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Supplemental Information

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Statistical mechanics of an elastically pinned membrane: Static profile and correlations

Josip Augustin Janeš,[1,2] Henning Stumpf,[1] Daniel Schmidt,[1,3] Udo Seifert,[3] and Ana-Sunčana Smith[1,2,*]

[1] *PULS Group, Institut für Theoretische Physik and Cluster of Excellence: Engineering of Advanced Materials, Friedrich Alexander Universität Erlangen-Nürnberg, 91052 Erlangen, Germany*

[2] *Institut Ruđer Bošković, 10000 Zagreb, Croatia*

[3] *II. Institut für Theoretische Physik, Universität Stuttgart, 70569 Stuttgart, Germany*

CONTENTS

I. HAMILTONIAN OF THE SYSTEM

Our model consists of one flexible pinning site (harmonic spring of an elastic constant λ and rest length l_0) that confines fluctuations of a tensed membrane (bending rigidity κ, tension σ). The latter resides in a harmonic non-specific potential (γ, minimum at h_0 above the substrate). Consequently, the membrane adopts a shape profile $h(\mathbf{r})$

* author to whom correspondence should be addressed: smith@physik.uni-erlangen.de

along the lateral position $\mathbf{r}$ around the pinning at $\mathbf{r}_0$, introduced by the delta distribution $\delta(\mathbf{r}-\mathbf{r}_0)$. This situation is captured in a usual way by the energy functional

$$\mathcal{H}=\int_A \mathrm{d}\mathbf{r}\left[\frac{\kappa}{2}\left(\nabla^2 h(\mathbf{r})\right)^2+\frac{\sigma}{2}\left(\nabla h(\mathbf{r})\right)^2+\frac{\gamma}{2}\left(h(\mathbf{r})-h_0\right)^2+\frac{\lambda}{2}\left(h(\mathbf{r})-l_0\right)^2\delta(\mathbf{r}-\mathbf{r}_0)\right], \tag{SI-I.1}$$

which combines the Helfrich-Hamiltonian for the membrane in the Monge gauge and a harmonic spring for the pinning site. The integration goes over the membrane surface A. Here and throughout the paper, the energy scale $k_{\mathrm{B}}T$ with Boltzmann constant k_{B} and absolute temperature T is set to unity.

To simplify further calculations we will convert to the coordinate system whose horizontal axes coincide with the minimum of the non-specific potential. This is done by the vertical translation

$$h(\mathbf{r})=u(\mathbf{r})+h_0. \tag{SI-I.2}$$

Eq. SI-I.1 then becomes

$$\mathcal{H}=\int_A \mathrm{d}\mathbf{r}\left[\frac{\kappa}{2}\left(\nabla^2 u(\mathbf{r})\right)^2+\frac{\sigma}{2}\left(\nabla u(\mathbf{r})\right)^2+\frac{\gamma}{2}\left(u(\mathbf{r})\right)^2+\frac{\lambda}{2}\left(u(\mathbf{r})-(l_0-h_0)\right)^2\delta(\mathbf{r}-\mathbf{r}_0)\right]. \tag{SI-I.3}$$

We can now simplify the Hamiltonian by the integral identities,

$$\begin{aligned}\int_A \mathrm{d}\mathbf{r}\left(\nabla u(\mathbf{r})\right)^2&=-\int_A \mathrm{d}\mathbf{r}\left(u(\mathbf{r})\nabla^2 u(\mathbf{r})\right)+\int_{\partial A}\mathrm{d}l\left(u(\mathbf{r})\frac{\partial u(\mathbf{r})}{\partial n}\right)\\ \int_A \mathrm{d}\mathbf{r}\left(\nabla^2 u(\mathbf{r})\right)^2&=\int_A \mathrm{d}\mathbf{r}\left(u(\mathbf{r})\nabla^4 u(\mathbf{r})\right)+\int_{\partial A}\mathrm{d}l\left(\nabla^2 u(\mathbf{r})\frac{\partial u(\mathbf{r})}{\partial n}-u(\mathbf{r})\frac{\partial\nabla^2 u(\mathbf{r})}{\partial n}\right),\end{aligned} \tag{SI-I.4}$$

where $\int_{\partial A}\mathrm{d}l$ is a line integral over the membrane boundary ∂A and $\partial/\partial n$ is a derivative normal to the boundary. Because the membrane is in the minimum of the non-specific potential far from the pinning site, the boundary terms in eq. SI-I.4 vanish. Inserting eq. SI-I.4 with vanishing boundary terms into Hamiltonian eq. SI-I.3 gives

$$\mathcal{H}=\frac{1}{2}\int_A \mathrm{d}\mathbf{r}\left[u(\mathbf{r})\left(\kappa\nabla^4-\sigma\nabla^2+\gamma+\lambda\delta(\mathbf{r}-\mathbf{r}_0)\right)u(\mathbf{r})-2u(\mathbf{r})\left(\lambda(l_0-h_0)\delta(\mathbf{r}-\mathbf{r}_0)\right)+\lambda(l_0-h_0)^2\delta(\mathbf{r}-\mathbf{r}_0)\right]. \tag{SI-I.5}$$

Thermalized membrane will fluctuate around the shape that minimizes the Hamiltonian, motivating the notation

$$u(\mathbf{r})=\langle u(\mathbf{r})\rangle+v(\mathbf{r}), \tag{SI-I.6}$$

where $\langle u(\mathbf{r})\rangle$ is the thermal equilibrium mean shape, where brackets $\langle ...\rangle$ indicate the canonical ensemble average, and $v(\mathbf{r})$ fluctuations around the mean shape. Inserting SI-I.6 into SI-I.5 and collecting the powers of $v(\mathbf{r})$ gives

$$\begin{aligned}\mathcal{H}=\mathcal{H}[\langle u(\mathbf{r})\rangle]&+\int_A \mathrm{d}\mathbf{r}\left\{\left[\kappa\nabla^4-\sigma\nabla^2+\gamma+\lambda\delta(\mathbf{r}-\mathbf{r}_0)\right]\langle u(\mathbf{r})\rangle-\lambda(l_0-h_0)\delta(\mathbf{r}-\mathbf{r}_0)\right\}v(\mathbf{r})+\\&+\frac{1}{2}\int_A \mathrm{d}\mathbf{r}\ v(\mathbf{r})\left[\kappa\nabla^4-\sigma\nabla^2+\gamma+\lambda\delta(\mathbf{r}-\mathbf{r}_0)\right]v(\mathbf{r}).\end{aligned} \tag{SI-I.7}$$

Naturally, for vanishing fluctuations $v(\mathbf{r})=0$, Hamiltonian SI-I.7 reduces to the energy of the mean shape $\mathcal{H}[\langle u(\mathbf{r})\rangle]$. Demanding that the term linear in fluctuations $v(\mathbf{r})$ (first variation of the Hamiltonian) vanishes, gives the equation for the mean shape $\langle u(\mathbf{r})\rangle$

$$\left[\kappa\nabla^4-\sigma\nabla^2+\gamma+\lambda\delta(\mathbf{r}-\mathbf{r}_0)\right]\langle u(\mathbf{r})\rangle=\lambda(l_0-h_0)\delta(\mathbf{r}-\mathbf{r}_0). \tag{SI-I.8}$$

Third term in SI-I.7 gives the energy contained in the fluctuations

$$\mathcal{H}_{\text{fluct}}=\frac{1}{2}\int_A \mathrm{d}\mathbf{r}\ v(\mathbf{r})\left[\kappa\nabla^4-\sigma\nabla^2+\gamma+\lambda\delta(\mathbf{r}-\mathbf{r}_0)\right]v(\mathbf{r}). \tag{SI-I.9}$$

For fluctuations satisfying the eigenequation

$$\left[\kappa\nabla^4 - \sigma\nabla^2 + \gamma + \lambda\delta(\mathbf{r}-\mathbf{r}_0)\right] v(\mathbf{r}) = Ev(\mathbf{r}), \tag{SI-I.10}$$

where E is the eigenvalue, system Hamiltonian becomes quadratic in fluctuations

$$\mathcal{H}_{\text{fluct}} = \mathcal{H} - \mathcal{H}[\langle u(\mathbf{r})\rangle] = \frac{1}{2}\int_A \mathrm{d}\mathbf{r}\; Ev^2(\mathbf{r}). \tag{SI-I.11}$$

A. Two-point correlation function in terms of normal modes

As is shown in the next section, there is in fact a family of functions $\psi_j(\mathbf{r})$, known as the normal modes of the system, satisfying eq. SI-I.10

$$\left[\kappa\nabla^4 - \sigma\nabla^2 + \gamma + \lambda\delta(\mathbf{r})\right] \psi_j(\mathbf{r}) = E_j\psi_j(\mathbf{r}), \tag{SI-I.12}$$

with eigenvalues E_j. Normal modes $\psi_j(\mathbf{r})$ form an orthonormal basis on a space of functions representing the profile of a pinned membrane. Orthonormality implies

$$\int_{\mathbb{R}^2} \mathrm{d}\mathbf{r}\; \psi_i(\mathbf{r})\psi_j(\mathbf{r}) = \delta_{ij} \tag{SI-I.13}$$

and being a basis implies that the fluctuations $v(\mathbf{r})$ can be expanded in the normal modes as

$$v(\mathbf{r}) = \sum_i \eta_i\psi_i(\mathbf{r}), \tag{SI-I.14}$$

where η_i are the coefficients of expansion. Inserting SI-I.14 into the Hamiltonian SI-I.11 and using eqs. SI-I.12 and SI-I.13 gives

$$\mathcal{H}_{fluct} = \frac{1}{2}\sum_i E_i\eta_i^2. \tag{SI-I.15}$$

Using the equipartition theorem

$$\left\langle \frac{\delta H_{fluct}}{\delta\eta_i}\eta_j \right\rangle = E_i\left\langle \eta_i\eta_j\right\rangle = \delta_{ij}, \tag{SI-I.16}$$

we find the correlation between the normal mode expansion coefficients

$$\langle\eta_i\eta_j\rangle = \frac{\delta_{ij}}{E_i}, \tag{SI-I.17}$$

which can be used to find the spatial two-point correlation function of the system as follows

$$\langle v(\mathbf{r})v(\mathbf{r}')\rangle = \sum_{i,j}\langle\eta_i\eta_j\rangle\,\psi_i(\mathbf{r})\psi_j(\mathbf{r}') = \sum_j \frac{\psi_j(\mathbf{r})\psi_j^*(\mathbf{r}')}{E_j}, \tag{SI-I.18}$$

where we have used eqs. SI-I.14 and SI-I.17. The explicit form of the normal modes ψ_j is found in the next section.

II. ORTHOGONAL BASIS OF A MEMBRANE PINNED BY A HARMONIC SPRING

A. Statement of the problem

The aim of this section is to find a complete set of orthogonal functions $\psi_j(\mathbf{r})$ by solving the eigenequation

$$\left[\kappa\nabla^4 - \sigma\nabla^2 + \gamma\right]\psi_j(\mathbf{r}) = E_j\psi_j(\mathbf{r}) \tag{SI-II.1}$$

on $\mathbb{R}^2$ with the following boundary conditions:

1.
$$\psi_j(\mathbf{r}=0) < \infty. \tag{SI-II.2a}$$

2.
$$\lim_{\epsilon\to 0}\int\limits_{D(\epsilon)} \mathrm{d}\mathbf{r}\left[\kappa\nabla^4 - \sigma\nabla^2 + \gamma\right]\psi_j(\mathbf{r}) + \lambda\psi_j(0) = 0, \tag{SI-II.2b}$$

where $D(\epsilon) = \{\mathbf{r}\in\mathbb{R}^2 | |\mathbf{r}| \le \epsilon\}$ is a disk of radius $\epsilon \in [0,\infty\rangle$.

3.
$$\psi_j(\mathbf{r}\to\infty) = 0. \tag{SI-II.2c}$$

4.
$$\Delta\psi_j(\mathbf{r}\to\infty) = 0. \tag{SI-II.2d}$$

This is equivalent to solving the eigenequation

$$\left[\kappa\nabla^4 - \sigma\nabla^2 + \gamma + \lambda\delta(\mathbf{r})\right]\psi_j(\mathbf{r}) = E_j\psi_j(\mathbf{r}), \tag{SI-II.3}$$

with boundary conditions given by eqs. SI-II.2a, SI-II.2c and SI-II.2d. To show this we integrate eq. SI-II.3 over $\Omega \setminus D(\epsilon)$, where $\Omega \subseteq \mathbb{R}^2$, and over $D(\epsilon)$ to get the following two equations

$$\lim_{\epsilon\to 0}\int\limits_{\Omega\setminus D(\epsilon)} \mathrm{d}\mathbf{r}\left[\kappa\nabla^4 - \sigma\nabla^2 + \gamma + \lambda\delta(\mathbf{r})\right]\psi_j(\mathbf{r}) = \lim_{\epsilon\to 0}\int\limits_{\Omega\setminus D(\epsilon)} \mathrm{d}\mathbf{r}E_j\ \psi_j(\mathbf{r}), \tag{SI-II.4}$$

$$\lim_{\epsilon\to 0}\int\limits_{D(\epsilon)} \mathrm{d}\mathbf{r}\left[\kappa\nabla^4 - \sigma\nabla^2 + \gamma + \lambda\delta(\mathbf{r})\right]\psi_j(\mathbf{r}) = \lim_{\epsilon\to 0}\int\limits_{D(\epsilon)} \mathrm{d}\mathbf{r}E_j\ \psi_j(\mathbf{r}). \tag{SI-II.5}$$

Eq. SI-II.4 gives

$$\int\limits_{\Omega\setminus\{0\}} \mathrm{d}\mathbf{r}\left[\kappa\nabla^4 - \sigma\nabla^2 + \gamma - E_j\right]\psi_j(\mathbf{r}) = 0, \tag{SI-II.6}$$

as delta distribution term does not contribute because $\mathbf{r}=0$ is not included in the domain of integration. Because Ω is arbitrary, eq. SI-II.6 defines the general solution to eq. SI-II.1. Assuming condition eq. SI-II.2a holds results in

$$\lim_{\epsilon\to 0}\int\limits_{D(\epsilon)} \mathrm{d}\mathbf{r}E_j\ \psi_j(\mathbf{r}) = 0 \tag{SI-II.7}$$

and eq. SI-II.5 gives the boundary condition eq. SI-II.2b. Therefore, we have arrived at the initial problem defined by eq. SI-II.1 and boundary conditions given by eqs. SI-II.2a-SI-II.2d. Physically, condition eq. SI-II.2a corresponds to the reasonable requirement that the height of the membrane profile is finite at the origin. Condition eq. SI-II.2b introduces the effect of the pinning at $\mathbf{r}=0$. Conditions given by eqs. SI-II.2c and SI-II.2d correspond to the assumption that the membrane is in the minimum of the non-specific potential far from the pinning and therefore has vanishing height and curvature there.

1. Hermiticity of the eigenoperator

We require Hermiticity of the operator $\left[\kappa\nabla^4 - \sigma\nabla^2 + \gamma + \lambda\delta(\mathbf{r})\right]$ as eigenvalues E_j in eq. SI-II.3 are connected with the energy and therefore have to be real. Consequently, the operator has to satisfy the well-known property

$$\left\langle \left[\kappa\nabla^4 - \sigma\nabla^2 + \gamma + \lambda\delta(\mathbf{r})\right]\psi_i(\mathbf{r}), \psi_j(\mathbf{r})\right\rangle = \left\langle \psi_j(\mathbf{r}), \left[\kappa\nabla^4 - \sigma\nabla^2 + \gamma + \lambda\delta(\mathbf{r})\right]\psi_i(\mathbf{r})\right\rangle, \tag{SI-II.8}$$

where $\psi_i(\mathbf{r})$ and $\psi_j(\mathbf{r})$ are two different eigenfunctions of the operator. This leads to conditions

$$\begin{aligned}\int\limits_{\partial A} \mathrm{d}l \left(\psi_i(\mathbf{r})\frac{\partial\psi_j(\mathbf{r})}{\partial n}\right) &= \int\limits_{\partial A} \mathrm{d}l \left(\psi_j(\mathbf{r})\frac{\partial\psi_i(\mathbf{r})}{\partial n}\right)\\ \int\limits_{\partial A} \mathrm{d}l \left(\nabla^2\psi_i(\mathbf{r})\frac{\partial\psi_j(\mathbf{r})}{\partial n} - \psi_j(\mathbf{r})\frac{\partial\nabla^2\psi_i(\mathbf{r})}{\partial n}\right) &= \int\limits_{\partial A} \mathrm{d}l \left(\nabla^2\psi_j(\mathbf{r})\frac{\partial\psi_i(\mathbf{r})}{\partial n} - \psi_i(\mathbf{r})\frac{\partial\nabla^2\psi_j(\mathbf{r})}{\partial n}\right).\end{aligned} \tag{SI-II.9}$$

Conditions given by eq. SI-II.9 are restricting the set of boundary conditions we can choose from when solving the eigenequation SI-II.3. Vanishing of the boundary terms in eq. SI-I.4 and boundary conditions given by eqs. SI-II.2c-SI-II.2d satisfy hermiticity conditions given by eq. SI-II.9.

2. Problem in terms of polar coordinates

Singularity at the origin due to the pinning and the radial symmetry with respect to the pinning make polar coordinates (r,ϕ) a natural choice for solving the problem. We will first derive the orthogonal basis $\psi_j(\mathbf{r})$ defined on a disk $D(P) = \{r \in \mathbb{R}^2 | |\mathbf{r}| \le P\}$, which enables us to take advantage of the radial symmetry in the system. Physically, disk $D(P)$ corresponds to a circular membrane of radius P. With the orthogonal basis for a finite system in hand, we can expand the membrane profile $u(\mathbf{r})$ in a generalized Fourier series

$$u(\mathbf{r}) = \sum_j \frac{\langle u(\mathbf{r}), \psi_j(\mathbf{r}, P)\rangle}{\langle \psi_j(\mathbf{r}, P), \psi_j(\mathbf{r}, P)\rangle}\psi_j(\mathbf{r}, P) \tag{SI-II.10}$$

where we have emphasized that the basis functions $\psi_j(\mathbf{r}, P)$ depend on the membrane radius P. $\langle u(\mathbf{r}), \psi_j(\mathbf{r}, P)\rangle$ are the projections of $u(\mathbf{r})$ on the basis elements $\psi_j(\mathbf{r}, P)$, $\langle \psi_j(\mathbf{r}, P), \psi_j(\mathbf{r}, P)\rangle$ are the normalization constants and the generalized scalar product is defined as

$$\langle x(\mathbf{r}), y(\mathbf{r})\rangle = \int\limits_{D(P)} \mathrm{d}\mathbf{r}\; x(\mathbf{r})y^*(\mathbf{r}), \tag{SI-II.11}$$

where * denotes complex conjugation. As will be shown, taking the limit $P \to \infty$ of eqs. SI-II.10 and SI-II.11 gives us the expansion of an infinite planar membrane in the corresponding orthogonal basis $\psi_j(\mathbf{r}, P \to \infty) = \psi_j(\mathbf{r})$ on $\mathbb{R}^2$.

B. Orthogonal series representation of the membrane profile - Solution to the finite radius problem

Motivated by the radial symmetry of the problem, we insert into eq. SI-II.1 the cylindrical wave ansatz $\psi_j(\mathbf{r}, P) = \psi_{nm}(\mathbf{r}, P) = J_m(k_{nm}(P)r)e^{im\phi}$, where n and m are integers and $J_m(r)$ is the m-th order Bessel function of the first kind, while scale factors $k_{nm}(P)$ contain the P dependence. Using

$$\nabla^2 J_m(k_{nm}r)\mathrm{e}^{im\phi} = -k_{nm}^2 J_m(k_{nm}r)e^{im\phi}, \tag{SI-II.12}$$

leads to the eigenvalues

$$E_{nm} = \kappa k_{nm}^4 + \sigma k_{nm}^2 + \gamma. \tag{SI-II.13}$$

There are three distinct cases:

- $E_{nm} < \gamma$, $k_{nm} \in i\mathbb{R}$,
- $E_{nm} = \gamma$, $k_{nm} = 0$ or $k_{nm} = \pm i\sqrt{\sigma/\kappa}$,

- $E_{nm} > \gamma$, $k_{nm} \in \mathbb{R}_{>0}$.

With boundary conditions eqs. SI-II.2c and SI-II.2d and $E_{nm} \leq \gamma$, solution is a zero-height, flat profile $\psi(\mathbf{r}) = 0$. Therefore only the case $E_{nm} > \gamma$ and $k_{nm} > 0$ is interesting. In this case, solving eq. SI-II.13 for k_{nm} gives four solutions

$$k_{nm,1} = q_{nm}, \quad k_{nm,2} = -q_{nm}, \quad k_{nm,3} = i\sqrt{q_{nm}^2 + \frac{\sigma}{\kappa}} := iQ_{nm}, \quad k_{nm,4} = -i\sqrt{q_{nm}^2 + \frac{\sigma}{\kappa}} := -iQ_{nm}, \tag{SI-II.14}$$

$$\text{with} \quad q_{nm} = \sqrt{\frac{1}{2}\left(-\frac{\sigma}{\kappa} + \sqrt{\left(\frac{\sigma}{\kappa}\right)^2 + \frac{4(E_{nm} - \gamma)}{\kappa}}\right)}. \tag{SI-II.15}$$

The general solution to eq. SI-II.1 should be a linear combination of four linearly independent terms $J_m(k_{nm,i}r)e^{im\phi}$, where $i = \{1, 2, 3, 4\}$. But as $J_m(x)$ and $J_m(-x)$ are linearly dependent for integer m, solution is a linear combination of Bessel functions of the first and second kind J_m and Y_m and modified Bessel functions of the first and second kind K_m and I_m:

$$\begin{aligned}\psi_{nm}(\mathbf{r}, P) =& \left(a_{nm}J_m(q_{nm}r) + b_{nm}Y_m(q_{nm}r) + c_{nm}K_m(Q_{nm}r) + d_{nm}I_m(Q_{nm}r)\right)e^{im\phi} \\ =& R_{nm}(r)e^{im\phi}.\end{aligned} \tag{SI-II.16}$$

$R_{nm}(r)$ is a shorthand notation for the radial part of $\psi_{nm}(\mathbf{r}, P)$, where P dependence of $R_{nm}(r)$ is not shown explicitly, but is assumed. Hence, expansion eq. SI-II.10 can be written more precisely as

$$u(\mathbf{r}) = \sum_{m=-\infty}^{\infty}\sum_{n}\frac{\langle u(\mathbf{r}), R_{nm}(r)e^{im\phi}\rangle}{\langle R_{nm}(r)e^{im\phi}, R_{nm}(r)e^{im\phi}\rangle}R_{nm}(r)e^{im\phi}. \tag{SI-II.17}$$

Coefficients $\{a_{nm}, b_{nm}, c_{nm}, d_{nm}\}$ are specified by boundary conditions. Boundary condition eq. SI-II.2a is given by requiring

- $R_{nm}(r)$ stays finite when $r \to 0$.

The Bessel functions of the first kind, J_m and I_m, fulfil this condition ($J_0(0) = I_0(0) = 1$ and $J_m(0) = I_m(0) = 0$ for $m > 0$). The remaining Bessel functions Y_m and K_m diverge for $r \to 0$. However, for $m = 0$, both Bessel functions diverge logarithmically such that the sum $b_{n0}Y_0(q_{nm}r) + c_{n0}K_0(q_{nm}r)$ stays finite with $c_{n0} = 2b_{n0}/\pi$. Consequently,

$$R_{nm}(r) = a_{nm}J_m(q_{nm}r) + d_{nm}I_m(Q_{nm}r) + \delta_{m0}b_{nm}\left(Y_m(q_{nm}r) + \frac{2}{\pi}K_m(Q_{nm}r)\right), \tag{SI-II.18}$$

where δ_{m0} is the Kronecker delta. Therefore, the term multiplied by δ_{m0} is contributing only for $m = 0$. Second boundary condition (eq. SI-II.2b) is given by requiring

- $$\lim_{\epsilon \to 0}\int_{D(\epsilon)} \mathrm{d}\mathbf{r}\left[\kappa\nabla^4 - \sigma\nabla^2 + \gamma\right]\psi_j(\mathbf{r}) + \lambda\psi_j(0) = 0. \tag{SI-II.19}$$

By solving the integral, one obtains

$$b_{n0} = \Pi_n(a_{n0} + d_{n0}), \tag{SI-II.20}$$

where

$$\Pi(q_{n0}) = \Pi_n = \frac{\lambda}{8\kappa(q_{n0}^2 + \frac{\sigma}{2\kappa}) + \frac{\lambda}{\pi}\ln(1 + \frac{\sigma}{\kappa q_{n0}^2})}\ . \tag{SI-II.21}$$

Inserting eq. SI-II.20 into eq. SI-II.18 leads to

$$R_{nm}(r) = a_{nm}J_m(q_{nm}r) + d_{nm}I_m(Q_{nm}r) + \delta_{m0}\Pi_n(a_{nm} + d_{nm})\left(Y_m(q_{nm}r) + \frac{2}{\pi}K_m(Q_{nm}r)\right). \tag{SI-II.22}$$

At the membrane boundary $r = P$, conditions are given by

- vanishing of the membrane profile (eq. SI-II.2c)

$$R_{nm}(P) = 0 \tag{SI-II.23}$$

and

- vanishing of the Laplacian of the membrane profile (eq. SI-II.2d)

$$\Delta R_{nm}(P) = 0. \tag{SI-II.24}$$

Conditions given by eqs.SI-II.23 and SI-II.24 lead to

$$d_{nm} = -\delta_{m0} X_n a_{nm}, \tag{SI-II.25}$$

where

$$X(q_{n0}) = X_n = \frac{\Pi_n \frac{2}{\pi} K_0(Q_{n0}P)/I_0(Q_{n0}P)}{1 + \Pi_n \frac{2}{\pi} K_0(Q_{n0}P)/I_0(Q_{n0}P)}. \tag{SI-II.26}$$

Inserting eq. SI-II.25 into eq. SI-II.22 leads to

$$R_{nm}(r) = a_{nm} \left\{ J_m(q_{nm}r) - \delta_{m0} \left[X_n I_m(Q_{nm}r) + \Pi_n(X_n - 1) \left(Y_m(q_{nm}r) + \frac{2}{\pi} K_m(Q_{nm}r) \right) \right] \right\}. \tag{SI-II.27}$$

Coefficient a_{nm} will be used to normalize the basis functions just before we take the limit $P \to \infty$. Conditions given by eqs. SI-II.23 and SI-II.24 lead to the condition for the allowed q_{mn}

$$\frac{Y_m(q_{nm}P)}{J_m(q_{nm}P)} = -\left(\frac{1}{\delta_{m0}\Pi_n} + \frac{2}{\pi} \frac{K_m(Q_{nm}P)}{I_m(Q_{nm}P)} \right). \tag{SI-II.28}$$

To summarize, the series representation of a circular pinned membrane of radius P is given in terms of orthogonal functions $R_{nm}(r)e^{im\phi}$ as

$$u(\mathbf{r}) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{\langle u(\mathbf{r}), R_{nm}(r)e^{im\phi} \rangle}{\langle R_{nm}(r)e^{im\phi}, R_{nm}(r)e^{im\phi} \rangle} R_{nm}(r)e^{im\phi}, \tag{SI-II.29}$$

where $R_{nm}(r)$ is given by eq. SI-II.27, while q_{nm} and eigenvalues E_{nm} are given by equations SI-II.15 and SI-II.28.

C. Integral representation of the membrane profile - Solution to the infinite radius problem

1. *Asymptotic behaviour of the series solution for large membrane radius*

If the membrane spatial dimension is much larger then the spatial dimension of the pinning effect, system behaves as if the membrane radius is infinite and the pinning is localized at the origin. To describe this situation we investigate the asymptotic behavior of the series representation (eq. SI-II.29) for large radius P. As $q_{nm} > 0$, we use asymptotic forms of Bessel functions for large P [1]

$$J_m(q_{nm}P) \sim \sqrt{\frac{2}{\pi q_{nm}P}} \cos(q_{nm}P - \frac{\pi}{4} - m\frac{\pi}{2}), \qquad Y_m(q_{nm}P) \sim \sqrt{\frac{2}{\pi q_{nm}P}} \sin(q_{nm}P - \frac{\pi}{4} - m\frac{\pi}{2}) \quad \text{as} \quad q_{nm}P \to \infty, \tag{SI-II.30}$$

$$I_m(Q_{nm}P) \sim \frac{1}{\sqrt{2\pi}} \frac{e^{Q_{nm}P}}{\sqrt{Q_{nm}P}}, \qquad K_m(Q_{nm}P) \sim \sqrt{\frac{\pi}{2}} \frac{e^{-Q_{nm}P}}{\sqrt{Q_{nm}P}} \quad \text{as} \quad Q_{nm}P \to \infty, \tag{SI-II.31}$$

and find the asymptotic form of (eq. SI-II.28)

$$q_{nm} \sim \frac{n\pi}{P} + \frac{1}{P} \left[\frac{\pi}{4} + m\frac{\pi}{2} - \arctan\left(\frac{1}{\delta_{m0}\Pi_n} \right) \right], \quad n \in \mathbb{N}, \quad P \to \infty. \tag{SI-II.32}$$

For any m, we let $n \to \infty$ and $P \to \infty$ such that the first term in eq. SI-II.32 dominates and we find

$$q_{nm} \sim \frac{n\pi}{P} := n\Delta q, \quad n \to \infty \quad \text{and} \quad P \to \infty. \tag{SI-II.33}$$

Changing notation from $R_{nm}(r)$ to $R_m(r, n\Delta q)$ and defining

$$u_m(r) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \; u(\mathbf{r})e^{-im\phi} \tag{SI-II.34}$$

$$U_m(n\Delta q, P) = 2\pi \int_0^P \mathrm{d}r \; r u_m(r) R_m^*(r, n\Delta q), \tag{SI-II.35}$$

eq. SI-II.29 can be written as

$$u(\mathbf{r}) = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} \frac{U_m(n\Delta q, P)}{2\pi\int_0^P \mathrm{d}r r|R_m(r, n\Delta q)|^2} R_m(r, n\Delta q)e^{im\phi}. \tag{SI-II.36}$$

Prefactors containing 2π in eqs. SI-II.34-SI-II.36 are distributed in a way to obtain the Fourier transform norm convention in the limit $\lambda \to 0$, as will be shown.

Using asymptotic forms (eq. SI-II.31) for large P and the fact that the Π_n is bounded we find

$$\lim_{P\to\infty} X_n I_0(Q_{n0}r) = 0, \quad \lim_{P\to\infty} X_n Y_0(q_{n0}r) = 0, \quad \lim_{P\to\infty} X_n K_0(Q_{n0}r) = 0. \tag{SI-II.37}$$

Therefore, asymptotic form of eq. SI-II.27 is given by

$$R_{nm}(r) \sim a_{nm}\left\{J_m(q_{nm}r) + \delta_{m0}\left[\Pi_n\left(Y_m(q_{nm}r) + \frac{2}{\pi}K_m(Q_{nm}r)\right)\right]\right\}, \quad P \to \infty. \tag{SI-II.38}$$

2. Normalization

We now calculate the normalization

$$2\pi\int_0^P \mathrm{d}r r|R_m(r, n\Delta q)|^2. \tag{SI-II.39}$$

($m = 0$)

$$\begin{aligned}
&\frac{1}{|a_{n0}|^2}\int_0^P \mathrm{d}r r|R_0(r, q_{n0})|^2 = \\
&\frac{P^2}{2}\Bigg[J_0^2(q_{n0}P) + J_1^2(q_{n0}P) + \Pi_n^2\left[Y_0^2(q_{n0}P) + Y_1^2(q_{n0}P)\right] + \frac{4\Pi_n^2}{\pi^2}\left[K_0^2(Q_{n0}P) + K_1^2(Q_{n0}P)\right] \\
&+ 2\Pi_n\left[J_0(q_{n0}P)Y_0(q_{n0}P) + J_1(q_{n0}P)Y_1(q_{n0}P)\right]\Bigg] + \frac{4}{\pi}\frac{\Pi_n}{Q_{n0}^2 + q_{n0}^2}\left[1 + \frac{2}{\pi}\Pi_n \ln\left(\frac{q_{n0}}{Q_{n0}}\right)\right] + \frac{q_n^2 - Q_{n0}^2}{Q_n^2 q_{n0}^2}\frac{2\Pi_n^2}{\pi} \\
&+ \frac{4\Pi_n P}{\pi(Q_{n0}^2 + q_n^2)}\left[q_{n0}J_1(q_{n0}P)K_0(Q_{n0}P) - Q_{n0}J_0(q_{n0}P)K_1(Q_{n0}P) + \Pi_n\left[q_{n0}Y_1(q_{n0}P)K_0(Q_{n0}P) - Q_{n0}Y_0(q_{n0}P)K_1(Q_{n0}P)\right]\right].
\end{aligned} \tag{SI-II.40}$$

Because $q_n > 0$, we can use asymptotic forms (eqs. SI-II.30 and SI-II.31) as $P \to \infty$ and after keeping the highest order terms of P we find

$$\frac{1}{|a_{n0}|^2}\int_0^P \mathrm{d}r r|R_0(r, q_{n0})|^2 \sim \frac{P}{\pi q_{n0}}(1 + \Pi_n^2) \quad \text{as} \quad P \to \infty. \tag{SI-II.41}$$

($m \neq 0$)

Eq. SI-II.38 gives

$$R_{nm}(r) = a_{nm}J_m(q_{nm}r), \tag{SI-II.42}$$

and eq. SI-II.32 gives

$$q_{nm} = \frac{1}{P}\left(n\pi + \frac{\pi}{4} + (m-1)\frac{\pi}{2}\right). \tag{SI-II.43}$$

Inserting eq. SI-II.42 into eq. SI-II.39 and solving the integral yields

$$\frac{1}{|a_{nm}|^2}\int_0^P \mathrm{d}r r|R_m(r,q_{nm})|^2 = \frac{P}{2q_{nm}}\left(q_{nm}PJ_{m-1}^2(q_{nm}P) - 2mJ_{m-1}(q_{nm}P)J_m(q_{nm}) + q_{nm}PJ_m^2(q_{nm}P)\right). \tag{SI-II.44}$$

By condition eq. SI-II.23, the second and third terms vanish leaving

$$\frac{1}{|a_{nm}|^2}\int_0^P \mathrm{d}r r|R_m(r,q_{nm})|^2 = \frac{P^2}{2}J_{m-1}^2(q_{nm}P). \tag{SI-II.45}$$

Using the asymptotic form of J_m (eq. SI-II.30) we find

$$\frac{1}{|a_{nm}|^2}\int_0^P \mathrm{d}r r|R_m(r,q_{nm})|^2 \sim \frac{P}{\pi q_{nm}}\cos^2(q_{nm}P - \frac{\pi}{4} - \frac{(m-1)\pi}{2}). \tag{SI-II.46}$$

Inserting eq. SI-II.43 into eq. SI-II.46 leads to

$$\frac{1}{|a_{nm}|^2}\int_0^P \mathrm{d}r r|R_m(r,q_{nm})|^2 \sim \frac{P}{\pi q_{nm}} \quad \text{as} \quad P\to\infty. \tag{SI-II.47}$$

Therefore, asymptotic behaviour of the normalization as $P\to\infty$ is given by

$$2\pi\int_0^P \mathrm{d}r r|R_m(r,n\Delta q)|^2 = |a_{nm}|^2\frac{2P}{n\Delta q}(1+\delta_{m0}(\Pi(n\Delta q))^2) = |a_{nm}|^2\frac{2\pi}{\Delta q(n\Delta q)}(1+\delta_{m0}(\Pi(n\Delta q))^2), \tag{SI-II.48}$$

where $\Pi(n\Delta q)$ is given by eq. SI-II.21.

3. Infinite radius limit of the series

Inserting eq. SI-II.48 into eq. SI-II.36 we find

$$u(\mathbf{r}) = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\Delta q(n\Delta q)\frac{U_m(n\Delta q,P)}{2\pi|a_{nm}|^2(1+\delta_{m0}(\Pi(n\Delta q))^2)}R_m(r,n\Delta q)e^{im\phi}. \tag{SI-II.49}$$

We now set

$$a_{nm} = \frac{i^m}{\sqrt{\left(1+\delta_{m0}\left(\Pi(n\Delta q)\right)^2\right)}} \tag{SI-II.50}$$

in eqs. SI-II.27 and SI-II.49, where factor i^m in eq. SI-II.50 was chosen to obtain the conventional form of the Fourier transform in polar coordinates [2] in the limit $\lambda\to 0$. With the use of eq. SI-II.50, eq. SI-II.49 simplifies to

$$u(\mathbf{r}) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\Delta q(n\Delta q)U_m(n\Delta q,P)R_m(r,n\Delta q)e^{im\phi}, \tag{SI-II.51}$$

where

$$R_{nm}(r,n\Delta q) = \frac{i^m}{\sqrt{\left(1+\delta_{m0}\left(\Pi(n\Delta q)\right)^2\right)}}\times$$
$$\times\left\{J_m(n\Delta qr) - \delta_{m0}\left[X(n\Delta q)I_m\left(\sqrt{(n\Delta q)^2+\frac{\sigma}{\kappa}}r\right) + \Pi(n\Delta q)(1-X(n\Delta q))\left(Y_m(n\Delta qr) + \frac{2}{\pi}K_m\left(\sqrt{(n\Delta q)^2+\frac{\sigma}{\kappa}}r\right)\right)\right]\right\}. \tag{SI-II.52}$$

By letting $P \to \infty$, $\Delta q \to dq$ and $n\Delta q \to q$, we find

$$u(\mathbf{r}) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dqq U_m(q) R_m(r,q) e^{im\phi}, \tag{SI-II.53}$$

where

$$U_m(q) = 2\pi \int_0^{\infty} \mathrm{d}rr \; u_m(r) R_m^*(r,q) \tag{SI-II.54}$$

and $R_m(r,q)$ is given by

$$R_m(r,q) = \frac{i^m}{\sqrt{\left(1 + \delta_{m0} \left(\Pi(q)\right)^2\right)}} \left[J_m(qr) + \delta_{m0}\Pi(q) \left(Y_m(qr) + \frac{2}{\pi} K_m(Qr)\right)\right]. \tag{SI-II.55}$$

4. Summary of the solution

To summarize, a profile of a pinned membrane can be expanded in the basis functions $\psi_m(\mathbf{r},q) = R_m(r,q)e^{im\phi}$ as

$$u(\mathbf{r}) = \sum_{m=-\infty}^{\infty} u_m(r) e^{im\phi} = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dqq U_m(q) R_m(r,q) e^{im\phi} = \frac{1}{(2\pi)^2} \sum_{m=-\infty}^{\infty} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{q} \; U(\mathbf{q}) e^{-im\theta} \psi_m(\mathbf{r},q), \tag{SI-II.56}$$

where $\mathbf{q} = (q,\theta)$ and $\psi_m(\mathbf{r},q)$ satisfy the orthogonality condition

$$\int_{\mathbb{R}^2} \mathrm{d}\mathbf{r} \psi_m(\mathbf{r},q) \psi_{m'}^*(\mathbf{r},q') = \frac{\delta(q-q')}{q} 2\pi \delta_{m,m'} \tag{SI-II.57}$$

and the eigenequation

$$\left[\kappa\nabla^4 - \sigma\nabla^2 + \gamma + \lambda\delta(\mathbf{r})\right] \psi_m(\mathbf{r},q) = E_q \psi_m(\mathbf{r},q), \tag{SI-II.58}$$

where continuous eigenvalues E_q are given by

$$E_q = \kappa q^4 + \sigma q^2 + \gamma. \tag{SI-II.59}$$

Multiplying eq. SI-II.54 with $e^{im\theta}$, where θ is the angle coordinate of a vector $\mathbf{q} = (q,\theta)$, and summing over m gives the q-space transform $U(\mathbf{q}) = U(q,\theta)$ of $u(\mathbf{r})$

$$U(\mathbf{q}) = \sum_{m=-\infty}^{\infty} U_m(q) e^{im\theta} = 2\pi \sum_{m=-\infty}^{\infty} \int_0^{\infty} \mathrm{d}rr \; u_m(r) R_m^*(r,q) e^{im\theta} = \sum_{m=-\infty}^{\infty} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r} \; u(\mathbf{r}) e^{im\theta} \psi_m^*(\mathbf{r},q). \tag{SI-II.60}$$

Eqs. SI-II.56 and SI-II.60 make a transform pair that enables us to switch between the $\mathbf{r}$ and $\mathbf{q}$ spaces during calculations.

5. Free membrane limit

For pinning stifness $\lambda \to 0$, we have $\Pi(q) \to 0$ and $R_m(r,q) \to i^m J_m(qr)$ and eqs. SI-II.56 and SI-II.60 reduce to

$$U(\mathbf{q}) = \sum_{m=-\infty}^{\infty} 2\pi i^{-m} e^{im\theta} \int_0^{\infty} \mathrm{d}rr \; u_m(r) J_m(qr) \tag{SI-II.61}$$

$$u(\mathbf{r}) = \sum_{m=-\infty}^{\infty} \frac{i^m}{2\pi} e^{im\phi} \int_0^{\infty} dqq U_m(q) J_m(qr), \tag{SI-II.62}$$

with

$$U_m(q) = 2\pi i^{-m} \int_0^\infty \mathrm{d}rr\; u_m(r) J_m(qr) \tag{SI-II.63}$$

$$u_m(r) = \frac{i^m}{2\pi} \int_0^\infty dqq U_m(q) J_m(qr), \tag{SI-II.64}$$

which is a 2D Fourier transform pair expresed in polar coordinates [2], which is in agreement with the fact that plane waves satisfy eigenequation SI-II.3 with $\lambda = 0$.

D. Static properties at the pinning position

1. *Fluctuations*

We now show a few calculations that verify that the found orthonormal functions $\psi_m(\mathbf{r}, q)$ satisfy

$$\langle v(\mathbf{r}_1) v^*(\mathbf{r}_2) \rangle = g(\mathbf{r}_1|\mathbf{r}_2) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty dqq \frac{\psi_m(\mathbf{r}_1, q)\psi_m^*(\mathbf{r}_2, q)}{E_q}. \tag{SI-II.65}$$

For $\lambda \to 0$, eq. SI-II.65 becomes

$$\begin{aligned}\langle v_f(\mathbf{r}_1) v_f^*(\mathbf{r}_2) \rangle = g_f(\mathbf{r}_1|\mathbf{r}_2) &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty dqq \frac{J_m(qr_1) J_m(qr_2) e^{im(\phi_1 - \phi_2)}}{E_q} \\ &= \frac{1}{2\pi} \int_0^\infty dqq \frac{J_0(q|\mathbf{r}_1 - \mathbf{r}_2|)}{E_q},\end{aligned} \tag{SI-II.66}$$

where we used the Graf/Neumman addition formula

$$J_0(|\mathbf{r}_1 - \mathbf{r}_2|) = J_0(\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\phi_1 - \phi_2)}) = \sum_{m=-\infty}^{\infty} J_m(r_1) J_m(r_2) e^{im(\phi_1 - \phi_2)}. \tag{SI-II.67}$$

We have therefore successfully reproduced the result for the free membrane two-point correlation function. Setting $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}$ in eq. SI-II.65 we find the fluctuation amplitude

$$\langle v^2(\mathbf{r}) \rangle = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty dqq \frac{|\psi_m(\mathbf{r}, q)|^2}{E_q}. \tag{SI-II.68}$$

At the pinning site $\mathbf{r} = 0$

$$\langle v^2(0) \rangle = \frac{1}{2\pi} \int_0^\infty \mathrm{d}q \; \frac{q}{\kappa q^4 + \sigma q^2 + \gamma} \frac{\left(8\kappa q^2 + 4\sigma\right)^2}{\lambda^2 + \left[8\kappa q^2 + 4\sigma + \frac{\lambda}{\pi} \ln\left(1 + \sigma/(\kappa q^2)\right)\right]^2} = \frac{1}{\lambda + \lambda_m}. \tag{SI-II.69}$$

2. *Mean shape*

Expanding the mean shape in $\psi_m(\mathbf{r}, q) = R_m(r, q) e^{im\phi}$ gives

$$\langle u(\mathbf{r}) \rangle = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty dqq \langle U_m(q) \rangle \psi_m(\mathbf{r}, q), \tag{SI-II.70}$$

where $\langle U_m(q) \rangle$ are the coefficients of the expansion yet to be determined. Inserting eq. SI-II.70 into the equation SI-I.8 for the mean shape with $\mathbf{r}_0 = 0$, and using eq. SI-II.58 we find

$$\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty dqq \langle U_m(q) \rangle E_q \psi_m(\mathbf{r}, q) = \lambda(l_0 - h_0)\delta(\mathbf{r}). \tag{SI-II.71}$$

Multiplying by $\psi_{m'}(\mathbf{r}, q')$, integrating over $\mathbf{r}$, using the orthonormality eq. SI-II.57 between different basis functions and $\delta(\mathbf{r}) = \delta(r)/(2\pi r)$ leads to the coefficients of expansion

$$\langle U_m(q)\rangle = \lambda(l_0 - h_0)\frac{\delta_{m0} R^*_m(0, q)}{E_q}. \tag{SI-II.72}$$

Expansion of the mean membrane shape is, therefore, given only by $m = 0$ modes:

$$\langle u(\mathbf{r})\rangle = \lambda(l_0 - h_0)\frac{1}{2\pi}\int_0^\infty dqq\frac{R_0(r, q)R^*_0(0, q)}{E_q}. \tag{SI-II.73}$$

At the pinning site $\mathbf{r} = 0$

$$\begin{aligned}\langle u(0)\rangle =&\lambda(l_0 - h_0)\frac{1}{2\pi}\int_0^\infty dqq\frac{|\psi_0(0, q)|^2}{E_q}\\ =&\frac{\lambda}{\lambda + \lambda_m}(l_0 - h_0),\end{aligned} \tag{SI-II.74}$$

where last equality follows from eq. SI-II.69.

III. PLANE WAVE MODE COUPLING

A. Gaussian integrals

For a symmetric $n \times n$-dimensional matrix A, with inverse A^{-1}, n-dimensional vector $\mathbf{B}$ and n-dimensional vector $\mathbf{x}$, Gaussian integral is given by

$$\int_{-\infty}^{\infty} \mathrm{d}^n\mathbf{x} \exp\left[-\frac{1}{2}\mathbf{x}^T A\mathbf{x} + \mathbf{B}^T\cdot\mathbf{x}\right] = \left(\int_{-\infty}^{\infty} \mathrm{d}^n\mathbf{x} \exp\left[-\frac{1}{2}\mathbf{x}^T A\mathbf{x}\right]\right) \exp\left[\frac{1}{2}\mathbf{B}^T A^{-1}\mathbf{B}\right] = \sqrt{\frac{(2\pi)^n}{\det A}} \exp\left[\frac{1}{2}\mathbf{B}^T A^{-1}\mathbf{B}\right], \tag{SI-III.1}$$

where ^{T} denotes the matrix transpose. First moment of x_i is given by

$$\begin{aligned} \langle x_i\rangle &\equiv \frac{1}{\mathcal{Z}} \int_{-\infty}^{\infty} \mathrm{d}^n\mathbf{x}\, x_i \exp\left[-\frac{1}{2}\mathbf{x}^T A\mathbf{x} + \mathbf{B}^T\cdot\mathbf{x}\right] \\ &= \sum_k A_{ik}^{-1} B_k \end{aligned} \tag{SI-III.2}$$

and the second moment is given by

$$\begin{aligned} \langle x_i x_j\rangle &\equiv \frac{1}{\mathcal{Z}} \int_{-\infty}^{\infty} \mathrm{d}^n\mathbf{x}\, x_i x_j \exp\left[-\frac{1}{2}\mathbf{x}^T A\mathbf{x} + \mathbf{B}^T\cdot\mathbf{x}\right] \\ &= A_{ij}^{-1} + \sum_{kl} A_{ik}^{-1} B_k A_{jl}^{-1} B_l. \end{aligned} \tag{SI-III.3}$$

Here, the partition function $\mathcal{Z}$ is used,

$$\mathcal{Z} \equiv \int_{-\infty}^{\infty} \mathrm{d}^n\mathbf{x} \exp\left[-\frac{1}{2}\mathbf{x}^T A\mathbf{x} + \mathbf{B}^T\cdot\mathbf{x}\right]. \tag{SI-III.4}$$

Hubbard-Stratonovich transformation is given by

$$\exp\left\{-\frac{a}{2}x^2\right\} = \sqrt{\frac{1}{2\pi a}} \int_{-\infty}^{\infty} \mathrm{d}y \exp\left[-\frac{y^2}{2a} - ixy\right]. \tag{SI-III.5}$$

B. Plane wave expansion of the Hamiltonian

Expanding the membrane profile into Fourier series

$$u(\mathbf{r}) = \sum_{\mathbf{k}} u_{\mathbf{k}} \mathrm{e}^{i\mathbf{k}\mathbf{r}}, \tag{SI-III.6}$$

and taking into account that $u(\mathbf{r}) \in \mathbb{R}$, $u_{\mathbf{k}} = u^*_{-\mathbf{k}}$ and thus $|u_{\mathbf{k}}|^2 = u^*_{\mathbf{k}} u_{\mathbf{k}} = u_{-\mathbf{k}} u_{\mathbf{k}}$, the Hamiltonian (eq. SI-I.3) writes

$$\mathcal{H} = \frac{A}{2} \sum_{\mathbf{k}} (\kappa k^4 + \sigma k^2 + \gamma)|u_{\mathbf{k}}|^2 + \frac{\lambda}{2}\left(\sum_{\mathbf{k}} u_{\mathbf{k}} \mathrm{e}^{i\mathbf{k}\mathbf{r}_0} + (h_0 - l_0)\right)^2. \tag{SI-III.7}$$

Exponential of the Hamiltonian becomes

$$\exp[-\mathcal{H}] = \exp\left[-\frac{A}{2}\sum_{\mathbf{k}}(\kappa k^4+\sigma k^2+\gamma)|u_{\mathbf{k}}|^2 - \frac{\lambda}{2}\left(\sum_{\mathbf{k}} u_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}_0} + (h_0-l_0)\right)^2\right]$$
$$\overset{eq.SI-III.5}{=} \int_{-\infty}^{\infty}\frac{d\phi}{(2\pi\lambda)^{1/2}}\exp\left[-\frac{A}{2}\sum_{\mathbf{k}}(\kappa k^4+\sigma k^2+\gamma)|u_{\mathbf{k}}|^2 - \left(\frac{\phi^2}{2\lambda} - i\phi\left(\sum_{\mathbf{k}} u_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}_0} + (h_0-l_0)\right)\right)\right]$$
$$= \int_{-\infty}^{\infty}\frac{d\phi}{(2\pi\lambda)^{1/2}}\exp\left[-\frac{1}{2}\mathbf{u}D\mathbf{u} + \mathbf{J}\cdot\mathbf{u} - \frac{1}{2}\frac{\phi^2}{\lambda} + K\phi\right]. \tag{SI-III.8}$$

Here, we use the definitions

$$D_{\mathbf{k},\mathbf{k}'} := A(\kappa k^4+\sigma k^2+\gamma)\delta(\mathbf{k}+\mathbf{k}'), \tag{SI-III.9}$$
$$J_{\mathbf{k}} := i\phi e^{i\mathbf{k}\mathbf{r}_0}, \tag{SI-III.10}$$
$$K := i(h_0-l_0) \tag{SI-III.11}$$

and we omitted the matrix transpose notation, but it is assumed where necessary.

C. Mean mode amplitude

In reality, membrane's motion is restricted inside some interval $u_{\mathbf{k}} \in (s_-, s_+)$, where s_- could be the position of a substrate or another cell and s_+ the position of internal cell structures that block further membrane motion. The mean Fourier amplitude is thus given by

$$\langle u_{\mathbf{k}}\rangle = \frac{\int_{s_-}^{s_+} d\mathbf{u}\; u_{\mathbf{k}}\exp[-\mathcal{H}]}{\int_{s_-}^{s_+} d\mathbf{u}\;\exp[-\mathcal{H}]}, \tag{SI-III.12}$$

But when the system's parameters $(\kappa, \sigma, \gamma, \lambda)$ are such that $\exp[-\mathcal{H}] \approx 0$ for $|u_{\mathbf{k}}| > s_-, s_+$ (regime of small fluctuations), we can extend the limits of integration in eq. SI-III.12 to infinity with the aim of obtaining Gaussian integrals. Therefore, with this approximation we obtain

$$\langle u_{\mathbf{k}}\rangle = \frac{\int_{-\infty}^{\infty} d\mathbf{u}\; u_{\mathbf{k}}\exp[-\mathcal{H}]}{\int_{-\infty}^{\infty} d\mathbf{u}\;\exp[-\mathcal{H}]}$$
$$\overset{eq.SI-III.8}{=} \frac{1}{\int_{-\infty}^{\infty} d\mathbf{u}\;\exp[-\mathcal{H}]}\int_{-\infty}^{\infty}\frac{d\phi}{(2\pi\lambda)^{1/2}}\exp\left[-\frac{1}{2}\frac{\phi^2}{\lambda}+K\phi\right]\int_{-\infty}^{\infty} d\mathbf{u}\; u_{\mathbf{k}}\exp\left[-\frac{1}{2}\mathbf{u}D\mathbf{u}+\mathbf{J}\cdot\mathbf{u}\right]$$
$$\overset{eq.SI-III.1,SI-III.2}{=} \frac{1}{\int_{-\infty}^{\infty} d\mathbf{u}\;\exp[-\mathcal{H}]}\int_{-\infty}^{\infty}\frac{d\phi}{(2\pi\lambda)^{1/2}}\exp\left[-\frac{1}{2}\frac{\phi^2}{\lambda}+K\phi\right]\int_{-\infty}^{\infty} d\mathbf{u}\;\exp\left[-\frac{1}{2}\mathbf{u}D\mathbf{u}\right]\exp\left[\frac{1}{2}\mathbf{J}D^{-1}\mathbf{J}\right]\sum_{\mathbf{k}} D^{-1}_{\mathbf{k},\mathbf{k}'}\mathbf{J}_{\mathbf{k}'}$$
$$= \frac{\int_{-\infty}^{\infty} d\mathbf{u}\;\exp\left[-\frac{1}{2}\mathbf{u}D\mathbf{u}\right]}{\int_{-\infty}^{\infty} d\mathbf{u}\;\exp[-\mathcal{H}]}\int_{-\infty}^{\infty}\frac{d\phi}{(2\pi\lambda)^{1/2}}\exp\left[-\frac{1}{2}\frac{\phi^2}{\lambda}+K\phi\right]\exp\left[-\frac{1}{2}\sum_{\mathbf{k}}\frac{\phi^2}{A(\kappa k^4+\sigma k^2+\gamma)}\right]\left(\frac{ie^{-i\mathbf{k}\mathbf{r}_0}\phi}{A(\kappa k^4+\sigma k^2+\gamma)}\right)$$
$$= \frac{\int_{-\infty}^{\infty} d\mathbf{u}\;\exp\left[-\frac{1}{2}\mathbf{u}D\mathbf{u}\right]}{\int_{-\infty}^{\infty} d\mathbf{u}\;\exp[-\mathcal{H}]}\frac{ie^{-i\mathbf{k}\mathbf{r}_0}}{A(\kappa k^4+\sigma k^2+\gamma)}\int_{-\infty}^{\infty}\frac{d\phi}{(2\pi\lambda)^{1/2}}\phi\exp\left[-\frac{1}{2}\left(\frac{1}{\lambda}+\sum_{\mathbf{k}}\frac{1}{A(\kappa k^4+\sigma k^2+\gamma)}\right)\phi^2+K\phi\right]$$
$$\overset{eq.SI-III.2}{=} \frac{\int_{-\infty}^{\infty} d\mathbf{u}\;\exp\left[-\frac{1}{2}\mathbf{u}D\mathbf{u}\right]}{\int_{-\infty}^{\infty} d\mathbf{u}\;\exp[-\mathcal{H}]}\frac{ie^{-i\mathbf{k}\mathbf{r}_0}}{A(\kappa k^4+\sigma k^2+\gamma)}\int_{-\infty}^{\infty}\frac{d\phi}{(2\pi\lambda)^{1/2}}\exp\left[-\frac{1}{2}\left(\frac{1}{\lambda}+\sum_{\mathbf{k}}\frac{1}{A(\kappa k^4+\sigma k^2+\gamma)}\right)\phi^2+K\phi\right]\times$$
$$\times K\left(\frac{1}{\lambda}+\sum_{\mathbf{k}}\frac{1}{A(\kappa k^4+\sigma k^2+\gamma)}\right)^{-1}$$
$$= -(h_0-l_0)\frac{e^{-i\mathbf{k}\mathbf{r}_0}}{A(\kappa k^4+\sigma k^2+\gamma)}\left(\frac{1}{\lambda}+\sum_{\mathbf{k}}\frac{1}{A(\kappa k^4+\sigma k^2+\gamma)}\right)^{-1}, \tag{SI-III.13}$$

where in the last step we used

$$\int_{-\infty}^{\infty} d\mathbf{u}\ \exp[-\mathcal{H}] \overset{eq.SI-III.8}{=} \int_{-\infty}^{\infty} \frac{\mathrm{d}\phi}{(2\pi\lambda)^{1/2}} \exp\left[-\frac{1}{2}\frac{\phi^2}{\lambda} + K\phi\right] \int_{-\infty}^{\infty} d\mathbf{u}\ \exp\left[-\frac{1}{2}\mathbf{u}D\mathbf{u} + \mathbf{J}\cdot\mathbf{u}\right]$$
$$\overset{eq.SI-III.1}{=} \int_{-\infty}^{\infty} \frac{\mathrm{d}\phi}{(2\pi\lambda)^{1/2}} \exp\left[-\frac{1}{2}\frac{\phi^2}{\lambda} + K\phi\right] \int_{-\infty}^{\infty} d\mathbf{u}\ \exp\left[-\frac{1}{2}\mathbf{u}D\mathbf{u}\right] \exp\left[\frac{1}{2}\mathbf{J}D^{-1}\mathbf{J}\right]$$
$$= \int_{-\infty}^{\infty} d\mathbf{u}\ \exp\left[-\frac{1}{2}\mathbf{u}D\mathbf{u}\right] \int_{-\infty}^{\infty} \frac{\mathrm{d}\phi}{(2\pi\lambda)^{1/2}} \exp\left[-\frac{1}{2}\left(\frac{1}{\lambda} + \sum_{\mathbf{k}} \frac{1}{A(\kappa k^4 + \sigma k^2 + \gamma)}\right)\phi^2 + K\phi\right] \tag{SI-III.14}$$

and inserted eq. SI-III.11.

D. Mean shape

For the mean shape we find from eq. SI-III.6

$$\langle u(\mathbf{r})\rangle = \sum_{\mathbf{k}} \langle u_{\mathbf{k}}\rangle \mathrm{e}^{i\mathbf{k}\mathbf{r}}. \tag{SI-III.15}$$

Combining eqs. SI-III.15 and SI-III.12 we obtain the mean shape for a pinned membrane of surface A

$$\langle u(\mathbf{r})\rangle = (l_0 - h_0)\left(\frac{1}{\lambda} + \frac{1}{A}\sum_{\mathbf{k}} \frac{1}{(\kappa k^4 + \sigma k^2 + \gamma)}\right)^{-1} \frac{1}{A}\sum_{\mathbf{k}} \frac{e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_0)}}{\kappa k^4 + \sigma k^2 + \gamma}. \tag{SI-III.16}$$

In the limit of an infinite membrane ($A \to \infty$), $\mathbf{k}$ becomes continuous and sums transforms to corresponding integrals, namely Fourier series (eq. SI-III.15) becomes a Fourier transform

$$\langle u(\mathbf{r})\rangle_{A\to\infty} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} d\mathbf{k} \langle u(\mathbf{k})\rangle \mathrm{e}^{i\mathbf{k}\mathbf{r}}, \tag{SI-III.17}$$

and mean shape eq. SI-III.16 becomes

$$\langle u(\mathbf{r})\rangle_{A\to\infty} = (l_0 - h_0)\left(\frac{1}{\lambda} + \frac{1}{\lambda_m}\right)^{-1} \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} d\mathbf{k} \frac{e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_0)}}{\kappa k^4 + \sigma k^2 + \gamma}$$
$$= (l_0 - h_0)\frac{\lambda\lambda_m}{\lambda + \lambda_m} G(\mathbf{r} - \mathbf{r}_0), \tag{SI-III.18}$$

where we used the notation

$$G(\mathbf{r} - \mathbf{r}') \coloneqq \lim_{A\to\infty} \frac{1}{A}\sum_{\mathbf{k}} \frac{e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}}{\kappa k^4 + \sigma k^2 + \gamma} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} d\mathbf{k} \frac{e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}}{\kappa k^4 + \sigma k^2 + \gamma}, \qquad 1/\lambda_m \coloneqq G(0). \tag{SI-III.19}$$

From eqs. SI-III.17 and SI-III.18 we find the mean Fourier amplitude for an infinite membrane

$$\langle u(\mathbf{k})\rangle = \frac{\lambda\lambda_m}{\lambda + \lambda_m}(l_0 - h_0)\ \frac{e^{-i\mathbf{k}\mathbf{r}_0}}{\kappa k^4 + \sigma k^2 + \gamma}. \tag{SI-III.20}$$

E. Mode coupling

In the regime of small fluctuations, mode coupling coefficients are given by

$$\begin{aligned}
\langle u_{\mathbf{k}} u_{\mathbf{k}'} \rangle_A &= \frac{\int_{-\infty}^{\infty} d\mathbf{u}\ u_{\mathbf{k}} u_{\mathbf{k}'} \exp[-\mathcal{H}]}{\int_{-\infty}^{\infty} d\mathbf{u}\ \exp[-\mathcal{H}]} \\
&= \frac{1}{\int_{-\infty}^{\infty} d\mathbf{u}\ \exp[-\mathcal{H}]} \int_{-\infty}^{\infty} \frac{\mathrm{d}\phi}{(2\pi\lambda)^{1/2}} \exp\left[-\frac{1}{2}\frac{\phi^2}{\lambda} + K\phi\right] \int_{-\infty}^{\infty} d\mathbf{u}\ u_{\mathbf{k}} u_{\mathbf{k}'} \exp\left[-\frac{1}{2}\mathbf{u}D\mathbf{u} + \mathbf{J}\cdot\mathbf{u}\right] \\
&\overset{eq. SI-III.1, SI-III.3}{=} \frac{\int_{-\infty}^{\infty} d\mathbf{u}\ \exp\left[-\frac{1}{2}\mathbf{u}D\mathbf{u}\right]}{\int_{-\infty}^{\infty} d\mathbf{u}\ \exp[-\mathcal{H}]} \int_{-\infty}^{\infty} \frac{\mathrm{d}\phi}{(2\pi\lambda)^{1/2}} \exp\left[-\frac{1}{2}\frac{\phi^2}{\lambda} + K\phi\right] \exp\left[\frac{1}{2}\mathbf{J}D^{-1}\mathbf{J}\right] \left(D^{-1}_{\mathbf{k},\mathbf{k}'} + \sum_{\mathbf{l},\mathbf{m}} D^{-1}_{\mathbf{k},\mathbf{l}}\mathbf{J}_{\mathbf{l}} D^{-1}_{\mathbf{k}',\mathbf{m}}\mathbf{J}_{\mathbf{m}}\right) \\
&= \frac{\int_{-\infty}^{\infty} d\mathbf{u}\ \exp\left[-\frac{1}{2}\mathbf{u}D\mathbf{u}\right]}{\int_{-\infty}^{\infty} d\mathbf{u}\ \exp[-\mathcal{H}]} \int_{-\infty}^{\infty} \frac{\mathrm{d}\phi}{(2\pi\lambda)^{1/2}} \exp\left[-\frac{1}{2}\left(\frac{1}{\lambda} + \sum_{\mathbf{k}} \frac{1}{A(\kappa k^4 + \sigma k^2 + \gamma)}\right)\phi^2 + K\phi\right] \times \\
&\times \left(\frac{\delta(\mathbf{k}+\mathbf{k}')}{A(\kappa k^4 + \sigma k^2 + \gamma)} - \frac{e^{-i\mathbf{k}\mathbf{r}_0}}{A(\kappa k^4 + \sigma k^2 + \gamma)} \frac{e^{-i\mathbf{k}'\mathbf{r}_0}}{A(\kappa k'^4 + \sigma k'^2 + \gamma)}\phi^2\right) \\
&= \frac{\int_{-\infty}^{\infty} d\mathbf{u}\ \exp\left[-\frac{1}{2}\mathbf{u}D\mathbf{u}\right]}{\int_{-\infty}^{\infty} d\mathbf{u}\ \exp[-\mathcal{H}]} \int_{-\infty}^{\infty} \frac{\mathrm{d}\phi}{(2\pi\lambda)^{1/2}} \exp\left[-\frac{1}{2}\left(\frac{1}{\lambda} + \sum_{\mathbf{k}} \frac{1}{A(\kappa k^4 + \sigma k^2 + \gamma)}\right)\phi^2 + K\phi\right] \times \\
&\times \left(\frac{\delta(\mathbf{k}+\mathbf{k}')}{A(\kappa k^4 + \sigma k^2 + \gamma)} - \frac{e^{-i\mathbf{k}\mathbf{r}_0}}{A(\kappa k^4 + \sigma k^2 + \gamma)} \frac{e^{-i\mathbf{k}'\mathbf{r}_0}}{A(\kappa k'^4 + \sigma k'^2 + \gamma)} \times \right. \\
&\times \left.\left(\left(\frac{1}{\lambda} + \sum_{\mathbf{k}} \frac{1}{A(\kappa k^4 + \sigma k^2 + \gamma)}\right)^{-1} + K^2 \left(\frac{1}{\lambda} + \sum_{\mathbf{k}} \frac{1}{A(\kappa k^4 + \sigma k^2 + \gamma)}\right)^{-2}\right)\right) \\
&= \frac{\delta(\mathbf{k}+\mathbf{k}')}{A(\kappa k^4 + \sigma k^2 + \gamma)} - \frac{e^{-i\mathbf{k}\mathbf{r}_0}}{A(\kappa k^4 + \sigma k^2 + \gamma)} \frac{e^{-i\mathbf{k}'\mathbf{r}_0}}{A(\kappa k'^4 + \sigma k'^2 + \gamma)} \left(\frac{1}{\lambda} + \sum_{\mathbf{k}} \frac{1}{A(\kappa k^4 + \sigma k^2 + \gamma)}\right)^{-1} - \\
&- \frac{e^{-i\mathbf{k}\mathbf{r}_0}}{A(\kappa k^4 + \sigma k^2 + \gamma)} \frac{e^{-i\mathbf{k}'\mathbf{r}_0}}{A(\kappa k'^4 + \sigma k'^2 + \gamma)} K^2 \left(\frac{1}{\lambda} + \sum_{\mathbf{k}} \frac{1}{A(\kappa k^4 + \sigma k^2 + \gamma)}\right)^{-2} \\
&= \frac{\delta(\mathbf{k}+\mathbf{k}')}{A(\kappa k^4 + \sigma k^2 + \gamma)} - \frac{e^{-i\mathbf{k}\mathbf{r}_0}}{A(\kappa k^4 + \sigma k^2 + \gamma)} \frac{e^{-i\mathbf{k}'\mathbf{r}_0}}{A(\kappa k'^4 + \sigma k'^2 + \gamma)} \left(\frac{1}{\lambda} + \sum_{\mathbf{k}} \frac{1}{A(\kappa k^4 + \sigma k^2 + \gamma)}\right)^{-1} + \\
&+ \langle u_{\mathbf{k}} \rangle \langle u_{\mathbf{k}'} \rangle.
\end{aligned} \tag{SI-III.21}$$

F. Correlation function

For the correlations we find from eq. SI-III.6

$$\langle v(\mathbf{r}) v(\mathbf{r}') \rangle = \sum_{\mathbf{k},\mathbf{k}'} \langle u_{\mathbf{k}} u_{\mathbf{k}'} \rangle_A \mathrm{e}^{i\mathbf{k}\mathbf{r}} \mathrm{e}^{i\mathbf{k}'\mathbf{r}'} - \sum_{\mathbf{k}} \langle u_{\mathbf{k}} \rangle_A \mathrm{e}^{i\mathbf{k}\mathbf{r}} \sum_{\mathbf{k}'} \langle u_{\mathbf{k}'} \rangle_A \mathrm{e}^{i\mathbf{k}'\mathbf{r}'}. \tag{SI-III.22}$$

Combining eqs. SI-III.21 and SI-III.22 we obtain the correlation function for a pinned membrane of surface A

$$\langle v(\mathbf{r}) v(\mathbf{r}') \rangle = \sum_{\mathbf{k}} \frac{e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}}{\kappa k^4 + \sigma k^2 + \gamma} - \left(\frac{1}{\lambda} + \sum_{\mathbf{k}} \frac{1}{A(\kappa k^4 + \sigma k^2 + \gamma)}\right)^{-1} \sum_{\mathbf{k}} \frac{e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_0)}}{\kappa k^4 + \sigma k^2 + \gamma} \sum_{\mathbf{k}} \frac{e^{i\mathbf{k}(\mathbf{r}'-\mathbf{r}_0)}}{\kappa k^4 + \sigma k^2 + \gamma}. \tag{SI-III.23}$$

In the limit of an infinite membrane ($A \to \infty$), eq. SI-III.22 becomes

$$\langle v(\mathbf{r}) v(\mathbf{r}') \rangle_{A\to\infty} = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} d\mathbf{k} \int_{\mathbb{R}^2} d\mathbf{k}' \langle u(\mathbf{k}) u(\mathbf{k}') \rangle \mathrm{e}^{i\mathbf{k}\mathbf{r}} \mathrm{e}^{i\mathbf{k}'\mathbf{r}'} - \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} d\mathbf{k} \int_{\mathbb{R}^2} d\mathbf{k}' \langle u(\mathbf{k}) \rangle \langle u(\mathbf{k}') \rangle \mathrm{e}^{i\mathbf{k}\mathbf{r}} \mathrm{e}^{i\mathbf{k}'\mathbf{r}'}. \tag{SI-III.24}$$

and eq. SI-III.23 becomes

$$\langle v(\mathbf{r})v(\mathbf{r}')\rangle_{A\to\infty} = G(\mathbf{r}-\mathbf{r}') - \frac{\lambda\lambda_m}{\lambda+\lambda_m}G(\mathbf{r}-\mathbf{r}_0)G(\mathbf{r}_0-\mathbf{r}'), \tag{SI-III.25}$$

where we have used notation from eq. SI-III.19. From eqs. SI-III.24 and SI-III.25 we find the mode coupling coefficients for an infinite membrane

$$\langle u(\mathbf{k})u(\mathbf{k}')\rangle = \frac{\delta(\mathbf{k}+\mathbf{k}')}{\kappa k^4+\sigma k^2+\gamma} + \langle u(\mathbf{k})\rangle\langle u(\mathbf{k}')\rangle - \frac{\lambda\lambda_m}{\lambda+\lambda_m}\frac{e^{-i\mathbf{k}\mathbf{r}_0}}{\kappa k^4+\sigma k^2+\gamma}\frac{e^{-i\mathbf{k}'\mathbf{r}_0}}{\kappa k'^4+\sigma k'^2+\gamma}. \tag{SI-III.26}$$

IV. FREE ENERGY OF A MEMBRANE WITH TWO EQUAL PINNINGS

Using the $\mathbf{k}$ space representation of the Hamiltonian (eq. SI-III.8) the partition function for a membrane with two equal pinnings is

$$\begin{aligned}
\mathcal{Z} &= \int_{-\infty}^{\infty} d\mathbf{u}\ \exp\left[-\mathcal{H}\right] \\
&= \int_{-\infty}^{\infty} d\mathbf{u}_\mathbf{k} \exp\left[-\frac{A}{2}\sum_\mathbf{k}(\kappa k^4+\sigma k^2+\gamma)|u_\mathbf{k}|^2 - \sum_{j=1}^{2}\frac{\lambda}{2}\left(\sum_\mathbf{k} u_\mathbf{k}\mathrm{e}^{i\mathbf{k}\mathbf{r}_j} + (h_0-l_0)\right)^2\right] \\
&\overset{eq.SI-III.5}{=} \int_{-\infty}^{\infty} d\mathbf{u}_\mathbf{k}\int_{-\infty}^{\infty}\frac{\mathrm{d}\phi_1}{(2\pi\lambda)^{1/2}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\phi_2}{(2\pi\lambda)^{1/2}}\exp\left[-\frac{A}{2}\sum_\mathbf{k}(\kappa k^4+\sigma k^2+\gamma)|u_\mathbf{k}|^2 - \sum_{j=1}^{2}\left(\frac{\phi_j^2}{2\lambda} - i\phi_j\left(\sum_\mathbf{k} u_\mathbf{k}\mathrm{e}^{i\mathbf{k}\mathbf{r}_j} + (h_0-l_0)\right)\right)\right] \\
&= \mathcal{N}\int_{-\infty}^{\infty}\frac{\mathrm{d}\phi_1}{(2\pi\lambda)^{1/2}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\phi_2}{(2\pi\lambda)^{1/2}}\exp\left[\sum_{j=1}^{2}\left(-\frac{\phi_j^2}{2\lambda} - i\phi_j(h_0-l_0)\right)\right]\exp\left[-\frac{1}{2A}\sum_\mathbf{k}\sum_{i,j=1}^{2}\phi_i\frac{e^{ik(\mathbf{r}_i-\mathbf{r}_j)}}{\kappa k^4+\sigma k^2+\gamma}\phi_j\right] \\
&= \mathcal{N}\int_{-\infty}^{\infty}\frac{\mathrm{d}\phi_1}{(2\pi\lambda)^{1/2}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\phi_2}{(2\pi\lambda)^{1/2}}\exp\left[\sum_{j=1}^{2}\left(-\frac{\phi_j^2}{2\lambda} - i\phi_j(h_0-l_0)\right)\right]\exp\left[-\frac{1}{2}\sum_{i,j=1}^{2}\phi_i g_f(\mathbf{r}_i-\mathbf{r}_j)\phi_j\right] \\
&= \mathcal{N}\int_{-\infty}^{\infty}\frac{\mathrm{d}\phi_1}{(2\pi\lambda)^{1/2}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\phi_2}{(2\pi\lambda)^{1/2}}\exp\left[-\frac{1}{2}\sum_{i,j=1}^{2}\phi_i\left(\frac{\delta_{ij}}{\lambda}+g_f(\mathbf{r}_i-\mathbf{r}_j)\right)\phi_j - i\sum_{j=1}^{2}\phi_j(h_0-l_0)\right] \\
&= \mathcal{N}\frac{1}{\lambda\sqrt{\det M}}\exp\left[-\frac{1}{2}\sum_{i,j=1}^{2}(h_0-l_0)^2 M_{ij}^{-1}\right],
\end{aligned} \tag{SI-IV.1}$$

where

$$M_{ij} = \frac{\delta_{ij}}{\lambda} + g_f(\mathbf{r}_i-\mathbf{r}_j) \tag{SI-IV.2}$$

and $\mathcal{N}$ is the partition function of a membrane without pinnings. Dividing $\mathcal{Z}$ with the partition function $\mathcal{Z}_f$ of the free membrane and unrestrained pinnings, given by

$$\mathcal{Z}_f = \int_{-\infty}^{\infty} d\mathbf{u}\int_{-\infty}^{\infty} d\mathbf{l}_1\int_{-\infty}^{\infty} d\mathbf{l}_2\ \exp\left[-\mathcal{H}\right]\exp\left[-\sum_{j=1}^{2}\frac{\lambda}{2}l_j^2\right] = \mathcal{N}\left(\frac{2\pi}{\lambda}\right), \tag{SI-IV.3}$$

we find the free energy of the membrane with two pinnings

$$\begin{aligned}
F = -\ln\left(\frac{\mathcal{Z}}{\mathcal{Z}_f}\right) &= \frac{1}{2}(h_0-l_0)^2\sum_{i,j=1}^{2}M_{ij}^{-1} + \frac{1}{2}\ln\det M \\
&= \frac{1}{2}(h_0-l_0)^2\frac{2\mathcal{K}}{1+\mathcal{K}g_f(\mathbf{r}_1-\mathbf{r}_2)} + \frac{1}{2}\ln\left(1/\mathcal{K}^2 - g_f^2(\mathbf{r}_1-\mathbf{r}_2)\right) \\
&= \frac{\mathcal{K}(h_0-l_0)^2}{1+\mathcal{K}g_f(\mathbf{r}_1-\mathbf{r}_2)} + \frac{1}{2}\ln\left(1-\mathcal{K}^2 g_f^2(\mathbf{r}_1-\mathbf{r}_2)\right) - \frac{1}{2}\ln\mathcal{K}^2,
\end{aligned} \tag{SI-IV.4}$$

which is, without the constant term, the interaction potential between two equal pinnings used in the manuscript.

V. ASYMPTOTIC BEHAVIOUR

Using $r = |\mathbf{r} - \mathbf{r}'|$, the correlation function of the free membrane is

$$g_f(r) = \frac{K_0(a_- r) - K_0(a_+ r)}{2\pi\sigma\sqrt{1 - \left(\frac{\lambda_m^0}{4\sigma}\right)^2}}, \tag{SI-V.1}$$

with

$$a_\pm = \left[\frac{\sigma}{2\kappa}\left(1 \pm \sqrt{1 - \left(\frac{\lambda_m^0}{4\sigma}\right)^2}\right)\right]^{1/2}. \tag{SI-V.2}$$

For the special values $\sigma = 0$ and $\sigma = \frac{1}{4}\lambda_m^0$, the formula can be written by the Kelvin function

$$g_f(r)|_{\sigma=0} = -\frac{4}{\pi\lambda_m^0}\mathrm{kei}_0\left(\frac{r}{\xi_\parallel}\right), \tag{SI-V.3}$$

or the Bessel function

$$g_f(r)|_{\sigma=\frac{1}{4}\lambda_m^0} = r\frac{\xi_\parallel}{4\pi\kappa}\mathrm{K}_1\left(\frac{r}{\xi_\parallel}\right), \tag{SI-V.4}$$

which can both be obtained from eq. SI-V.1 as limiting behaviour.
In the case of $\sigma > \frac{1}{4}\lambda_m^0$, both a_+ and a_- are real, so we can immediately use the asymptotic expansion of the Bessel functions [3]. Using its property

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\sum_{k=0}^{\infty}\frac{a_k(\nu)}{z^k} \tag{SI-V.5}$$

for $|\mathrm{ph}\ z| \leq \frac{3}{2}\pi - \delta$ and

$$a_k(\nu) = \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)\dots(4\nu^2 - (2k-1)^2)}{k!8^k}, \tag{SI-V.6}$$

with $a_0(\nu) = 1$ and only looking at the slowest decaying term, one arrives at

$$g_f(r)|_{\sigma>\frac{1}{4}\lambda_m^0} \sim \frac{1}{2\pi\sigma\sqrt{1 - \left(\frac{\lambda_m^0}{4\sigma}\right)^2}}\sqrt{\frac{\pi}{2a_- r}}e^{-a_- r}\left(1 - \sqrt{\frac{a_-}{a_+}}e^{-(a_+ - a_-)r}\right)\left(1 + \mathcal{O}\left(\frac{1}{r}\right)\right). \tag{SI-V.7}$$

In the case of $0 < \sigma < \frac{1}{4}\lambda_m^0$, a_+ and a_- are imaginary, so we will rewrite the terms to form the real valued expansion. One can rewrite the parameters to $a_\pm = \frac{1}{\xi_\parallel}e^{\pm\frac{i}{2}\psi}$ with $\psi = \arctan\left(\sqrt{\left(\frac{\lambda_m^0}{4\sigma}\right)^2 - 1}\right)$. Plugging this into the asymptotic expansion yields

$$g_f(r)|_{0<\sigma<\frac{1}{4}\lambda_m^0} \sim \frac{1}{\pi\sigma\sqrt{\left(\frac{\lambda_m^0}{4\sigma}\right)^2 - 1}}\sqrt{\frac{\pi\xi_\parallel}{2r}}\sin\left(\frac{\psi}{4} + \frac{r}{\xi_\parallel}\sin\left(\frac{\psi}{2}\right)\right)e^{-\frac{r}{\xi_\parallel}\cos\left(\frac{\psi}{2}\right)}\left(1 + \mathcal{O}\left(\frac{1}{r}\right)\right), \tag{SI-V.8}$$

where all the terms are real valued. For vanishing surface tension, the asymptotic decay needs the asymptotics of the Kelvin function [3], which are

$$\mathrm{kei}_\nu(x) \sim -e^{-\frac{x}{\sqrt{2}}}\left(\frac{\pi}{2x}\right)^{\frac{1}{2}}\sum_{k=0}^{\infty}\frac{a_k(\nu)}{x^k}\sin\left(\frac{x}{\sqrt{2}} + \left(\frac{\nu}{2} + \frac{k}{4} + \frac{1}{8}\right)\pi\right), \tag{SI-V.9}$$

where $a_k(\nu)$ is the same as in eq. SI-V.6. This allows to check the asymptotics for $\sigma = 0$,

$$g_f(r)|_{\sigma=0} \sim \frac{4}{\pi\lambda_m^0}\sqrt{\frac{\pi\xi_\parallel}{2r}}e^{-\frac{r}{\sqrt{2}\xi_\parallel}}\left(\sin\left(\frac{r}{\sqrt{2}\xi_\parallel}+\frac{\pi}{8}\right)+\mathcal{O}\left(\frac{1}{r}\right)\right). \tag{SI-V.10}$$

For the case $\sigma = \frac{1}{4}\lambda_m^0$, the asymptotics are given by

$$g_f(r)|_{\sigma=\frac{1}{4}\lambda_m^0} \sim \frac{\xi_\parallel}{4\kappa}\sqrt{\frac{\xi_\parallel r}{2\pi}}e^{-\frac{r}{\xi_\parallel}}\left(1+\mathcal{O}\left(\frac{1}{r}\right)\right). \tag{SI-V.11}$$

The derivatives can all be calculated from the asymptotics of the Bessel and Kelvin functions. The leading order of the derivatives is the same as the correlation function itself, as the argument of exponential and sine function are linear in r.

SUPPORTING REFERENCES

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