar{Q}_{b,0}\right\|_{2,P_0} \mathrm{Pr}_0(0 < |\bar{Q}_{b,0}| \leq t)^{1/2} + \frac{\left\|\bar{Q}_{b,n} - \bar{Q}_{b,0}\right\|_{2,P_0}^2}{t} \\
&\quad\leq \left\|\bar{Q}_{b,n} - \bar{Q}_{b,0}\right\|_{2,P_0} C_0^{1/2} t^{\alpha/2} + \frac{\left\|\bar{Q}_{b,n} - \bar{Q}_{b,0}\right\|_{2,P_0}^2}{t},
\end{aligned}$$

where the first inequality holds because $d_0^* \neq d_n$ implies that $|\bar{Q}_{b,n} - \bar{Q}_{b,0}| > |\bar{Q}_{b,0}|$, the second inequality holds by the Cauchy-Schwarz and Markov inequalities, and the third inequality holds by (16). The first result follows by optimizing over t to find that the upper bound is minimized when $t = C\left\|\bar{Q}_{b,n} - \bar{Q}_{b,0}\right\|_{2,P_0}^{2(1+\alpha)/(2+\alpha)}$ for a constant C which depends on C_0 and α.

Now suppose that $\left\|\bar{Q}_{b,n} - \bar{Q}_{b,0}\right\|_{\infty,P_0} = o_{P_0}(1)$. Note that

$$\begin{aligned}
&|\Psi_{d_n}(P_0) - \Psi_{d_0^*}(P_0)| \\
&\quad= E_0\left|I(d_n \neq d_0^*)\bar{Q}_{b,0}\right| \\
&\quad\leq E_0\left[I(0 < |\bar{Q}_{b,0}| \leq |\bar{Q}_{b,n} - \bar{Q}_{b,0}|)|\bar{Q}_{b,0}|\right] \\
&\quad\leq E_0\left[I\left(0 < |\bar{Q}_{b,0}| \leq \left\|\bar{Q}_{b,n} - \bar{Q}_{b,0}\right\|_{\infty,P_0}\right)|\bar{Q}_{b,0}|\right] \\
&\quad\leq \left\|\bar{Q}_{b,n} - \bar{Q}_{b,0}\right\|_{\infty,P_0} \mathrm{Pr}_0\left(0 < |\bar{Q}_{b,0}| \leq \left\|\bar{Q}_{b,n} - \bar{Q}_{b,0}\right\|_{\infty,P_0}\right).
\end{aligned}$$

By (16), $|\Psi_{d_n}(P_0) - \Psi_{d_0^*}(P_0)| \lesssim \left\|\bar{Q}_{b,n} - \bar{Q}_{b,0}\right\|_{\infty,P_0}^{1+\alpha}$. □

SUPPLEMENTARY APPENDIX B: MULTIPLE TIME POINT CASE

We now present an extension of our approach to the multiple time point case. We give rigorous conditions under which our approach will work at the end of this section, but we do not give interpretable sufficient conditions under which they hold as we did for the single time point in Section 7 of the main text. The primary challenge is showing that a condition like C5) holds for the multiple time point case, i.e. that the estimated rule has nearly optimal value. While we believe that interpretable sufficient conditions for the analogue of C5) exist, they are beyond the scope of this work so leave their existence as a conjecture.

For simplicity we will consider a two time point treatment with baseline covariates $L(0)$, a treatment $A(0)$, intermediate covariate $L(1)$, a treatment $A(1)$, and an outcome Y which comes after all treatments and covariates. The extension to the more general multiple time point case follows the same general arguments. We use the notation $\bar{A}(1) = (A(0), A(1))$ and $\bar{L}(1) = (L(0), L(1))$. The presentation in this section parallels that given in van der Laan and Luedtke [2014b], and we refer to the reader to that work for a more detailed description of the two time point problem. For the sake of simplicity we do not consider censoring, though censoring can easily be incorporated using the techniques in the referenced paper. The notation is similar in spirit to that of the rest of the paper, though there is some notational overload (e.g. d now used to represent a two time point rule, $\Psi(P_0)$ now the mean outcome under a two time point treatment).

A dynamic rule $d = (d_{A(0)}, d_{A(1)})$ consists of two rules, one for each time point. The first time point rule $d_{A(0)}$ may be a function of $L(0)$, while the second time point rule $d_{A(1)}$ may rely on $L(0)$, $A(0)$, and $L(1)$. Notationally, we use $d(O)$ to mean $(d_{A(0)}(L(0)), d_{A(1)}(A(0), \bar{L}(1)))$. For a rule d, define

$$\Psi_d(P_0) \triangleq E_0 E_0\Big[E_0\left[Y|\bar{A}(1) = d(O), \bar{L}(1)\right] \Big| A(0) = d_{A(0)}(L(0)), L(0)\Big].$$

A (possibly non-unique) optimal rule is given by $d_0^* \triangleq \arg\max_d \Psi_d(P_0)$. Our parameter of interest is

$$\Psi(P_0) \triangleq \Psi_{d_0^*}(P_0).$$

For a distribution P, define the treatment mechanisms $g_{A(0)}(P)(O) \triangleq Pr_P(A(0)|L(0))$ and $g_{A(1)}(P)(O) \triangleq Pr_P(A(1)|A(0), \bar{L}(1))$. Also define

$$\begin{aligned}\tilde{D}(d, P)(O) \triangleq & D_2^*(d, P)(O) + D_1^*(d, P)(O) \\ & + E_0\Big[E_0\left[Y|\bar{A}(1) = d(O), \bar{L}(1)\right] \Big| A(0) = d_{A(0)}(L(0)), L(0)\Big],\end{aligned}$$

where

$$D_1^*(d,P) = \frac{I(A(0)=d_{A(0)}(L(0)))}{g_{A(0)}(P)(O)}\Big(E_P\left[Y \mid \bar{A}(1)=d(O),\bar{L}(1)\right]$$
$$- E_0\Big[E_0\left[Y|\bar{A}(1)=d(O),\bar{L}(1)\right]\Big|A(0)=d_{A(0)}(L(0)),L(0)\Big]\Big),$$
$$D_2^*(d,P) = \frac{I(\bar{A}(1)=d(O))}{\prod_{k=0}^{1} g_{A(k)}(P)(O)}\left(Y - E_P\left[Y \mid \bar{A}(1)=d(O),\bar{L}(1)\right]\right).$$

We can now generalize the CI presented in Section 5 to the two time point case. Let $\{\ell_n\}$ be some sequence of natural numbers. For each $j > \ell_n$, let $\hat{P}_{n,j}$ represent some estimate of P_0 and $d_{n,j}$ some estimate of d_0^*, each based only on the observations $O_1, ..., O_{j-1}$. We really only need estimates of $Pr_P(A(0)|L(0))$, $Pr_P(A(1)|A(0),\bar{L}(1))$, and the two conditional regressions in the definition of $D_1^*(d,P)$. Define

$$\tilde{\sigma}_{0,n,j}^2 \triangleq \mathrm{Var}_{P_0}\left(\tilde{D}(d_{n,j},\hat{P}_{n,j})\Big| O_1,...,O_{j-1}\right).$$

Let $\tilde{\sigma}_{n,j}^2$ represent an estimate of $\tilde{\sigma}_{0,n,j}^2$ based on (some subset of) the observations $(O_1,...,O_{j-1})$. Also define

$$\Gamma_n \triangleq \frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n}\tilde{\sigma}_{n,j}^{-1}.$$

Define our estimate $\hat{\Psi}(P_n)$ of $\Psi(P_0)$ as

$$\hat{\Psi}(P_n) \triangleq \Gamma_n^{-1}\frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n}\tilde{\sigma}_{n,j}^{-1}\tilde{D}_{n,j}(O_j) = \frac{\sum_{j=\ell_n+1}^{n}\tilde{\sigma}_{n,j}^{-1}\tilde{D}_{n,j}(O_j)}{\sum_{j=\ell_n+1}^{n}\tilde{\sigma}_{n,j}^{-1}},$$

where $\tilde{D}_{n,j}(o) \triangleq \tilde{D}(d_{n,j},\hat{P}_{n,j})(o)$. The following $1-\alpha$ CI for $\Psi(P_0)$ is asymptotically valid under conditions similar to C1) through C5) presented in the main text:

$$\hat{\Psi}(P_n) \pm z_{1-\alpha/2}\frac{\Gamma_n^{-1}}{\sqrt{n-\ell_n}}.$$

We now state a formal theorem establishing the validity of this CI. To avoid stating the somewhat messy analogue of $R_{1n} = o_{P_0}(n^{-1/2})$ in Condition C4) in the two time point case we assume that the treatment mechanisms in each $\hat{P}_{n,j}$ is correctly specified, though this assumption is not necessary since we really only need to control a double robust remainder term.

THEOREM 1. *Suppose that*

A.C1) $n - \ell_n$ *diverges to infinity as* n *diverges to infinity.*
A.C2) Lindeberg-like condition: for all $\epsilon > 0$,

$$\frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n} E_0\Big[\left(\frac{\tilde{D}_{n,j}(O)}{\tilde{\sigma}_{n,j}}\right)^2 T_{n,j}(O)\Big| O_1,...,O_{j-1}\Big] = o_{P_0}(1),$$

where $T_{n,j}(O) \triangleq I\left(\frac{|\tilde{D}_{n,j}(O)|}{\tilde{\sigma}_{n,j}} > \epsilon\sqrt{n-\ell_n}\right)$.
A.C3) $\frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n}\frac{\tilde{\sigma}_{0,n,j}^2}{\tilde{\sigma}_{n,j}^2}$ *converges to* 1 *in probability.*
A.C4) $g_{A(0)}(\hat{P}_{n,j}) = g_{A(0)}(P_0)$ *and* $g_{A(1)}(\hat{P}_{n,j}) = g_{A(1)}(P_0)$ *for all* n, j.
A.C5) $\frac{1}{n-\ell_n}\sum_{j=\ell_n+1}^{n}\frac{\Psi_{d_{n,j}}(P_0)-\Psi(P_0)}{\tilde{\sigma}_{n,j}} = o_{P_0}(n^{-1/2})$.

Then,

$$\Gamma_n\sqrt{n-\ell_n}\left(\hat{\Psi}(P_n) - \Psi(P_0)\right) \rightsquigarrow N(0,1).$$

It follows that an asymptotically valid $1-\alpha$ *CI for* $\Psi(P_0)$ *is given by*

$$\hat{\Psi}(P_n) \pm z_{1-\alpha/2}\frac{\Gamma_n^{-1}}{\sqrt{n-\ell_n}},$$

where $z_{1-\alpha/2}$ *denotes the* $1-\alpha/2$ *quantile of a standard normal random variable.*

Readers will notice that, except for Condition A.C4), the conditions in the above theorem are, notationally, identical to those stated in the main text for Theorem 2. We have restated these conditions to emphasize that the notation in these conditions now refers to the two time point objects defined in this section, rather than to the single time point objects from the main text.

PROOF OF THEOREM 1. We can follow the Proof of Theorem 2 through (A.8). From Bang and Robins [2005], we know that, for any treatment rule d, the correct treatment mechanism specification from Condition A.C4) yields an exact first-order representation: $\Psi_d(P_0) = E_0[\tilde{D}(d, \hat{P}_{n,j})(O)]$ for all n, j. Thus the notationally identical (A.10) holds. The remainder of the proof goes through without any further changes. □

Eliciting simple sufficient conditions under which A.C5) holds is an area of future work.

The generalization to problems with more time points follows along the same lines as the generalization to the two time point problem.

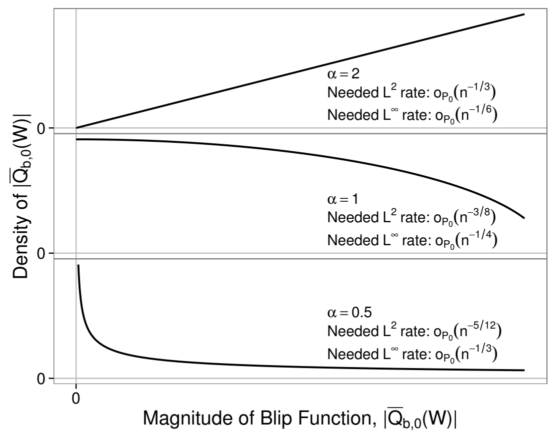

FIG A.1. *Examples of three densities of $|\bar{Q}_{b,0}(W)|$ whose corresponding cumulative distribution functions satisfy (16). If the rate of convergence of $\bar{Q}_{b,n} - \bar{Q}_{b,0}$ to zero in $L^2(P_0)$ or $L^\infty(P_0)$ attains the rates indicated above indicated above, then Condition C5) will be satisfied for the plug-in optimal rule estimate considered in Theorem 8.*

SUPPLEMENTARY APPENDIX C: SUPPLEMENTARY FIGURES

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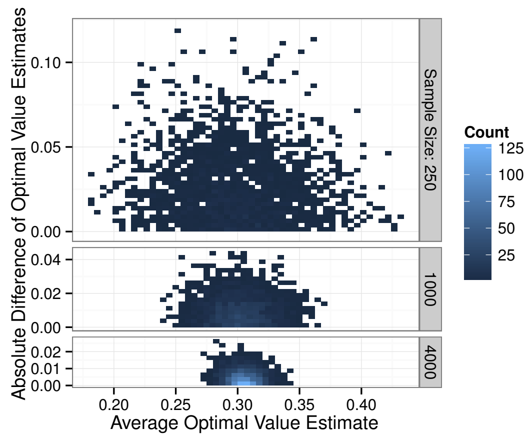

FIG A.2. *Comparison of optimal value estimates given two different permutations of a data set generated according to C-E. The horizontal axis shows the average of the optimal value estimates across the two permutations, and the vertical axis shows the absolute difference between these two optimal value estimates. Squares represent number of observations (across 2000 Monte Carlo draws) which have a given average optimal value-absolute difference combination. The difference between these two estimates decreases as sample size grows.*

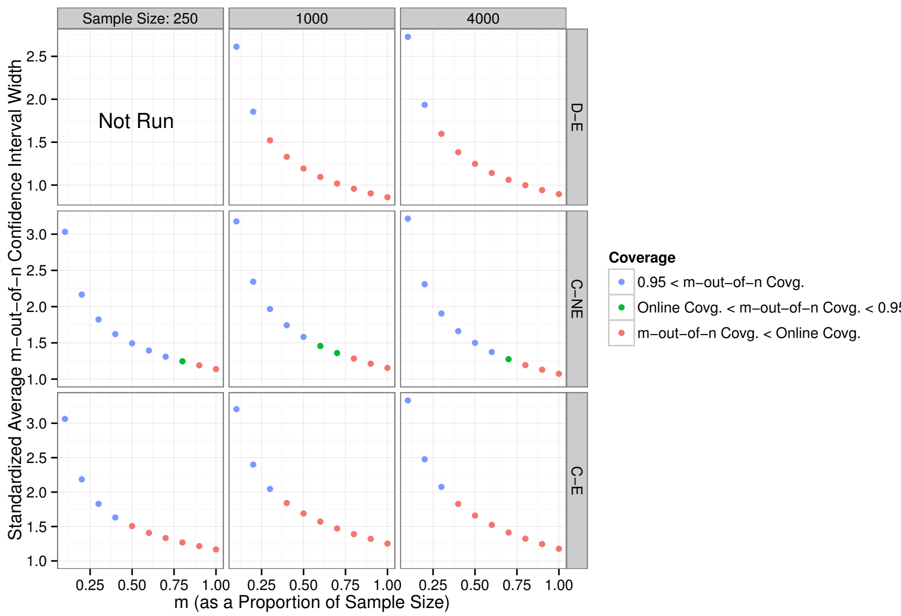

FIG A.3. *Performance of the m-out-of-n bootstrap at sample sizes 250, 1000, and 4000 (NPMLE not run at sample size 250). The vertical axis shows the average CI width divided by the average CI width of the online one-step CI. That is, any vertical axis value above 1 indicates the m-out-of-n bootstrap has on average wider CIs than the online one-step CI.*