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Supplementary Appendix: Robust Regression for Optimal Individualized Treatment Rules

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Abstract

Because different patients may respond quite differently to the same drug or treatment, there is increasing interest in discovering individualized treatment rules. In particular, there is an emerging need to find optimal individualized treatment rules which would lead to the "best" clinical outcome. In this paper, we propose a new class of loss functions and estimators based on robust regression to estimate the optimal individualized treatment rules. Compared to existing estimation methods in the literature, the new estimators are novel and advantageous in the following aspects: first, they are robust against skewed, heterogeneous, heavy-tailed errors or outliers in data; second, they are robust against a misspecification of the baseline function; third, under some general situations, the new estimator coupled with the pinball loss approximately maximizes the outcome's conditional quantile instead of the conditional mean, which leads to a more robust optimal individualized treatment rule than traditional mean-based estimators. Consistency and asymptotic normality of the proposed estimators are established. Their empirical performance is demonstrated via extensive simulation studies and an analysis of an AIDS data set.

KEYWORDS:
Optimal individualized treatment rules; Personalized medicine; Quantile regression; Robust regression

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APPENDIX

A PROOF OF ASYMPTOTIC PROPERTIES

We rewrite model (1) in the main paper as

$$Y_i = \varphi_0(X_i) + \{A_i - \pi(X_i)\}C(X_i;\beta_0) + \epsilon_i,$$

where $\varphi_0(X_i) = h_0(X_i) + \pi(X_i)C(X_i;\beta_0)$. Define $\Delta C(x;\beta) = C(x;\beta) - C(x;\beta_0)$. Assume $\gamma \in \Gamma$, $\beta \in \mathcal{B}$ and γ' is any fix point in Γ. To establish the asymptotic properties, we assume the following regularity conditions:

(C1) $\{(Y_i, X_i, A_i, \epsilon_i), i = 1, \dots, n\}$ are independent and identically distributed random variables.

(C2) $\epsilon_i \perp A_i \mid X_i$ for all $i = 1, \dots, n$.

(C3) $E|\Delta C(X_i;\beta)| < \infty$ for all $\beta \in \mathcal{B}$.

(C4) $\mathrm{pr}\{x \in \mathcal{X} : \ \Delta C(x;\beta) \neq 0\} > 0$ for all $\beta \neq \beta_0$.

(C5) $E|\varphi(X_i;\gamma)| < \infty$ for all $\gamma \in \Gamma$.

(C6) $G_2(\gamma)$ has a unique minimizer γ^*, where $G_2(\gamma)$ is the pointwise limit of $L_n(\beta_0,\gamma) - L_n(\beta_0,\gamma')$ in probability.

(C7) $L_n(\beta,\gamma)$ is strictly convex with respect to (β,γ).

(C8) $\epsilon \mid X = x$ has a nonzero density on $\mathbb{R}$ for almost all $x \in \mathcal{X}$.

Lemma 1. $\left|\rho_\tau(x-y) - \rho_\tau(x)\right| \leq |y|$, for all $\tau \in (0,1)$.

Proof of Lemma 1. We have

$$\begin{aligned}\left|\rho_\tau(x-y) - \rho_\tau(x)\right| &= \left|\tau\left\{(x-y)_+ - x_+\right\} + (1-\tau)\left\{(x-y)_- - x_-\right\}\right| \\ &\leq |(x-y)_+ - x_+| + |(x-y)_- - x_-| = |y|.\end{aligned}$$

Thus, Lemma 1 is proved. □

Lemma 2.

$$\begin{aligned}\rho_\tau(x-y) - \rho_\tau(x) =& -\tau y I\{x \geq 0\} + (1-\tau) y I\{x < 0\} + (y-x)I\{x \geq 0\}I\{y > x\} \\ &+ (x-y)I\{x < 0\}I\{y < x\},\end{aligned}$$

for all $\tau \in (0,1)$.

Proof of Lemma 2. Denote $D = \rho_\tau(x-y) - \rho_\tau(x)$.

1. If $x \geq 0$, $y \leq 0 \Rightarrow D = -\tau y$;
2. If $x \geq 0$, $y > 0$, $|x| \geq |y| \Rightarrow D = -\tau y$;
3. If $x \geq 0$, $y > 0$, $|x| < |y| \Rightarrow D = -\tau y + (y - x)$;
4. If $x < 0$, $y \geq 0 \Rightarrow D = (1-\tau)y$;
5. If $x < 0$, $y < 0$, $|x| \geq |y| \Rightarrow D = (1-\tau)y$;
6. If $x < 0$, $y < 0$, $|x| < |y| \Rightarrow D = (1-\tau)y + (x-y)$;

Combining the above 6 cases, Lemma 2 is proved. □

Proof of Theorem 1. Recall that the loss function defined in (2) of the main paper takes the form

$$L_n(\beta,\gamma) = \frac{1}{n}\sum_{i=1}^{n} \rho_\tau\left[\varphi_0(X_i) - \varphi(X_i;\gamma) + \epsilon_i - \{A_i - \pi(X_i)\}\Delta C(X_i;\beta)\right].$$

By definition,

$$\begin{aligned}(\hat\beta^R_{\rho(\tau)}, \hat\gamma^R_{\rho(\tau)}) =& \mathrm{argmin}_{(\beta,\gamma)}\{L_n(\beta,\gamma) - L_n(\beta_0,\gamma')\}\\ =& \mathrm{argmin}_{(\beta,\gamma)}[\left\{L_n(\beta,\gamma) - L_n(\beta_0,\gamma)\right\} + \left\{L_n(\beta_0,\gamma) - L_n(\beta_0,\gamma')\right\}],\end{aligned}$$

Define

$$\begin{aligned}S_{1n}(\beta,\gamma) =& L_n(\beta,\gamma) - L_n(\beta_0,\gamma) = 1/n\sum_{i=1}^{n} d_{1i};\\ S_{2n}(\beta,\gamma) =& L_n(\beta_0,\gamma) - L_n(\beta_0,\gamma') = 1/n\sum_{i=1}^{n} d_{2i},\end{aligned}$$

where

$$\begin{aligned}d_{1i} =& \rho_\tau\left[\varphi_0(X_i) - \varphi(X_i;\gamma) + \epsilon_i - \{A_i - \pi(X_i)\}\Delta C(X_i;\beta)\right] - \rho_\tau\left\{\varphi_0(X_i) - \varphi(X_i;\gamma) + \epsilon_i\right\},\\ d_{2i} =& \rho_\tau\left\{\varphi_0(X_i) - \varphi(X_i;\gamma) + \epsilon_i\right\} - \rho_\tau\left\{\varphi_0(X_i) - \varphi(X_i;\gamma') + \epsilon_i\right\}.\end{aligned}$$

By Lemma 1, C3 and C5, $E|d_{1i}| \leq E|\{A_i - \pi(X_i)\}\Delta C(X_i;\beta)| \leq E|\Delta C(X_i;\beta)| < \infty$ and $E|d_{2i}| \leq E|\varphi(X_i;\gamma) - \varphi(X_i;\gamma')| \leq E|\varphi(X_i;\gamma)| + E|\varphi(X_i;\gamma')| < \infty$. Then, by the law of large number, for all $\beta \in \mathcal{B}$, $\gamma \in \Gamma$, we have $S_{1n}(\beta,\gamma)$ converges in probability to $G_1(\beta,\gamma)$, and $S_{2n}(\beta,\gamma)$ converges in probability to $G_2(\gamma)$, where

$$G_1(\beta,\gamma) = E(D);$$

$$D = \rho_\tau\left[\varphi_0(X) - \varphi(X;\gamma) + \epsilon - \{A - \pi(X)\}\Delta C(X;\beta)\right] - \rho_\tau\left\{\varphi_0(X) - \varphi(X;\gamma) + \epsilon\right\}.$$

Below we show that a) (β_0, γ^*) is the minimizer of $G_1(\beta,\gamma) + G_2(\gamma)$, b) (β_0, γ^*) is the unique minimizer. The consistency then follows from the argmax continuous mapping theorem under Assumption (C7).

Denote $K_1 = \varphi_0(X) - \varphi(X;\gamma) + \epsilon$, $K_2 = \{A - \pi(X)\}\Delta C(X;\beta)$. By Lemma 2,

$$\begin{aligned} D =& -\tau K_2 I\{K_1 \geq 0\} + (1-\tau)K_2 I\{K_1 < 0\} + (K_2 - K_1) I\{K_1 \geq 0\} I\{K_2 > K_1\} \\ &+ (K_1 - K_2) I\{K_1 < 0\} I\{K_2 < K_1\}. \end{aligned}$$

Since $\epsilon \perp A \mid X$ and $\mathrm{pr}(A \mid X) = \pi(X)$, applying double expectation rule with X, we have $E[-\tau K_2 I\{K_1 \geq 0\}] = E[(1-\tau)K_2 I\{K_1 < 0\}] = 0$. Thus,

$$G_1(\beta,\gamma) = E[(K_2 - K_1) I\{K_1 \geq 0\} I\{K_2 > K_1\}] + E[(K_1 - K_2) I\{K_1 < 0\} I\{K_2 < K_1\}]. \tag{A1}$$

It is easy to check $G_1(\beta,\gamma) \geq 0$ and achieves minimal value 0 at point (β_0,γ) for all $\gamma \in \Gamma$. In addition, by A6, we know $G_2(\gamma)$ has unique minimizer γ^*. Combining the above two facts, a) is proved.

Combining C4, C8 and (A1), we could prove $G_1(\beta,\gamma) > 0$ for all $\beta \neq \beta_0$ and $\gamma \in \Gamma$. So b) holds. □

Proof of Theorem 3. (a) When $M(x) = H_\alpha(x)$, the proof follows similar steps as Theorem 1. The only difference is that $G_1(\beta,\gamma)$ takes a different expression now and we need to redo the proof of 1) $G_1(\beta,\gamma) > 0$ for all $\beta \neq \beta_0, \gamma \in \Gamma$, and 2) $G_1(\beta_0,\gamma) = 0$ for all $\gamma \in \Gamma$. By definition, $G_1(\beta,\gamma) \triangleq E(D)$, where

$$D = H_\alpha \left[\varphi_0(X) - \varphi(X;\gamma) + \epsilon - \{A - \pi(X)\}\Delta C(X;\beta)\right] - H_\alpha \left\{\varphi_0(X) - \varphi(X;\gamma) + \epsilon\right\}.$$

Then, 2) holds immediately.

Denote $K_1 = \varphi_0(X) - \varphi(X;\gamma) + \epsilon$, $K_2 = \{A - \pi(X)\}\Delta C(X;\beta)$. We have the following four cases:

1. If $K_1 > \alpha$ then $H_\alpha(K_1 - K_2) \geq \alpha(K_1 - K_2) - 0.5\alpha^2$. Thus, $D \geq -\alpha K_2$;
2. If $K_1 < -\alpha$ then $H_\alpha(K_1 - K_2) \geq \alpha(K_2 - K_1) - 0.5\alpha^2$. Thus, $D \geq \alpha K_2$;
3. If $K_1 \in [-\alpha,\alpha]$ and $K_1 - K_2 \in [-\alpha,\alpha]$ then
$$D = 1/2(K_1 - K_2)^2 - 1/2K_1^2 = -K_1K_2 + 1/2K_2^2;$$
4. If $K_1 \in [-\alpha,\alpha]$ and $K_1 - K_2 \notin [-\alpha,\alpha]$ then
$$\begin{aligned} H_\alpha(K_1 - K_2) \geq& 1/2(K_1 - K_2)^2 - \left\{1/2(\alpha + |K_2|)^2 - \left[\alpha(\alpha + |K_2|) - 1/2\alpha^2\right]\right\} \\ =& 1/2(K_1 - K_2)^2 - 1/2K_1^2. \end{aligned}$$
Thus, $D \geq 1/2(K_1 - K_2)^2 - 1/2K_1^2 - 1/2K_2^2 = -K_1K_2$.

Combining the above four equalities and inequalities,

$$\begin{aligned} G_1(\beta,\gamma) \geq & E[-\alpha K_2 I\{K_1 > \alpha\}] + E[\alpha K_2 I\{K_1 < -\alpha\}] + E[-K_1 K_2 I\{K_1 \in [-\alpha,\alpha]\}] \\ & + E\left[1/2 K_2^2 I\left(\{K_1 \in [-\alpha,\alpha]\} \cup \{K_1 - K_2 \in [-\alpha,\alpha]\}\right)\right]. \end{aligned}$$

Since $\epsilon \perp A \mid X$ and $\mathrm{pr}(A \mid X) = \pi(X)$, applying double expectation rule with X, we have $E[-\alpha K_2 I\{K_1 > \alpha\}] = E[\alpha K_2 I\{K_1 < -\alpha\}] = E[-K_1 K_2 I\{K_1 \in [-\alpha,\alpha]\}] = 0$. Thus,

$$G_1(\beta;\gamma) \geq E\left[1/2 K_2^2 I\left(\{K_1 \in [-\alpha,\alpha]\} \cup \{K_1 - K_2 \in [-\alpha,\alpha]\}\right)\right]. \tag{A2}$$

Combining (A2), C4 and C8, we can check that 1) holds. Thus, part (a) is proved.

(b) When $M(x) = J_\epsilon(x)$, similarly $D = J_\epsilon\left(K_1 - K_2\right) - J_\epsilon\left(K_1\right)$. Notice that we have the following three cases:

1. If $K_1 > \epsilon$ then $D \geq -K_2$;
2. If $K_1 < -\epsilon$ then $D \geq K_2$;
3. If $K_1 \in [-\epsilon,\epsilon]$ then $D \geq 0$;

The rest of the proof follows similar steps as part (a). □

Proof of Theorem 5. From Theorem 1, $\beta_\tau = \beta_0$. Plugging this into Theorem 4 and applying double expectation rules, we have

$$J(\tau) = E\left[f_\epsilon\left\{(\tilde{X}^{\mathrm{T}}\gamma(\tau) - \varphi_0(X) \mid X\right\}\begin{pmatrix} \pi(X)\{1-\pi(X)\}\tilde{X}\tilde{X}^{\mathrm{T}} & 0 \\ 0 & \tilde{X}\tilde{X}^{\mathrm{T}} \end{pmatrix}\right]$$

and

$$\Sigma(\tau,\tau) = E\left\{\left[\tau - I\left\{\epsilon < \tilde{X}^{\mathrm{T}}\gamma(\tau) - \varphi_0(X)\right\}\right]^2 \begin{pmatrix} \pi(X)\{1-\pi(X)\}\tilde{X}\tilde{X}^{\mathrm{T}} & 0 \\ 0 & \tilde{X}\tilde{X}^{\mathrm{T}} \end{pmatrix}\right\}.$$

Thus, $\sqrt{n}\{\hat{\beta}(\tau) - \beta_0\}$ converges in distribution to $N(0, J_{11}^{-1}(\tau)\Sigma_{11}(\tau,\tau)J_{11}^{-1}(\tau))$, where $J_{11}^{-1}(\tau)$ and $\Sigma_{11}(\tau,\tau)$ are defined as in Theorem 5.

Conditional on X, $I\left\{\epsilon < \tilde{X}^{\mathrm{T}}\gamma(\tau) - \varphi_0(X)\right\}$ is a binomial random variable with

$$p = \mathrm{pr}\left(\epsilon < \tilde{X}^{\mathrm{T}}\gamma(\tau) - \varphi_0(X)\right).$$

Then,

$$E\left(\left[\tau - I\{\epsilon < \tilde{X}^{\mathrm{T}}\gamma(\tau) - \varphi_0(X)\}\right]^2 | X\right) = (p-\tau)^2 + p(1-p) \leq \min\{\tau^2, (1-\tau)^2\} + 0.25.$$

Thus, $\Sigma_{11}(\tau,\tau) \leq \left[\min\{\tau^2,(1-\tau)^2\} + 0.25\right] E\left[\pi(X)\{1-\pi(X)\}\tilde{X}\tilde{X}^{\mathrm{T}}\right]$. □

B ADDITIONAL SIMULATION RESULTS: THE ERROR TERMS INTERACTIVE WITH THE TREATMENT

We conducted additional simulations under the condition that the error terms interactive with the treatment and the conditional independent error assumption does not hold. We consider the following model with $p = 2$

$$Y_i = 1 + 0.5\sin\{\pi(X_{i1} - X_{i2})\} + 0.25(1 + X_{i1} + 2X_{i2})^2 + \{A_i - \pi(X_i)\}\theta_0^{\mathrm{T}}\tilde{X}_i + \sigma(X_i, A_i)\epsilon_i,$$

where $X_i = (X_{i1}, X_{i2})^{\mathrm{T}}$, $\tilde{X}_i = (1, X_i^{\mathrm{T}})^{\mathrm{T}}$, $\sigma(X_i, A_i) = 1 + A_i d_0 X_{i1}^2$, $\theta_0^{\mathrm{T}} = (0.5, 2, -1)$ and X_{ik} are independent and identically distributed Uniform[-1,1]. To distinguish from the simulation study that we have done in the main paper, we call this simulation study Simulation Study II and the simulation study that we have done in the main paper Simulation Study I.

Similar to Section 4 of the main paper, we take linear forms for both the baseline and the contrast functions, where $\varphi(X;\gamma) = \gamma^{\mathrm{T}}\tilde{X}$, $C(X;\beta) = \beta^{\mathrm{T}}W$ and $W = (\tilde{X}, X_1^2, X_2^2, X_1X_2)$. We consider $d_0 = 5$, 10 or 15. The error terms ϵ_i's are independent and identically distributed from N(0,1) or Gamma(1,1)-1 distribution. Two propensity score models are considered: the constant case $\pi(X_i) = 0.5$ and the non-constant case $\pi(X_i) = \mathrm{logit}(X_{i1} - X_{i2})$. For the non-constant case, it is estimated from the data based on the standard logistic regression. Since the findings are similar, we only report the results of the constant case in Table B1 and allocate those of the non-constant case to the Supplementary Appendix B for saving space.

The simulation results for the constant and non-constant propensity scores are similar, so we focus on the result based on constant propensity score. We compare the performance of four methods: lsA-learning, robust regression with $\rho_{0.5}$ (RR($\rho_{0.5}$)), robust regression with $\rho_{0.25}$ (RR($\rho_{0.25}$)) and robust regression with Huber loss (RR(H)). We consider four different sample sizes 100, 200, 400 and 800. For each scenario, we again simulate 1000 replications. When the error terms are interactive with treatment, the true β_0 associated with g_μ^{opt} and g_τ^{opt} are different. Specifically, under our model, $\beta_0 = (\theta_0^{\mathrm{T}}, 0, 0, 0)^{\mathrm{T}}$ for g_μ^{opt}, $\beta_0 = (\theta_0^{\mathrm{T}}, d_0F_\epsilon^{-1}(0.5), 0, 0)^{\mathrm{T}}$ for $g_{0.5}^{\mathrm{opt}}$ and $\beta_0 = (\theta_0^{\mathrm{T}}, d_0F_\epsilon^{-1}(0.25), 0, 0)^{\mathrm{T}}$ for $g_{0.25}^{\mathrm{opt}}$. Consequently, the two criteria, MSE and PCD used in Simulation Study I, are no longer meaningful. So we evaluate the performance of methods in this simulation study based on value differences δ_μ, $\delta_{0.5}$ and $\delta_{0.25}$.

Based on Theorem 6 of the main paper, we can prove that $\hat{g}_{LS}^A(x)$ is consistent and converges to g_μ^{opt} as the sample size goes to infinity. This is also shown in Table B1 such that the δ_μ column for the lsA-learning method converges to 0 as the sample size increases. We also know under Normal error terms, $\delta_{0.5} = \delta_\mu$. Thus, the $\delta_{0.5}$ column for the lsA-learning method with Normal error also converges to 0. However, all other columns in Table B1 converge to a positive constant instead of 0 as the sample size goes to infinity.

Another observation from Table B1 is that RR(H) and RR($\rho_{0.5}$) perform similarly. One additional observation we have is even though lsA-learning outperform all other methods in δ_μ when the sample size is large, it may be worse than RR($\rho_{0.5}$) and

TABLE B1 Simulation results with the constant propensity score when the errors interact with treatment.

			Least Square			Pinball(0.5)			Pinball(0.25)			Huber		
Error	d_0	n	δ_μ	$\delta_{0.5}$	$\delta_{0.25}$	δ_μ	$\delta_{0.5}$	$\delta_{0.25}$	δ_μ	$\delta_{0.5}$	$\delta_{0.25}$	δ_μ	$\delta_{0.5}$	$\delta_{0.25}$
Normal	5	100	0.16	0.16	0.31	0.16	0.16	0.27	0.25	0.25	0.17	0.14	0.14	0.26
		200	0.09	0.09	0.24	0.10	0.10	0.19	0.18	0.18	0.09	0.08	0.08	0.19
		400	0.05	0.05	0.18	0.07	0.07	0.12	0.15	0.15	0.05	0.05	0.05	0.13
		800	0.02	0.02	0.14	0.05	0.05	0.09	0.14	0.14	0.04	0.03	0.03	0.09
	10	100	0.28	0.28	0.92	0.22	0.22	0.81	0.39	0.39	0.40	0.21	0.21	0.82
		200	0.19	0.19	0.85	0.15	0.15	0.71	0.33	0.33	0.28	0.13	0.13	0.72
		400	0.12	0.12	0.79	0.10	0.10	0.60	0.30	0.30	0.23	0.09	0.09	0.63
		800	0.06	0.06	0.73	0.07	0.07	0.50	0.27	0.27	0.22	0.06	0.06	0.54
	15	100	0.35	0.35	1.55	0.25	0.25	1.40	0.47	0.47	0.62	0.26	0.26	1.43
		200	0.27	0.27	1.48	0.18	0.18	1.31	0.45	0.45	0.45	0.18	0.18	1.34
		400	0.19	0.19	1.47	0.13	0.13	1.17	0.44	0.44	0.37	0.12	0.12	1.23
		800	0.12	0.12	1.39	0.09	0.09	1.03	0.41	0.41	0.35	0.08	0.08	1.07
Gamma	5	100	0.15	0.18	0.31	0.15	0.11	0.16	0.22	0.12	0.09	0.12	0.09	0.15
		200	0.09	0.12	0.26	0.10	0.06	0.10	0.18	0.08	0.05	0.08	0.05	0.09
		400	0.05	0.07	0.21	0.08	0.03	0.07	0.16	0.06	0.04	0.06	0.02	0.07
		800	0.02	0.04	0.17	0.07	0.03	0.06	0.15	0.06	0.03	0.05	0.02	0.07
	10	100	0.26	0.33	0.90	0.22	0.16	0.54	0.39	0.13	0.27	0.22	0.14	0.50
		200	0.19	0.29	0.88	0.17	0.08	0.44	0.37	0.10	0.22	0.17	0.07	0.41
		400	0.12	0.24	0.87	0.13	0.04	0.39	0.35	0.08	0.19	0.14	0.03	0.36
		800	0.06	0.17	0.78	0.12	0.03	0.37	0.33	0.07	0.19	0.13	0.02	0.35
	15	100	0.36	0.57	1.52	0.30	0.31	0.98	0.53	0.19	0.40	0.32	0.28	0.89
		200	0.28	0.53	1.51	0.22	0.19	0.81	0.55	0.16	0.29	0.24	0.16	0.71
		400	0.19	0.47	1.50	0.17	0.13	0.73	0.57	0.15	0.26	0.21	0.11	0.63
		800	0.11	0.43	1.50	0.15	0.11	0.71	0.58	0.15	0.24	0.18	0.09	0.62

Least square stands for lsA-learning. Pinball(0.5) stands for robust regression with pinball loss and parameter $\tau = 0.5$. Pinball(0.25) stands for robust regression with pinball loss and parameter $\tau = 0.25$. Huber stands for robust regression with Huber loss, where parameter α is tuned automatically with R function rlm.

RR(H) when the sample size is small. This is due to the fact that lsA-learning is inefficient under heteroscedastic or skewed errors. The last observation we have is that, overall lsA-learning, RR($\rho_{0.5}$) and RR($\rho_{0.25}$) perform best at the columns δ_μ, $\delta_{0.5}$ and $\delta_{0.25}$, respectively. The reason is given in the Remark 2 under Theorem 2, which shows that $\hat{g}^R_{\rho(\tau)}$ ($\triangleq I\{C(x; \hat{\beta}^R_{\rho(\tau)}) > 0\}$) in general approximates the unknown optimal individualized treatment rule g^{opt}_τ even when the conditional independence error assumption $\epsilon \perp A \mid X$ does not hold.

C SIMULATION RESULTS WITH EXTREME τ VALUES

We have also examined the performance of the proposed robust regression using the pinball loss RR(ρ_τ) with extreme τ values ($\tau = 0.1$ ($P(0.10)$) and 0.9 ($P(0.90)$)) for Model I of simulation study of the main paper (conditional independence error assumption is satisfied), under the constant propensity score case. The results are summarized in Table C3 . We find that when

TABLE B2 Simulation results with the non-constant propensity scores when the errors interacted with treatment.

			Least Square			**Pinball(0.5)**			**Pinball(0.25)**			**Huber**		
Error	d_0	n	δ_μ	$\delta_{0.5}$	$\delta_{0.25}$	δ_μ	$\delta_{0.5}$	$\delta_{0.25}$	δ_μ	$\delta_{0.5}$	$\delta_{0.25}$	δ_μ	$\delta_{0.5}$	$\delta_{0.25}$
Normal	5	100	0.19	0.19	0.36	0.20	0.20	0.34	0.30	0.30	0.23	0.17	0.17	0.33
		200	0.11	0.11	0.28	0.13	0.13	0.25	0.21	0.21	0.12	0.11	0.11	0.23
		400	0.06	0.06	0.21	0.08	0.08	0.17	0.17	0.17	0.06	0.06	0.06	0.16
		800	0.03	0.03	0.16	0.06	0.06	0.11	0.15	0.15	0.04	0.04	0.04	0.11
	10	100	0.29	0.29	0.93	0.24	0.24	0.88	0.44	0.44	0.50	0.24	0.24	0.88
		200	0.21	0.21	0.92	0.18	0.18	0.84	0.37	0.37	0.34	0.17	0.17	0.83
		400	0.13	0.13	0.87	0.14	0.14	0.75	0.32	0.32	0.25	0.12	0.12	0.75
		800	0.08	0.08	0.80	0.11	0.11	0.64	0.28	0.28	0.21	0.08	0.08	0.64
	15	100	0.35	0.35	1.58	0.27	0.27	1.51	0.53	0.53	0.72	0.26	0.26	1.51
		200	0.29	0.29	1.56	0.21	0.21	1.47	0.50	0.50	0.54	0.20	0.20	1.47
		400	0.21	0.21	1.58	0.17	0.17	1.37	0.48	0.48	0.39	0.15	0.15	1.38
		800	0.14	0.14	1.52	0.14	0.14	1.26	0.45	0.45	0.31	0.12	0.12	1.27
Gamma	5	100	0.18	0.21	0.34	0.20	0.17	0.24	0.28	0.18	0.14	0.18	0.15	0.21
		200	0.10	0.14	0.29	0.13	0.10	0.15	0.21	0.11	0.07	0.11	0.07	0.13
		400	0.06	0.09	0.23	0.10	0.05	0.10	0.18	0.07	0.04	0.07	0.03	0.08
		800	0.03	0.06	0.19	0.08	0.03	0.06	0.16	0.06	0.03	0.06	0.02	0.07
	10	100	0.27	0.34	0.90	0.28	0.25	0.67	0.46	0.21	0.33	0.28	0.22	0.62
		200	0.20	0.32	0.94	0.21	0.16	0.57	0.43	0.14	0.24	0.21	0.13	0.49
		400	0.13	0.27	0.92	0.16	0.09	0.46	0.38	0.10	0.18	0.15	0.06	0.39
		800	0.08	0.21	0.85	0.13	0.05	0.40	0.35	0.07	0.16	0.13	0.03	0.35
	15	100	0.34	0.55	1.49	0.33	0.37	1.09	0.59	0.25	0.46	0.33	0.33	0.99
		200	0.27	0.54	1.57	0.26	0.29	1.00	0.60	0.19	0.31	0.27	0.23	0.85
		400	0.19	0.50	1.56	0.20	0.21	0.88	0.61	0.15	0.21	0.22	0.14	0.70
		800	0.12	0.47	1.58	0.17	0.14	0.76	0.62	0.15	0.18	0.19	0.09	0.63

Least square stands for lsA-learning. Pinball(0.5) stands for robust regression with pinball loss and parameter $\tau = 0.5$. Pinball(0.25) stands for robust regression with pinball loss and parameter $\tau = 0.25$. Huber stands for robust regression with Huber loss, where parameter α is tuned automatically with R function rlm. Columns δ_μ, $\delta_{0.5}$ and $\delta_{0.25}$ are multiplied by 10.

τ is extremely small or large, the performance of the estimated individualized treatment rules (ITRs) is worse (has a smaller PCD), compared with the estimated ITRs for τ close to 0.5. The SE corresponding to the average MSE for P(0.10) and P(0.90) are also larger than those of P(0.5) and P(0.25), which indicates the estimates are less stable with extreme τ values. We note that the Huber loss works much better than the pinball loss with $\tau = 0.1$ (P(0.10)) and 0.9 (P(0.90)). When checking across different types of errors, we observe that quantile regression with extreme small or large τ works much worse with the Cauchy error than the Normal error. This is expected, since Cauchy distribution has more uncertainty in the tail than the normal distribution. Based on these observations, we do not recommend using the pinball loss with the extreme τ value in practice. A reasonable range of τ may be from 0.2 to 0.8.

TABLE C3 Simulation results for Model I with extreme τ values added.

		Homogeneous Error								
		Normal			**Log-Normal**			**Cauchy**		
n	method	MSE	PCD	$\delta_{0.5}$	MSE	PCD	$\delta_{0.5}$	MSE	PCD	$\delta_{0.5}$
100	LS	1.32 (0.040)	80.7	1.06	2.36 (0.081)	75.7	1.57		58.4	3.75
	P(0.5)	1.44 (0.042)	80.1	1.13	1.73 (0.051)	78.0	1.31	2.69 (0.077)	75.2	1.63
	P(0.25)	1.90 (0.057)	78.3	1.34	1.63 (0.051)	79.0	1.29	5.29 (0.168)	70.4	2.25
	P(0.10)	3.86 (0.122)	72.2	2.05	3.53 (0.114)	72.8	2.00	53.85 (5.883)	59.6	3.59
	P(0.90)	4.10 (0.112)	70.1	2.18	11.14 (0.328)	63.2	3.04	43.33 (3.077)	59.1	3.60
	Huber	1.15 (0.034)	81.9	0.93	1.45 (0.044)	79.9	1.13	2.61 (0.072)	74.9	1.66
200	LS	0.68 (0.021)	85.6	0.59	1.10 (0.033)	82.0	0.91		58.7	3.70
	P(0.5)	0.73 (0.021)	85.3	0.62	0.78 (0.021)	84.1	0.70	1.23 (0.037)	81.3	0.99
	P(0.25)	0.92 (0.028)	84.0	0.75	0.70 (0.023)	86.0	0.59	2.48 (0.079)	75.7	1.64
	P(0.10)	1.80 (0.058)	78.9	1.28	1.62 (0.053)	79.5	1.21	16.38 (0.598)	61.0	3.42
	P(0.90)	2.06 (0.055)	75.6	1.52	5.09 (0.136)	68.2	2.37	15.21 (0.842)	62.7	3.10
	Huber	0.58 (0.017)	86.8	0.50	0.66 (0.018)	85.5	0.58	1.24 (0.035)	80.8	1.03
400	LS	0.33 (0.009)	90.3	0.26	0.56 (0.016)	87.1	0.46		59.2	3.61
	P(0.5)	0.35 (0.010)	90.0	0.29	0.37 (0.010)	89.0	0.34	0.56 (0.016)	87.1	0.48
	P(0.25)	0.43 (0.013)	89.1	0.34	0.33 (0.010)	90.7	0.25	1.16 (0.037)	82.9	0.86
	P(0.10)	0.86 (0.027)	85.0	0.67	0.84 (0.027)	85.4	0.64	7.73 (0.268)	67.3	2.63
	P(0.90)	1.06 (0.027)	81.2	0.93	2.56 (0.068)	74.4	1.61	5.84 (0.166)	67.7	2.41
	Huber	0.28 (0.008)	91.1	0.22	0.31 (0.009)	90.2	0.27	0.58 (0.017)	86.7	0.49
800	LS	0.17 (0.005)	93.2	0.13	0.26 (0.008)	90.9	0.23		59.4	3.59
	P(0.5)	0.17 (0.005)	93.1	0.13	0.19 (0.005)	92.1	0.17	0.29 (0.009)	90.7	0.24
	P(0.25)	0.22 (0.007)	92.4	0.16	0.18 (0.006)	93.6	0.12	0.59 (0.019)	87.3	0.48
	P(0.10)	0.45 (0.014)	89.4	0.34	0.44 (0.014)	90.0	0.30	3.49 (0.110)	72.5	2.00
	P(0.90)	0.51 (0.013)	86.1	0.52	1.21 (0.031)	79.6	1.06	2.81 (0.073)	73.2	1.74
	Huber	0.14 (0.004)	93.8	0.11	0.16 (0.005)	93.1	0.14	0.29 (0.008)	90.5	0.25
		Heterogeneous Error								
		Normal			**Log-Normal**			**Cauchy**		
n	method	MSE	PCD	$\delta_{0.5}$	MSE	PCD	$\delta_{0.5}$	MSE	PCD	$\delta_{0.5}$
100	LS	3.24 (0.110)	74.7	1.70	8.98 (0.561)	68.6	2.44		56.2	4.05
	P(0.5)	1.70 (0.060)	80.5	1.08	1.80 (0.064)	80.1	1.08	3.45 (0.124)	75.1	1.69
	P(0.25)	2.50 (0.085)	77.4	1.42	2.51 (0.079)	76.8	1.46	9.13 (0.341)	67.2	2.66
	P(0.10)	7.54 (0.243)	67.4	2.58	7.43 (0.231)	67.3	2.61	157.12 (20.111)	56.8	3.95
	P(0.90)	10.01 (0.323)	64.9	2.81	27.29 (1.178)	60.4	3.43	127.34 (14.213)	56.9	3.90
	Huber	1.70 (0.057)	80.4	1.10	1.87 (0.063)	79.2	1.16	4.27 (0.155)	72.8	1.93
200	LS	1.54 (0.050)	80.6	1.06	4.71 (0.244)	73.4	1.85		55.2	4.17
	P(0.5)	0.78 (0.028)	86.7	0.53	0.90 (0.032)	85.3	0.63	1.49 (0.052)	81.9	0.95
	P(0.25)	1.16 (0.039)	83.5	0.81	1.23 (0.039)	82.0	0.91	3.95 (0.150)	73.2	1.90
	P(0.10)	3.75 (0.121)	72.6	1.96	4.03 (0.114)	70.7	2.17	37.84 (1.772)	59.1	3.66
	P(0.90)	4.66 (0.133)	69.9	2.18	12.02 (0.384)	64.1	2.92	33.40 (1.372)	60.0	3.47
	Huber	0.77 (0.025)	86.4	0.55	0.94 (0.032)	84.5	0.69	1.94 (0.071)	79.3	1.19
400	LS	0.80 (0.026)	86.0	0.58	2.69 (0.136)	77.8	1.34		54.7	4.26
	P(0.5)	0.39 (0.013)	90.5	0.27	0.44 (0.017)	89.6	0.32	0.71 (0.024)	86.9	0.50
	P(0.25)	0.56 (0.019)	88.8	0.37	0.66 (0.020)	86.9	0.50	1.70 (0.055)	79.6	1.17
	P(0.10)	1.87 (0.055)	78.5	1.28	2.00 (0.054)	77.6	1.35	14.40 (0.470)	61.0	3.41
	P(0.90)	2.24 (0.061)	75.5	1.51	5.84 (0.180)	68.5	2.32	13.98 (0.456)	62.5	3.12
	Huber	0.38 (0.012)	90.4	0.27	0.48 (0.017)	88.8	0.36	0.91 (0.029)	84.9	0.65
800	LS	0.41 (0.013)	89.9	0.29	1.35 (0.150)	83.1	0.82		56.5	4.00
	P(0.5)	0.18 (0.006)	93.6	0.12	0.20 (0.007)	92.6	0.16	0.36 (0.013)	91.0	0.25
	P(0.25)	0.28 (0.009)	92.2	0.18	0.31 (0.010)	90.8	0.24	0.89 (0.031)	85.8	0.60
	P(0.10)	0.91 (0.029)	84.5	0.69	0.99 (0.027)	82.9	0.82	7.16 (0.222)	66.6	2.72
	P(0.90)	1.13 (0.030)	80.9	0.94	2.71 (0.073)	74.1	1.64	6.49 (0.178)	67.2	2.50
	Huber	0.19 (0.006)	93.3	0.13	0.22 (0.007)	92.1	0.18	0.47 (0.017)	89.2	0.34

D COMPARISONS WITH THE METHOD THAT MAXIMIZES THE MARGINAL QUANTILE

We have compared our proposed method with the method of Wang et al[1], which maximizes the marginal quantile by directly optimizing an estimate of the marginal quantile-based value function $\tilde{V}_\tau(g) = \inf\{y : F_{Y_i^*(g)}(y) \geq \tau\}$, and does not require to specify an outcome regression model. We directly use the R package *quantoptr*, which is developed by the authors, for computing the ITR that maximizes the marginal quantiles. The method of Wang et al[1] is denoted by MAR_P(0.25) and MAR_P(0.5), respectively for $\tau = 0.25$ or 0.5. All results in this section are based on 50 replications.

We first consider Model I of Simulation Study I where the error terms are independent of the treatment. Under this setting, it is easy to verify $g_\mu^{\text{opt}} = g_\tau^{\text{opt}} = g_\tau^{\text{opt-mar}} = I\{\beta_0^{\mathrm{T}} \tilde{X}_i > 0\}$, where $g_\tau^{\text{opt-mar}}$ is the optimal individualized treatment rule which maximizes the τth marginal quantile ($g_\tau^{\text{opt-mar}} = \text{argmax}_g \tilde{V}_\tau(g)$). The results are shown in Table D4 . We find that our robust regression method generally gives better individualized treatment rules than Wang et al's method in terms of PCD and value, especially when the sample size is relatively large ($n \geq 200$). This is expected since our robust regression method make more use of the model information.

Next, we consider the Simulation Study II where the error terms are interactive with the treatment. It is easy to verify that under this setting, $g_\tau^{\text{opt-mar}}$ does not have an analytic form. Thus, instead of reporting the value difference $\tilde{\delta}_\tau = \tilde{V}_\tau(g_\tau^{\text{opt-mar}}) - \tilde{V}_\tau(\hat{g})$, we report $\tilde{V}_\tau(\hat{g})$, where a large value is more desirable. The results are given in Table D5 . We find that MAR_P(0.5) and MAR_P(0.25) give the largest $\tilde{V}_{0.5}(\hat{g})$ and the $\tilde{V}_{0.25}(\hat{g})$ values, respectively, as expected. However, they are much worse than P(0.5) in terms of $\delta_{0.5}$ or P(0.25) in terms of $\delta_{0.25}$. In summary, when the conditional independence error assumption is not satisfied, the optimal individualized treatment rules that maximize the conditional quantile and that maximize the marginal quantile are different, and each method maximizes their own target value function.

E TWO TOY EXAMPLES: SHOWING DIFFERENCES BETWEEN ITRS THAT MAXIMIZES CONDITIONAL AND MARGINAL QUANTILES

In the following, we use two toy examples to illustrate the differences between ITRs that maximize conditional and marginal quantiles, and show that the individualized treatment rule that maximizes the marginal quantile may not be the optimal choice for an individual patient.

Consider a continuous outcome variable Y, a binary treatment indicator $A = 1/0$, and a binary covariate X taking value 1 or 0. Let $g(X)$ denote the optimal ITR. In the first example we consider the following data distribution: when $X = 0$, Y follows a uniform distribution $U(1,2)$ if $A = 0$, otherwise Y follows $U(0,1)$ if $A = 1$; when $X = 1$, Y follows $U(3,4)$ if $A = 0$, otherwise

TABLE D4 Comparisons with the method that maximizes marginal quantiles: Model I of Simulation Study I.

		Homogeneous Error								
		Normal			Log-Normal			Cauchy		
n	method	MSE	PCD	$\delta_{0.5}$	MSE	PCD	$\delta_{0.5}$	MSE	PCD	$\delta_{0.5}$
100	LS	1.69 (0.206)	78.2	1.35	2.51 (0.317)	75.6	1.56		58.1	3.73
	P(0.5)	2.28 (0.339)	74.9	1.76	1.86 (0.233)	78.0	1.28	2.47 (0.289)	74.4	1.74
	P(0.25)	2.39 (0.338)	75.0	1.65	1.52 (0.183)	80.1	1.11	5.89 (0.871)	66.7	2.69
	MAR_P(0.5)	1.28 (0.137)	76.9	1.40	1.63 (0.164)	72.4	1.89	1.92 (0.223)	69.7	2.20
	MAR_P(0.25)	1.61 (0.178)	73.6	1.85	1.34 (0.161)	76.1	1.53	1.90 (0.206)	68.7	2.41
	Huber	1.76 (0.246)	77.6	1.42	1.59 (0.229)	79.9	1.05	2.72 (0.344)	74.5	1.76
200	LS	0.66 (0.091)	86.8	0.48	0.92 (0.110)	82.6	0.80		61.0	3.38
	P(0.5)	0.73 (0.106)	84.6	0.64	0.82 (0.088)	84.3	0.66	1.29 (0.154)	82.0	0.92
	P(0.25)	0.95 (0.137)	84.3	0.71	0.61 (0.081)	87.1	0.48	2.69 (0.374)	77.2	1.51
	MAR_P(0.5)	1.07 (0.122)	80.7	1.01	1.23 (0.131)	76.9	1.29	1.34 (0.138)	77.3	1.35
	MAR_P(0.25)	0.96 (0.075)	81.4	0.96	0.89 (0.069)	82.8	0.81	1.67 (0.234)	75.0	1.76
	Huber	0.60 (0.089)	86.7	0.48	0.66 (0.082)	86.0	0.55	1.30 (0.152)	82.4	0.88
400	LS	0.39 (0.043)	89.4	0.31	0.58 (0.078)	87.1	0.43		55.7	4.13
	P(0.5)	0.40 (0.046)	89.7	0.28	0.42 (0.056)	89.3	0.31	0.61 (0.080)	86.7	0.53
	P(0.25)	0.39 (0.044)	88.3	0.38	0.26 (0.040)	91.9	0.19	1.11 (0.142)	81.4	0.97
	MAR_P(0.5)	0.76 (0.024)	85.5	0.54	0.90 (0.046)	82.2	0.79	1.07 (0.115)	80.5	1.02
	MAR_P(0.25)	0.84 (0.057)	84.6	0.62	0.71 (0.027)	86.8	0.48	1.19 (0.153)	78.3	1.32
	Huber	0.30 (0.034)	90.7	0.23	0.35 (0.052)	90.8	0.23	0.68 (0.087)	86.8	0.49
800	LS	0.14 (0.016)	93.6	0.11	0.27 (0.033)	91.4	0.20		61.1	3.41
	P(0.5)	0.14 (0.017)	93.5	0.12	0.19 (0.020)	92.1	0.17	0.28 (0.036)	91.1	0.24
	P(0.25)	0.25 (0.037)	92.6	0.15	0.14 (0.018)	93.9	0.10	0.58 (0.080)	86.9	0.50
	MAR_P(0.5)	0.68 (0.017)	87.7	0.38	0.73 (0.026)	86.1	0.51	0.75 (0.041)	86.3	0.52
	MAR_P(0.25)	0.66 (0.016)	89.3	0.30	0.62 (0.009)	90.7	0.23	0.81 (0.056)	84.9	0.63
	Huber	0.13 (0.014)	94.0	0.10	0.16 (0.017)	92.9	0.13	0.28 (0.038)	90.9	0.23
		Heterogeneous Error								
		Normal			**Log-Normal**			**Cauchy**		
n	method	MSE	PCD	$\delta_{0.5}$	MSE	PCD	$\delta_{0.5}$	MSE	PCD	$\delta_{0.5}$
100	LS	3.36 (0.537)	74.1	1.76	8.91 (1.685)	65.5	2.87		49.6	4.90
	P(0.5)	1.90 (0.273)	80.4	1.08	2.25 (0.328)	78.6	1.24	3.44 (0.466)	75.9	1.59
	P(0.25)	2.65 (0.326)	77.9	1.23	2.85 (0.320)	74.6	1.66	7.85 (1.385)	68.1	2.62
	MAR_P(0.5)	1.44 (0.153)	75.4	1.51	1.86 (0.177)	72.6	1.84	1.98 (0.220)	70.3	2.15
	MAR_P(0.25)	1.16 (0.120)	78.4	1.21	1.40 (0.182)	78.7	1.29	1.96 (0.215)	71.0	2.15
	Huber	1.76 (0.210)	80.3	1.04	2.27 (0.290)	77.1	1.38	4.15 (0.571)	72.6	2.01
200	LS	1.34 (0.210)	79.6	1.14	4.77 (0.968)	72.7	1.93		56.6	4.05
	P(0.5)	0.83 (0.152)	85.2	0.65	0.90 (0.117)	84.0	0.74	1.59 (0.299)	82.6	0.86
	P(0.25)	1.21 (0.227)	80.2	1.11	1.09 (0.123)	79.6	1.09	5.71 (1.071)	71.1	2.28
	MAR_P(0.5)	1.07 (0.113)	81.2	0.96	1.36 (0.125)	77.4	1.25	1.36 (0.140)	77.6	1.32
	MAR_P(0.25)	0.96 (0.075)	81.9	0.87	1.12 (0.105)	80.5	1.04	1.85 (0.227)	72.0	2.07
	Huber	0.79 (0.135)	84.6	0.67	0.95 (0.123)	83.2	0.80	2.38 (0.452)	79.1	1.26
400	LS	0.89 (0.142)	85.7	0.70	1.76 (0.270)	79.3	1.15		50.8	4.72
	P(0.5)	0.36 (0.072)	91.2	0.25	0.37 (0.076)	90.1	0.27	0.69 (0.122)	87.0	0.49
	P(0.25)	0.55 (0.103)	88.9	0.41	0.58 (0.102)	88.2	0.39	1.68 (0.297)	79.8	1.21
	MAR_P(0.5)	0.94 (0.104)	83.9	0.72	0.89 (0.067)	83.4	0.70	1.00 (0.089)	82.6	0.79
	MAR_P(0.25)	0.75 (0.046)	86.8	0.48	0.70 (0.028)	87.9	0.39	1.36 (0.174)	78.5	1.31
	Huber	0.43 (0.078)	90.7	0.27	0.44 (0.086)	89.7	0.30	0.84 (0.111)	84.6	0.67
800	LS	0.27 (0.024)	90.7	0.23	1.16 (0.291)	83.7	0.75		56.3	4.00
	P(0.5)	0.14 (0.021)	94.2	0.10	0.21 (0.035)	92.3	0.17	0.50 (0.083)	90.6	0.30
	P(0.25)	0.20 (0.027)	93.0	0.14	0.32 (0.049)	90.7	0.22	1.12 (0.197)	84.1	0.75
	MAR_P(0.5)	0.66 (0.029)	90.0	0.27	0.76 (0.030)	86.2	0.49	0.89 (0.082)	84.2	0.65
	MAR_P(0.25)	0.63 (0.016)	90.6	0.25	0.69 (0.029)	88.6	0.35	0.90 (0.094)	85.2	0.64
	Huber	0.14 (0.019)	94.1	0.10	0.24 (0.034)	91.6	0.20	0.62 (0.098)	88.1	0.45

TABLE D5 Comparisons with the method that maximizes marginal quantiles: Simulation Study II.

			LS					P(0.5)					P(0.25)					MAR_P(0.5)					MAR_P(0.25)				
Error	d_0	n	δ_μ	$\delta_{0.5}$	$\delta_{0.25}$	$\tilde{V}_{0.5}$	$\tilde{V}_{0.25}$	δ_μ	$\delta_{0.5}$	$\delta_{0.25}$	$\tilde{V}_{0.5}$	$\tilde{V}_{0.25}$	δ_μ	$\delta_{0.5}$	$\delta_{0.25}$	$\tilde{V}_{0.5}$	$\tilde{V}_{0.25}$	δ_μ	$\delta_{0.5}$	$\delta_{0.25}$	$\tilde{V}_{0.5}$	$\tilde{V}_{0.25}$	δ_μ	$\delta_{0.5}$	$\delta_{0.25}$	$\tilde{V}_{0.5}$	$\tilde{V}_{0.25}$
Normal	5	100	0.18	0.18	1.27	1.95	0.80	0.18	0.18	1.22	1.97	0.84	0.27	0.27	1.11	1.86	0.83	0.23	0.23	1.43	1.97	0.71	0.28	0.28	1.26	1.90	0.78
		200	0.07	0.07	1.18	2.02	0.87	0.12	0.12	1.17	2.01	0.88	0.16	0.16	1.03	1.97	0.93	0.19	0.19	1.36	2.02	0.76	0.20	0.20	1.17	1.99	0.87
		400	0.04	0.04	1.13	2.03	0.89	0.06	0.06	1.07	2.03	0.94	0.16	0.16	1.01	1.98	0.96	0.17	0.17	1.41	2.06	0.77	0.19	0.19	1.12	2.00	0.92
		800	0.02	0.02	1.13	2.03	0.89	0.05	0.05	1.05	2.04	0.95	0.14	0.14	1.00	1.99	0.97	0.15	0.15	1.41	2.10	0.79	0.18	0.18	1.05	2.02	0.97
	10	100	0.24	0.24	1.65	1.82	0.52	0.20	0.20	1.58	1.91	0.60	0.37	0.37	1.15	1.78	0.68	0.28	0.28	2.00	1.93	0.38	0.29	0.29	1.44	1.92	0.65
		200	0.23	0.23	1.77	1.86	0.47	0.14	0.14	1.54	1.97	0.65	0.32	0.32	1.11	1.82	0.74	0.25	0.25	2.06	2.05	0.37	0.23	0.23	1.30	1.96	0.75
		400	0.12	0.12	1.66	1.94	0.57	0.10	0.10	1.41	2.00	0.73	0.32	0.32	1.04	1.82	0.77	0.21	0.21	1.95	2.10	0.47	0.25	0.25	1.17	1.98	0.84
		800	0.07	0.07	1.53	1.96	0.66	0.07	0.07	1.31	1.99	0.78	0.27	0.27	1.03	1.88	0.84	0.19	0.19	2.02	2.15	0.46	0.21	0.21	1.16	2.00	0.87
Gamma	5	100	0.17	0.18	0.30	1.63	0.76	0.15	0.11	0.15	1.70	0.86	0.24	0.14	0.10	1.66	0.84	0.26	0.28	0.40	1.64	0.68	0.28	0.22	0.22	1.65	0.78
		200	0.07	0.09	0.20	1.69	0.85	0.11	0.06	0.10	1.73	0.89	0.19	0.08	0.05	1.71	0.91	0.23	0.22	0.32	1.71	0.78	0.25	0.16	0.15	1.69	0.86
		400	0.04	0.07	0.20	1.71	0.86	0.09	0.03	0.06	1.75	0.93	0.18	0.07	0.03	1.73	0.93	0.19	0.23	0.37	1.73	0.78	0.22	0.13	0.12	1.71	0.90
		800	0.02	0.04	0.16	1.71	0.88	0.08	0.03	0.06	1.75	0.93	0.15	0.06	0.03	1.74	0.93	0.15	0.16	0.27	1.76	0.84	0.18	0.08	0.06	1.74	0.93
	10	100	0.27	0.37	0.96	1.46	0.36	0.24	0.16	0.53	1.58	0.62	0.41	0.15	0.27	1.53	0.64	0.32	0.44	1.06	1.53	0.29	0.38	0.23	0.51	1.58	0.61
		200	0.20	0.29	0.86	1.52	0.45	0.22	0.11	0.44	1.63	0.69	0.38	0.10	0.21	1.55	0.69	0.28	0.31	0.82	1.63	0.49	0.31	0.13	0.36	1.63	0.72
		400	0.10	0.21	0.82	1.53	0.51	0.12	0.04	0.39	1.67	0.75	0.34	0.08	0.21	1.58	0.72	0.19	0.25	0.78	1.69	0.58	0.25	0.07	0.30	1.67	0.79
		800	0.05	0.17	0.79	1.57	0.55	0.12	0.03	0.36	1.69	0.77	0.32	0.06	0.18	1.60	0.75	0.19	0.27	0.84	1.73	0.57	0.25	0.05	0.24	1.67	0.81

LS stands for lsA-learning. P(0.5) stands for robust regression with pinball loss and parameter $\tau = 0.5$. P(0.25) stands for robust regression with pinball loss and parameter $\tau = 0.25$. Huber stands for robust regression with Huber loss, where parameter α is tuned automatically with R function rlm. MAR_P(0.5) and MAR_P(0.25) stand for the individualized treatment rules of Wang et al[1] that maximize the 50% and 25% marginal quantiles, respectively.

Y follows $U(4,5)$ if $A = 1$. Furthermore, assume $\pi_1 \equiv \Pr(X = 1) \geq 0.5$. Then, the marginal CDF of $Y^*(g(X))$ is given by

$$F^*(t) = \pi_0 \left[g(0)F_1(t) + \{1 - g(0)\}F_2(t)\right] + \pi_1 \left[g(1)F_4(t) + \{1 - g(1)\}F_3(t)\right],$$

where $\pi_0 = 1 - \pi_1$, and $F_1(t)$, $F_2(t)$, $F_3(t)$ and $F_4(t)$ are the CDFs of $U(0,1)$, $U(1,2)$, $U(3,4)$ and $U(4,5)$, respectively. In this situation, the ITR maximizing the marginal quantile of $Y^*(g(X))$ with $\tau = 0.5$ is given by: $g(1) = 1$ and $g(0)$ undetermined. Here $g(0)$ is undetermined because the maximal marginal quantile of $Y^*(g(X))$ with $\tau = 0.5$ is completely determined by $g(1)$. On the other hand, the ITR maximizing the conditional quantiles (no matter which τ we use) is always: $g(1) = 1$ and $g(0) = 0$, which is the optimal treatment decision for any individual.

In the second example, we demonstrate that the ITR that maximizes the marginal quantile and that maximizes the conditional quantile may lead to completely opposite decisions. Suppose when $X = 0$, Y follows $U(9,10)$ if $A = 0$, otherwise Y follows $U(0,11)$ if $A = 1$; when $X = 1$, Y follows $U(9,10)$ if $A = 0$, otherwise Y follows $U(11,12)$ if $A = 1$. Let $\pi_1 = 0.45$. Then, the marginal CDF $F^*(t)$ of $Y^*(g(X))$ can be similarly defined. In this situation, the ITR maximizing the marginal median is given by $g(0) = g(1) = 1$, which gives the maximal median of 10. However, the ITR maximizing the conditional median is given by $g(0) = 0$ and $g(1) = 1$.

References

1. Wang L, Zhou Y, Song R, Sherwood B. Quantile-Optimal Treatment Regimes. *Journal of the American Statistical Association, accepted.* 2017;.